A sphere of density $\rho_s$ and diameter $D$ is dropped from rest in a fluid of density $\rho$ and viscosity $\mu$. Assuming a constant drag coefficient $C_{d_0}$, derive a differential equation for the fall velocity $V(t)$ and show that the solution is

$$V = \left[ \frac{4gD(S-1)}{3C_{d_0}} \right]^{1/2} \tanh Ct \quad C = \left[ \frac{3gC_{d_0} (S-1)}{4S^2D} \right]^{1/2}$$

where $S = \rho_s/\rho$ is the specific gravity of the sphere material.

**Solution:**

Free body diagram showing the weight, buoyancy, and drag forces:

![Free body diagram](image)

Newton’s law for downward motion:

$$\sum F = ma$$

$$W - B - F_d = \frac{W \, dV}{g \, dt}$$

$$F_d = C_{d_0} \left( \frac{1}{2} \rho A V^2 \right); \quad \text{where} \quad A = \frac{\pi}{4} D^2 \quad \Rightarrow \quad F_d = C_{d_0} \frac{1}{2} \rho \frac{\pi}{4} D^2 V^2$$

$$B = \rho g \forall \quad \text{and} \quad W = \rho_s g \forall; \quad \text{where} \quad \forall = \frac{\pi}{6} D^3 \quad \Rightarrow \quad W - B = (\rho_s - \rho) g \forall = \rho (S - 1) g \frac{\pi}{6} D^3$$

Substitute and get:

$$\rho (S - 1) g \frac{\pi}{6} D^3 - C_{d_0} \frac{1}{2} \rho \frac{\pi}{4} D^2 V^2 = S \rho \frac{\pi}{6} D^3 \frac{dV}{dt}$$
Rearrange to:

\[
\frac{dV}{dt} = g\left(1 - \frac{1}{S}\right) - \frac{3C_d}{4SD} V^2
\]

\[
\frac{dV}{dt} = \beta - \alpha V^2 \quad \text{where} \quad \beta = g\left(1 - \frac{1}{S}\right) \quad \text{and} \quad \alpha = \frac{3C_d}{4SD}
\]

Separate the variables and integrate from rest:

\[
\int_{t=0}^{t=t} dt = \int_{v(0)=0}^{v(t)} \frac{dV}{\beta - \alpha V^2}
\]

\[
V(t) = \frac{\beta}{\sqrt{\alpha}} \tanh\left(\sqrt{\alpha/\beta} \cdot t\right) = V_{\text{final}} \tanh(\sqrt{Ct})
\]

Where:

\[
V_{\text{final}} = \left[\frac{4gD(S - 1)}{3C_d}\right]^{\frac{1}{2}} \quad \text{and} \quad C = \left[\frac{3gC_d}{4S^2D}\right]^{\frac{1}{2}}
\]