An incompressible, Newtonian liquid of density $\rho$ and dynamic viscosity $\mu$ is sheared between concentric cylinders with radius $R_1$ (inner) and $R_2$ (outer) rotating at angular velocity $\omega_1$ and $\omega_2$, as shown below. (a) Simplify the momentum equation and then solve the differential equation to get the velocity profile $u_\theta$ using appropriate boundary conditions. (b) Show that for inviscid flow ($\mu=0$), the flow would be unstable for the instability condition $\frac{d\gamma}{dr} > 0$ where $\gamma = u_\theta r$.
Flow between rotating cylinders

\[ \text{Cont: } \frac{1}{r^2} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{u}{\sin \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial z^2} = 0 \]

\[ v \theta = \text{constant; } \frac{\partial \theta}{\partial z} = 0 \text{ at } z = 0 \]

\[ \frac{u_0^2}{v} = -\frac{1}{r} \frac{\partial u}{\partial z} \quad \frac{r \theta}{v} = \frac{c}{v} \]

\[ u_r(r_1) = u_r(r_2) = 0 \]

\[ 0 = r \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{u_0}{v} \right) - \frac{u_0}{v^2} \right] \implies \frac{\partial u_0}{\partial z} = 0 \]

\[ u = u_0(r) \hat{e}_\theta \quad p = p(r) \]

\[ u_0(r_1) = v_1 w_1 \quad u_0(r_2) = v_2 w_2 \]

\[ \text{Div mass radially outward due to centrifugal force (outward)} \]

\[ \frac{1}{v} \frac{d}{d\theta} \left( v \frac{du}{d\theta} \right) = \frac{c}{v} \]

\[ \frac{d^2 u}{dv^2} + \frac{d}{dv} \left[ \frac{1}{v} \frac{du}{d\theta} \right] = \frac{1}{v^2} \frac{d}{dv} \left( v \frac{du}{d\theta} \right) \]

\[ \frac{1}{v} \frac{d}{d\theta} (v u_0) = 2 A \]

\[ \frac{d}{d\theta} (v u_0) = 2 A v \]

\[ V u_0 = A v^2 + B \]

\[ A v = \text{solid body rotation} \]

\[ B v = \text{potential vortex} \]

\[ u_0(r_1) = A v_1 + B \quad v_1 = v_1 w_1 \]

\[ u_0(r_2) = A v_2 + B / v \quad v_2 = v_2 w_2 \]

\[ A v_1^2 + B = v_1^2 w_1 \]

\[ A v_2^2 + B = v_2^2 w_2 \]

\[ A (v_2^2 - v_1^2) = v_2^2 w_2 - v_1^2 w_1 \]
\[ A = \frac{v_2^2 \omega_2 - v_1^2 \omega_1}{v_2^2 - v_1^2} \]

\[ B = v_1^2 \omega_1 - v_1^2 \left[ \frac{v_2^2 \omega_2 - v_1^2 \omega_1}{v_2^2 - v_1^2} \right] \]

\[ = v_1^2 \omega_1 \left( v_2^2 - v_1^2 \right) - v_1^2 v_2^2 \omega_2 + v_1^2 v_2^2 \omega_1 \]

\[ = \frac{v_2^2 \omega_1 - v_1^2 \omega_1}{v_2^2 - v_1^2} \]  

\[ \frac{d^2}{dv} = \frac{A^2 v^4 + 2AB + B^2}{v^3} \]

\[ \frac{d}{dv} = \frac{A^2 v^2 + 2AB\alpha v - B^2}{v^3} + \frac{c}{v^4} \]

Special case:

\[ P(v = R_1 \times R_2) \text{ given } c \]

\[ (v_1, \omega_1) \to 0 \text{ } \Rightarrow \omega = \omega_2 v = \text{ constant } \]

steady rotation of cylinder filled with fluid induces solid body rotation.

\[ (v_2 \to 0, \omega_2 = 0) \text{ } \Rightarrow \omega = \frac{v_1^2 \omega_1}{v} \]

potential vortex driven by steadily cylinder with no-slip condition.
Small clearance $v_2 - v_1 \ll v_1, \quad \omega_2 = 0$

\[ \frac{v_2}{v_1} = 1 - \frac{v-v_1}{v_2-v_1} = 1 - \frac{v-v_1}{v_2-v_1} \]

more generally

\[ \eta = \frac{v-v_1}{v_2-v_1} = \frac{v_2-v_1}{v_2-v_1} \]

\[ \mu \rho \omega^2 / \rho \omega_1 \] 

\[ s = \frac{v-v_1}{v_2-v_1} \]

Recall HW example 1.49 $M_{\infty} = \sqrt{\frac{1}{2}} \frac{dF}{d\theta}$

\[ M = \frac{M (v_2 - v_1)}{2\pi \omega_1 v_1^3} \]

Viscometer

Inviscid

Centripetal Instability -- Rayleigh 1888

For $\mu = 0$: $u_0(x)$ not restricted by radial viscosity alone, so some

\[ \frac{-u_0^2}{v} = -\frac{1}{2} T \frac{\partial \theta}{\partial r} \]

\[ \frac{\partial \theta}{\partial r} = 0 \]

Circulation

\[ \Gamma = \int \nu_0 \, d\theta = 2\pi v_0 \]

Circulation

\[ \Delta = v_1 \mu_0 = v^2 \omega \]

$\omega = \text{angular velocity}$
Next consider restriction on \( \frac{d\theta}{dt} \)

inviscid

in a steady incompressible

stream. \\

\[
K_E = \frac{1}{2} \rho \left( U_0^2 - \frac{1}{2} \rho \frac{d^2}{d^2} U_0^2 \right)
\]

\[
K_{E_A} = \frac{1}{2} \rho \left[ \left( \frac{U_1}{U_1} \right)^2 + \left( \frac{U_2}{U_1} \right)^2 \right]
\]

\[
K_{E_B} = \frac{1}{2} \rho \left[ \left( \frac{U_2}{U_1} \right)^2 + \left( \frac{U_1}{U_1} \right)^2 \right] \text{ exchange elements between } U_1 \text{ and } U_2
\]

\[
\Delta K = K_E - K_{E_A} = \frac{1}{2} \rho \left[ \frac{U_2^2 - U_1^2}{U_1^2} \right] \Delta U^2 \Delta U^2
\]

let assume \( \Delta U \) remains constant

\[
\frac{\Delta U}{U_1} \text{ for which}
\]

\[
\frac{U_2^2}{U_1^2} > \frac{U_1^2}{U_1^2} \Delta K > 0 \text{ stable or energy needed for exchange}
\]

\[
\frac{U_2^2}{U_1^2} < \frac{U_1^2}{U_1^2} \Delta K < 0 \text{ unstable or energy released is available to make disturbance grow}
\]

\[
(U_2^2 U_0^2) > (U_1^2 U_0^1)
\]

\[
(U_2^2 U_0^2) > (U_1^2 U_0^1)
\]

angular velocity \( \times \) intensity
 generally

\[ \theta(v) = \frac{1}{v^2} \frac{d}{dv} (v^2 \omega) = \frac{2}{v} \omega \frac{d}{dv} (v^2 \omega) \]

\[ \theta(v) > 0 \text{ for stability} \]

Apply

\[ u_0 = A v + B / v \]

\[ \omega = u_0 v = A + B / v^2 \]

\[ \delta = 4 A (A + B / v^2) \]

\[ A = \frac{\omega_1 (1 - \mu)}{1 - \mu / \omega_1} \]

\[ r = r / v_2 \]

\[ \gamma = r / v_2 \]

\[ \mu = \omega_2 / \omega_1 \]

\[ B = \frac{\omega_1 (1 - \mu)}{1 - \mu / \omega_1} \]

\[ \delta = -4 \omega_1^2 \gamma^4 (1 - \mu) (1 - \mu / \omega_1^2) \frac{1}{(1 - \mu)^2} \]

\[ \kappa = \frac{A v_2^2}{A \gamma} = \frac{v_2^2 / v_1^2}{1 - \mu} \]

\[ \text{For } \delta > 0 \quad \gamma \leq r \leq 1 \]

\[ \mu = \frac{\omega_2}{\omega_1} \quad \gamma^2 = \left( \frac{v_2}{v_1} \right)^2 \]

\[ v_2^2 \omega_2 > v_1^2 \omega_1 \]

For stability, outer cylinder must rotate at greater angular velocity than \( \gamma^2 \) times angular speed of the inner cylinder.
A cylinder's stable in opposite direction if \( w_2 < 0 \) and \( w_1 > 0 \) if

is always unstable.

Taylor 1923 extended analysis for viscous instability theory, which stabilizes the flow.