P7.91 A cup anemometer uses two 5-cm-diameter hollow hemispheres connected to two 15-cm rods, as in Fig. P7.91. Rod drag is neglected, and the central bearing has a retarding torque of 0.004 N·m. With simplifying assumptions, estimate and plot rotation rate $\Omega$ versus wind velocity in the range $0 < U < 25$ m/s.

Solution:

For any instantaneous angle $\theta$, as shown, the drag forces are assumed to depend on the relative velocity normal to the cup:

\[ F_{d1} = C_{D1} \frac{1}{2} \rho A (V_{Rel1})^2 = C_{D1} \frac{1}{2} \rho \left( \frac{\pi}{4} D^2 \right) (U \cos \theta - R\Omega)^2 \]

\[ F_{d2} = C_{D2} \frac{1}{2} \rho A (V_{Rel2})^2 = C_{D2} \frac{1}{2} \rho \left( \frac{\pi}{4} D^2 \right) (U \cos \theta + R\Omega)^2 \]

Where from Table 7.3 for 3D cups $C_{D1} \approx 1.4$ and $C_{D2} \approx 0.4$.

The moment around the central bearing is:

\[ M_0 = R(F_{d1} - F_{d2}) = R\rho \frac{\pi}{8} D^2 [1.4(U \cos \theta - R\Omega)^2 - 0.4(U \cos \theta + R\Omega)^2] \]

The generalized form of moment equation for U velocity with either positive or negative value would be:

\[ M_0 = R(F_{d1} - F_{d2}) = R\rho \frac{\pi}{8} D^2 [1.4(|U| \cos \theta - R\Omega)^2 - 0.4(|U| \cos \theta + R\Omega)^2] \]
Note the torque mirror itself over 90 deg increments.

\[ \overline{M_o} = \frac{2}{\pi} \int_{0}^{\pi} M_o d\theta \]

\[ = \frac{R \rho D^2}{4} \left[ 0.7 \left( \frac{\pi}{2} (2R^2 \Omega^2 + U^2) - 4R\Omega |U| \right) - 0.2 \left( \frac{\pi}{2} (2R^2 \Omega^2 + U^2) + 4R\Omega |U| \right) \right] \]

\[ \overline{M_o} = \frac{R \rho D^2}{4} \left[ \frac{\pi}{4} (2R^2 \Omega^2 + U^2) - 3.6R\Omega |U| \right] = 0.004 \]

\[ \left( \frac{R^2 \pi}{2} \right) \Omega^2 - (3.6RU|U|) \Omega + \frac{\pi}{4} U^2 - \frac{0.004}{R \rho D^2} = 0 \]

\[ \Omega = \sqrt{\frac{(3.6RU)^2 - 4 \times \left( \frac{R^2 \pi}{2} \right) \left( \frac{\pi}{4} U^2 - \frac{0.004}{R \rho D^2} \right)}{R^2 \pi}} \]

For sea-level air, take \( \rho = 1.225 \text{ kg/m}^3 \). Then:

\[ \Omega_1 = \frac{0.54|U| + \sqrt{0.18U^2 + 4.92}}{0.07} \]

\[ \Omega_2 = \frac{0.54|U| - \sqrt{0.18U^2 + 4.92}}{0.07} \]

\( \Omega_1 \) is not acceptable since \( R \Omega \) has to be less than \( U \) i.e. \( \Omega < \frac{U}{0.15} \) or \( \Omega < 6.67U \)

\[ \Omega_1 > \frac{0.54|U| + \sqrt{0.18U^2 + 4.92}}{0.07} > 13.7|U| \]

Therefore:

\[ \Omega = \frac{0.54|U| - \sqrt{0.18U^2 + 4.92}}{0.07} \]
It should be noted that anemometer turns counterclockwise as shown in the figure if:

\[ 0.54|U| \geq \sqrt{0.18U^2 + 4.92} \]

\[ 0.11U^2 \geq 4.92 \rightarrow |U| \geq 6.69 \text{ m/s} \]

The rotation rate \( \Omega \) versus wind velocity \( U \) can be plotted:

In addition, \( M_0 \) against \( \theta \) can be plotted for series of \( U \) and \( \Omega \) found from above solution. The plots will show mean value of 0.004 for \( M_0 \) as expected.