Chapter 8: Flow in Conduits

Entrance and developed flows

\[ \Pi_1 \text{ theorem } \Rightarrow \frac{L_e}{D} = f(Re) \]

Laminar flow: \( \text{Re} \text{crit} \sim 2000 \), i.e., for \( \text{Re} < \text{Re} \text{crit} \) laminar

\[ \text{Re} > \text{Re} \text{crit} \text{ turbulent} \]

\[ \frac{L_e}{D} = 0.06\text{Re} \quad \text{from experiments} \]

\[ L_{e\text{max}} = 0.06\text{Re} \text{crit}D \sim 138D \]

maximum \( L_e \) for laminar flow
Turbulent flow:

\[ \frac{Le}{D} \sim 4.4 \text{Re}^{1/6} \]

from experiment

<table>
<thead>
<tr>
<th>Re</th>
<th>Le/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>18</td>
</tr>
<tr>
<td>10^4</td>
<td>20</td>
</tr>
<tr>
<td>10^5</td>
<td>30</td>
</tr>
<tr>
<td>10^6</td>
<td>44</td>
</tr>
<tr>
<td>10^7</td>
<td>65</td>
</tr>
<tr>
<td>10^8</td>
<td>95</td>
</tr>
</tbody>
</table>

i.e., relatively shorter than for laminar flow

Laminar vs. Turbulent Flow

Reynolds 1883 showed difference depends on \( \text{Re} = \frac{VD}{\nu} \)
Shear-Stress Distribution Across a Pipe Section

Continuity: \( Q_1 = Q_2 = \text{constant}, \text{i.e.,} \ V_1 = V_2 \) since \( A_1 = A_2 \)

Momentum: \[ \sum F_s = \sum \rho u (V \cdot A) \]
\[ = \rho V_1 (V_1 A_1) - \rho V_2 (V_2 A_2) \]
\[ = \rho Q (V_2 - V_1) = 0 \]

\[ p A - \left( p + \frac{dp}{ds} \right) A - \Delta W \sin \alpha - \tau (2 \pi r) ds = 0 \]

\[ \Delta W = \gamma A ds \]
\[ \sin \alpha = \frac{dz}{ds} \]
\[ - \frac{dp}{ds} ds A - \gamma A ds \frac{dz}{ds} - \tau (2 \pi r) ds = 0 \]

\[ \tau = \frac{r}{2} \left[ - \frac{d}{ds} (p + \gamma z) \right] \]

\[ \tau_w = \frac{r_0}{2} \left[ - \frac{d}{ds} (p + \gamma z) \right] \]

\( \tau \) varies linearly from 0.0 at \( r = 0 \) (centerline) to \( \tau_{\max} = \tau_w \) at \( r = r_0 \) (wall), which is valid for laminar and turbulent flow.
Laminar Flow in Pipes

\[ \tau = \mu \frac{dV}{dy} = -\mu \frac{dV}{dr} = \frac{r}{2} \left[ -\frac{d}{ds} (p + \gamma z) \right] \]

\[ y = \text{wall coordinate} = r_0 - r \Rightarrow \frac{dV}{dr} = \frac{dV}{dy} \frac{dy}{dr} = -\frac{dV}{dy} \]

\[ \frac{dV}{dr} = -\frac{r^2}{2\mu} \left[ -\frac{d}{ds} (p + \gamma z) \right] \]

\[ V = -\frac{r^2}{4\mu} \left[ -\frac{d}{ds} (p + \gamma z) \right] + C \]

\[ V(r_0) = 0 \Rightarrow C = \frac{r_0^2}{4\mu} \left[ -\frac{d}{ds} (p + \gamma z) \right] \]

Exact solution to Navier-Stokes equations for laminar flow in circular pipe

\[ Q = \int V \cdot dA \]

\[ = \int_{r_0}^{r} V(r) 2\pi r dr \]

\[ dA = r dr d\theta = r dr (2\pi) \]
For a horizontal pipe,

\[
V_c = \frac{r_o^2}{4\mu} \left[ -\frac{d}{ds} (p + \gamma z) \right] = \frac{r_o^2 \Delta p}{4\mu L}
\]

where \( L = \) length of pipe \( = ds \)

\[
V(r) = \frac{r_o^2}{2\mu L} \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] = \frac{\Delta p}{4\mu L} (r_o^2 - r^2)
\]

\[
Q = \int_0^{r_o} \frac{\Delta p}{2\mu L} (r_o^2 - r^2) dr = \frac{\pi D^4 \Delta p}{128\mu L}
\]
Define friction factor \( f = \frac{8\tau_w}{\rho V^2} \) for turbulent flow and \( C_f = \frac{\tau_w}{\frac{1}{2}\rho V^2} \) for boundary layer flow.

\[
h_L = h_f = \frac{L}{\gamma} \left[ \frac{2\tau_w}{r_0} \right] = \frac{L}{\gamma} \left[ \frac{2\left( f \rho V^2 / 8 \right)}{r_0} \right] = f \frac{L}{D} \frac{V^2}{2g}
\]

Darcy – Weisbach Equation, which is valid for both laminar and turbulent flow.

Friction factor definition based on turbulent flow analysis where \( \tau_w = \tau_w(r_o, \bar{V}, \mu, \rho, k) \) thus \( n=6, m=3 \) and \( r=3 \) such that \( \Pi_{i=1,2,3} = f = \frac{8\tau_w}{\rho V^2} \),

\[
\text{Re} = \frac{\bar{V}D\rho}{\mu} = \frac{\bar{V}D}{v}, \text{ k/D; or } f=f(\text{Re}, k/D) \text{ where k=roughness height. For turbulent flow } f \text{ determined from turbulence modeling since exact solutions not known, as will be discussed next.}
\]

For laminar flow \( f \) not affected \( k \) and \( f(\text{Re}) \) determined from exact analytic solution to Navier-Stokes equations.

Exact solution:

\[
\tau_w = \frac{r_o}{2} \left[ -\frac{d}{ds} (p + \gamma z) \right] = \frac{r_o}{2} \left[ \frac{8\mu\bar{V}}{r_o^2} \right] = \frac{4\mu\bar{V}}{r_o}
\]
For laminar flow $\tau_w = \tau_w(r_o, \bar{V}, \mu)$ thus $n=4, m=3$ and $r=1$ such that

$$\pi_i = \frac{\tau_w r_o}{\mu \bar{V}} \text{constant.}$$

The constant depends on duct shape (circular, rectangular, etc.) and is referred to as Poiseuille number=$P_o$. $P_o=4$ for circular duct.

$$f = \frac{32\mu}{\rho r_o \bar{V}} = \frac{64\mu}{\rho V D} = \frac{64}{\text{Re}}$$

or

$$h_f = h_L = \frac{32\mu L \bar{V}}{\gamma D^2} \quad h_f = \text{head loss due to friction}$$

for $\Delta z=0: \Delta \rho \propto \bar{V}$ as per Hagen!
Stability and Transition

Stability: can a physical state withstand a disturbance and still return to its original state.

In fluid mechanics, there are two problems of particular interest: change in flow conditions resulting in (1) transition from one to another laminar flow; and (2) transition from laminar to turbulent flow.

(1) Example of transition from one to another laminar flow: Centrifugal instability for Couette flow between two rotating cylinders when centrifugal force > viscous force $Ta = \frac{r_i c^3 (\Omega_i^2 - \Omega_o^2)}{\nu^2} > Ta_{cr} = 1708 (c = r_0 - r_i << r_i)$, which is predicted by small-disturbance/linear stability theory.

![Diagram of Couette flow](image)
(2) Transition from laminar to turbulent flow

Not all laminar flows have different equilibrium states, but all laminar flows for sufficiently large Re become unstable and undergo transition to turbulence.

Transition: change over space and time and Re range of laminar flow into a turbulent flow.

\[ \text{Re}_c = \frac{U \delta}{\nu} \sim 1000, \delta = \text{transverse viscous thickness} \]

\[ \text{Re}_{\text{trans}} > \text{Re}_c \quad \text{with} \quad x_{\text{trans}} \sim 10-20 x_c \]

Small-disturbance/linear stability theory also predicts \( \text{Re}_c \) with some success for parallel viscous flow such as plane Couette flow, plane or pipe Poiseuille flow, boundary layers without or with pressure gradient, and free shear flows (jets, wakes, and mixing layers).

No theory for transition, but recent Direct Numerical Simualtions is helpful.

In general: \( \text{Re}_{\text{trans}} = \text{Re}_{\text{trans}}(\text{geometry, Re, pressure gradient/velocity profile shape, free stream turbulence, roughness, etc.}) \)
Criterion for Laminar or Turbulent Flow in a Pipe

Re\text{crit} \approx 2000 \quad \text{flow becomes unstable}

Re\text{trans} \approx 3000 \quad \text{flow becomes turbulent}

Re = \overline{V}D/\nu

Turbulent Flow in Pipes

Continuity and momentum:

\[ \tau(r = r_0) = \tau_w = \frac{r_o}{2} \left[ -\frac{d}{ds} (p + \gamma z) \right] \]

Energy:

\[ h_f = \frac{L}{\gamma} \left[ -\frac{d}{ds} (p + \gamma z) \right] \]

Combining:

\[ h_f = \frac{L}{r_o} \cdot \frac{2\tau_w}{r_o} \quad \text{define} \quad f = \frac{\tau_w}{\frac{1}{8}\rho\overline{V}^2} = \text{friction factor} \]

\[ h_f = f \cdot \frac{L}{D} \cdot \frac{\overline{V}^2}{2g} \quad \text{Darcy – Weisbach Equation} \]

\[ f = f(Re, k/D) = \text{still must be determined!} \]

\[ Re = \frac{\overline{V}D}{\nu} \quad k = \text{roughness} \]
Description of Turbulent Flow

Most flows in engineering are turbulent: flows over vehicles (airplane, ship, train, car), internal flows (heating and ventilation, turbo-machinery), and geophysical flows (atmosphere, ocean).

\( \mathbf{V}(x, t) \) and \( p(x, t) \) are random functions of space and time, but statistically stationary flaws such as steady and forced or dominant frequency unsteady flows display coherent features and are amendable to statistical analysis, i.e. time and space (conditional) averaging. RMS and other low-order statistical quantities can be modeled and used in conjunction with averaged equations for solving practical engineering problems.

Turbulent motions range in size from the width in the flow \( \delta \) to much smaller scales, which come progressively smaller as the \( \text{Re} = U\delta/\nu \) increases.
Fig. 1.1. A photograph of the turbulent plume from the ground test of a Titan IV rocket motor. The plume is seen to rise vertically due to buoyancy. Move to McGee and Hellingworth (1999). With permission of San Jose Mercury News.

Fig. 1.2. Planar images of concentration in a turbulent jet: (a) Re = 5,000 and (b) Re = 20,000. From Dulin and Dimotakis (1990).

Fig. 1.3. The time history of the axial component of velocity $U_1(t)$ on the centerline of a turbulent jet. From the experiment of Tong and Warhaft (1995).

Fig. 1.4. The mean axial velocity profile in a turbulent jet. The mean velocity $\langle U_1 \rangle$ is normalized by its value on the centerline, $\langle U_1 \rangle_0$; and the cross-stream (radial) coordinate $x_2$ is normalized by the distance from the nozzle $x_1$. The Reynolds number is 95,500. Adapted from Hussein, Capp, and George (1994).
Physical description:

(1) Randomness and fluctuations:

Turbulence is irregular, chaotic, and unpredictable. However, for statistically stationary flows, such as steady flows, can be analyzed using Reynolds decomposition.

\[
\bar{u} = \bar{u} + u' \quad \bar{u} = \frac{1}{T} \int_{t_0}^{t_0+T} u \, dt \quad \bar{u}' = 0 \quad \bar{u}'' = \frac{1}{T} \int_{t_0}^{t_0+T} u'' \, dt \
\]

\( \bar{u} \) = mean motion
\( u' \) = superimposed random fluctuation
\( u'' \) = Reynolds stresses; RMS = \( \sqrt{\bar{u}''} \)

Triple decomposition is used for forced or dominant frequency flows

\[
u = \bar{u} + u'' + u'
\]

Where \( u'' \) = organized component

(2) Nonlinearity

Reynolds stresses and 3D vortex stretching are direct result of nonlinear nature of turbulence. In fact, Reynolds stresses arise from nonlinear convection term after substitution of Reynolds decomposition into NS equations and time averaging.
(3) Diffusion

Large scale mixing of fluid particles greatly enhances diffusion of momentum (and heat), i.e.,

\[ \text{Reynolds Stresses: } - \rho u'_i u'_j \gg \tau_{ij} = \mu \varepsilon \]

Isotropic eddy viscosity:

\[ -u'_i u'_j = \nu e_{ij} - \frac{2}{3} \delta_{ij} k \]

(4) Vorticity/eddies/energy cascade

Turbulence is characterized by flow visualization as eddies, which vary in size from the largest \( L_\delta \) (width of flow) to the smallest. The largest eddies have velocity scale \( U \) and time scale \( L_\delta / U \). The orders of magnitude of the smallest eddies (Kolmogorov scale or inner scale) are:

\[ L_K = \text{Kolmogorov micro-scale} = \left[ \frac{\nu^3 L_\delta}{U^3} \right]^{\frac{1}{4}} = (\nu^3 / \varepsilon)^{1/4} \]

\[ L_K = \text{O(mm)} >> \text{L mean free path} = 6 \times 10^{-8} \text{ m} \]

Velocity scale = \( (\nu \varepsilon)^{1/4} = \text{O}(10^{-2} \text{ m/s}) \)

Time scale = \( (\nu / \varepsilon)^{1/2} = \text{O}(10^{-2} \text{ s}) \)

Largest eddies contain most of energy, which break up into successively smaller eddies with energy transfer to yet smaller eddies until \( L_K \) is reached and energy is dissipated at rate \( \varepsilon \) by molecular viscosity.
Richardson (1922):
L_δ  Big whorls have little whorls
Which feed on their velocity;
And little whorls have lesser whorls,
L_K  And so on to viscosity (in the molecular sense).

(5) Dissipation

\[ \ell_0 = L_δ \]
\[ u_0 = \sqrt{k} \quad k = \overline{u'^2} + \overline{v'^2} + \overline{w'^2} \]
\[ = 0 (U) \]
\[ \text{Re}_δ = u_0 \ell_0 / \nu = \text{big} \]

\[ \varepsilon = \text{rate of dissipation} = \text{energy/time} \]
\[ = \frac{u_0^2}{\tau_o} \quad \tau_o = \frac{\ell_0}{u_0} \]
\[ = \frac{u_0^3}{l_0} \quad \text{independent } \nu \]

The mathematical complexity of turbulence entirely precludes any exact analysis. A statistical theory is well developed; however, it is both beyond the scope of this course and not generally useful as a predictive tool. Since the time of Reynolds (1883) turbulent flows have been analyzed by considering the mean (time averaged) motion
and the influence of turbulence on it; that is, we separate the velocity and pressure fields into mean and fluctuating components.

It is generally assumed (following Reynolds) that the motion can be separated into a mean \((\overline{u}, \overline{v}, \overline{w}, \overline{p})\) and superimposed turbulent fluctuating \((u', v', w', p')\) components, where the mean values of the latter are 0.

\[
\begin{align*}
  u &= \overline{u} + u' \\
  v &= \overline{v} + v' \\
  w &= \overline{w} + w' \\
  p &= \overline{p} + p'
\end{align*}
\]

For compressible flow,

\[
\begin{align*}
  \rho &= \overline{\rho} + \rho' \quad \text{and} \quad T = \overline{T} + T'
\end{align*}
\]

where (for example)

\[
\overline{u} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} u \, dt
\]

and \(t_1\) sufficiently large that the average is independent of time.

Thus by definition \(\overline{u'} = 0\), etc. Also, note the following rules which apply to two dependent variables \(f\) and \(g\)

\[
\begin{align*}
  \overline{f} &= \overline{f} \\
  \overline{f + g} &= \overline{f} + \overline{g} \\
  \overline{f} \cdot \overline{g} &= \overline{f} \cdot \overline{g} \\
  \frac{\partial \overline{f}}{\partial s} &= \frac{\partial \overline{f}}{\partial s} \\
  \int f d\overline{s} &= \int f d\overline{s}
\end{align*}
\]

The most important influence of turbulence on the mean motion is an increase in the fluid stress due to what are
called the apparent stresses. Also known as Reynolds
stresses:

\[ \tau'_{ij} = -\rho u'_i u'_j \]

\[
= \begin{bmatrix}
-\rho u'^2 & -\rho u'v' & -\rho u'w' \\
-\rho u'v' & -\rho v'^2 & -\rho v'w' \\
-\rho u'w' & -\rho v'w' & -\rho w'^2 \\
\end{bmatrix}
\]

The mean-flow equations for turbulent flow are derived by
substituting \( \mathbf{V} = \overline{\mathbf{V}} + \mathbf{V}' \) into the Navier-Stokes equations
and averaging. The resulting equations, which are called
the Reynolds-averaged Navier-Stokes (RANS) equations
are:

**Continuity** \( \nabla \cdot \overline{\mathbf{V}} = 0 \) i.e. \( \nabla \cdot \overline{\mathbf{V}} = 0 \) and \( \nabla \cdot \mathbf{V}' = 0 \)

**Momentum** \( \rho \frac{D\overline{\mathbf{V}}}{Dt} + \rho \frac{\partial}{\partial x_j} (u'_i u'_j) = -\rho g \hat{k} - \nabla \overline{p} + \mu \nabla^2 \overline{\mathbf{V}} \)

or

\[
\tau_{ij} = \mu \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] - \rho u'_i u'_j
\]

\[
\tau'_{ij}
\]

\[
\begin{array}{c}
u_1 = u \\
u_2 = v \\
u_3 = w
\end{array}
\]

\[
\begin{array}{c}
x_1 = x \\
x_2 = y \\
x_3 = z
\end{array}
\]
Comments:

1) equations are for the mean flow
2) differ from laminar equations by Reynolds stress terms = $-\bar{\rho}u_iu_j$
3) influence of turbulence is to transport momentum from one point to another in a similar manner as viscosity
4) since $u_i'u_j'$ are unknown, the problem is indeterminate: the central problem of turbulent flow analysis is closure!

4 equations and $4 + 6 = 10$ unknowns

**FIGURE 5-35**
Hot-wire measurements showing turbulent velocity fluctuations: (a) typical trace of a single velocity component in a turbulent flow; (b) trace showing intermittent turbulence at the edge of a jet.

**FIGURE 5-36**
Flat-plate measurements of the fluctuating velocities $u'$ (streamwise), $w'$ (lateral), and $u'v'$ (normal), and the turbulent shear $u'v'$. [After Klebanoff (1953).]
FIGURE 8.37
The phenomenon of intermittency in a turbulent boundary layer: (a) measured intermittency factors [after Kibonoff (1955)]; (b) the superlayer interface between turbulent and nonturbulent fluid.

\[
\psi = \frac{\sqrt{v^2}}{\sqrt{u^2} - \sqrt{v^2}}
\]

Fig. 8.23. Measurement of fluctuating turbulent components in a wind tunnel, at maximum velocity \( U = 100 \text{ cm/sec} \) after Reichardt [41]

Non-streamwise of longitudinal fluctuation \( \sqrt{v^2} \), transverse fluctuation \( \sqrt{v^2} \), mean velocity \( \bar{u} \)

Fig. 8.4. Measurement of fluctuating components in a channel, after Reichardt [41]
The product \( u^2 \), the shearing stress \( \tau \), and the correlation coefficient \( \psi \)
Turbulence Modeling

Closure of the turbulent RANS equations require the determination of $-\rho u'v'$, etc. Historically, two approaches were developed: (a) eddy viscosity theories in which the Reynolds stresses are modeled directly as a function of local geometry and flow conditions; and (b) mean-flow velocity profile correlations, which model the mean-flow profile itself. The modern approaches, which are beyond the scope of this class, involve the solution for transport PDE’s for the Reynolds stresses, which are solved in conjunction with the momentum equations.

(a) eddy-viscosity: theories

$$-\rho u'v' = \mu_t \frac{\partial \bar{u}}{\partial y}$$

The problem is reduced to modeling $\mu_t$, i.e.,

$$\mu_t = \mu_t(x, \text{flow at hand})$$

Various levels of sophistication presently exist in modeling $\mu_t$

$$\mu_t = \rho V_t L_t$$

where $V_t$ and $L_t$ are based on large scale turbulent motion

The total stress is

$$\tau_{total} = (\mu + \mu_t) \frac{\partial \bar{u}}{\partial y}$$

molecular viscosity  eddy viscosity

(for high Re flow $\mu_t >> \mu$)
Mixing-length theory (Prandtl, 1920)

\[- \rho u'v' = c \rho \sqrt{u'^2} \sqrt{v'^2}\]

\[\sqrt{u'^2} = \ell_1 \frac{\partial \bar{u}}{\partial y}\]

\[\sqrt{v'^2} = \ell_2 \frac{\partial \bar{u}}{\partial y}\]

\[\Rightarrow -\rho u'v' = \rho \ell^2 \left[ \frac{\partial \bar{u}}{\partial y} \right] \frac{\partial \bar{u}}{\partial y} \frac{1}{\mu_t}\]

\[\ell = \ell(y)\]

\[\mu_t = \frac{C \rho k}{\varepsilon}\]

Known as a zero equation model since no additional PDE’s are solved, only an algebraic relation

Although mixing-length theory has provided a very useful tool for engineering analysis, it lacks generality. Therefore, more general methods have been developed.

One and two equation models
C = constant

\[ k = \text{turbulent kinetic energy} = u'^2 + v'^2 + w'^2 \]

\[ \varepsilon = \text{turbulent dissipation rate} \]

Governing PDE’s are derived for \( k \) and \( \varepsilon \) which contain terms that require additional modeling. Although more general than the zero-equation models, the \( k-\varepsilon \) model also has definite limitation; therefore, relatively recent work involves the solution of PDE’s for the Reynolds stresses themselves. Difficulty is that these contain triple correlations that are very difficult to model. Most recent work involves direct and large eddy simulation of turbulence.

(b) mean-flow velocity profile correlations

As an alternative to modeling the Reynolds stresses one can model mean flow profile directly for wall bounded flows such as pipes/channels and boundary layers. For simple 2-D flows this approach is quite good and will be used in this course. For complex and 3-D flows generally not successful. Consider the shape of a turbulent velocity profile for wall bounded flow.
Note that very near the wall $\tau_{\text{laminar}}$ must dominate since $-\rho u_i u_j = 0$ at the wall ($y = 0$) and in the outer part turbulent stress will dominate. This leads to the three-layer concept:

**Inner layer:**  viscous stress dominates

**Outer layer:**  turbulent stress dominates
Overlap layer: both types of stress important

1. **laminar sub-layer** (viscous shear dominates)

\[ \bar{u} = f(\mu, \tau_w, \rho, y) \]

Note: not \( f(\delta) \)

and \( \delta = D \) and

\( y = r_o - r \)

for pipe flow

From dimensional analysis

\[ u^+ = f(y^+) \]

law-of-the-wall

where:

\[ u^+ = \frac{\bar{u}}{u^*} \]

\[ u^* = \text{friction velocity} = \sqrt{\frac{\tau_w}{\rho}} \]

\[ y^+ = \frac{yu^*}{v} \]

very near the wall:

\[ \tau \sim \tau_w \sim \text{constant} = \mu \frac{du}{dy} \implies \bar{u} = cy \]

i.e.,

\[ u^+ = y^+ \quad 0 < y^+ < 5 \]
2. outer layer (turbulent shear dominates)

\[
\left( \bar{u}_e - \bar{u} \right)_{outer} = g(\delta, \tau_w, \rho, y)
\]

note: independent of \( \mu \) and actually also depends on \( \frac{dp}{dx} \)

From dimensional analysis

\[
\frac{\bar{u}_e - \bar{u}}{u^*} = g \left( \frac{y}{\delta} \right) \frac{\eta}{\eta}
\]

velocity defect law

3. overlap layer (viscous and turbulent shear important)

It is not that difficult to show that for both laws to overlap, \( f \) and \( g \) are logarithmic functions:

Inner region:

\[
\frac{\bar{u} - \bar{u}^*}{\nu} \frac{df}{dy^+} = \frac{u^*}{\nu} \frac{df}{dy^+}
\]

Outer region:

\[
\frac{y}{u^*} \frac{u^*}{\nu} \frac{dg}{d\eta} = \frac{y}{u^*} \frac{u^*}{\delta} \frac{dg}{d\eta}
\]

valid at large \( y^+ \) and small \( \eta \).
Therefore, both sides must equal universal constant, $\kappa^{-1}$

$$f(y^+) = \kappa^{-1} \ln y^+ + B = \frac{\bar{u}}{u^*} \quad \text{(inner variables)}$$

$$g(\eta) = \kappa^{-1} \ln \eta + A = \frac{u_e - \bar{u}}{u^*} \quad \text{(outer variables)}$$

\(\kappa\), A, and B are pure dimensionless constants

<table>
<thead>
<tr>
<th>(\kappa)</th>
<th>0.41</th>
<th>Von Karman constant</th>
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<tbody>
<tr>
<td>B</td>
<td>5.5 (or 5.0)</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>2.35</td>
<td>BL flow</td>
</tr>
<tr>
<td>0.65</td>
<td>pipe flow</td>
<td></td>
</tr>
</tbody>
</table>

Values vary somewhat depending on different exp. arrangements

The difference is due to loss of intermittency in duct flow. A = 0 means small outer layer
FIGURE 10.5
Velocity distribution for smooth pipes. [After Schlichting (36)]

\[ \frac{u}{u_*} \quad [\text{Eq. (10-19)}] \]

\[ \frac{u}{u_*} = 5.75 \log_{10} \left( \frac{x}{x_c} \right) + 5.5 \]

Range of experimental data for smooth pipes

FIGURE 9.9
Velocity distribution in a turbulent boundary layer.

\[ \frac{u}{u_*} = 5.75 \log_{10} \left( \frac{x}{x_c} \right) + 5.56 \]

Logarithmic velocity distribution

Law of the wall

Viscous sublayer

Buffer zone

Velocity defect line applies

Range of experimental data
FIGURE 9.11
Velocity distribution in a turbulent boundary layer—linear scales.

FIGURE 9.12
Velocity-defect law for boundary layers. [After Rouse (10)].
Note that the $y^+$ scale is logarithmic and thus the inner law only extends over a very small portion of $\delta$

Inner law region $< .2\delta$

And the log law encompasses most of the pipe/boundary-layer. Thus as an approximation one can simply assume

$$\frac{-u}{u^*} = \frac{1}{\kappa} \ln y^+ + B$$

is valid all across the shear layer. This is the approach used in this course for turbulent flow analysis. The approach is a good approximation for simple and 2-D flows (pipe and flat plate), but does not work for complex and 3-D flows.
FIGURE 6-5
Replot of the velocity profiles of Fig. 6-4 using inner-law variables $y^+$ and $u^+$.

FIGURE 6-6
Comparison of Spalding's inner-law expression with the pipe-flow data of Lindgren (1965).
Velocity Distribution and Resistance in Smooth Pipes

Assume log-law is valid across entire pipe

\[
\frac{\bar{u}(r)}{u^*} = \frac{1}{\kappa} \ln \left( \frac{r_o - r}{r_o} \right) + B
\]

\[
u^* = \sqrt{\frac{\tau_w}{\rho}} = \text{friction velocity}
\]

\[
\kappa = .41 \quad B = 5.0
\]

\[
\bar{V} = \frac{Q}{A} = \frac{\int_0^{r_o} \bar{u}(r) 2\pi r dr}{\pi r_o^2} = \frac{1}{2} u^* \left\{ \frac{2}{\kappa} \ln \frac{r_o u^*}{v} + 2B - \frac{3}{\kappa} \right\}
\]

\[
drop\ over\ bar: \quad \frac{V}{u^*} = 2.44 \ln \frac{r_o u^*}{v} + 1.34 = \left( \frac{\rho V^2}{\tau_o} \right)^{1/2} = \left( \frac{8}{f} \right)^{1/2}
\]

\[
\frac{1}{2} \text{Re} \left( \frac{f}{8} \right)^{1/2}
\]

\[
\frac{1}{\sqrt{f}} = 1.99 \log(\text{Re} \ f^{1/2}) - 1.02 \quad \text{Re} > 3000
\]

Since f equation is implicit, it is not easy to see dependency on \( \rho, \mu, V, \) and \( D \)

\[
f(\text{pipe}) = 0.316 \text{Re}_D^{-1/4} \quad 4000 < \text{Re}_D < 10^5
\]

Blasius (1911) power law curve fit to data
for $\Delta z=0$ (horizontal)

$$h_f = \frac{\Delta p}{\gamma} = f \frac{L V^2}{D^2 g}$$

$$\Delta p = 0.158L \rho^{3/4} \mu^{1/4} D^{-5/4} V^{7/4}$$

$$= 0.241L \rho^{3/4} \mu^{1/4} D^{-4.75} Q^{1.75}$$

laminar flow: $\Delta p = 8 \mu L Q / \pi R^4$

$\Delta p$ (turbulent) increases more sharply than $\Delta p$ (laminar) for same $Q$; therefore, increase $D$ for smaller $\Delta p$. 2D decreases $\Delta p$ by 27 for same $Q$.

$$\frac{u_{max}}{u^*} = u(r = 0) \frac{u}{u^*} = \frac{1}{\kappa} \ln \frac{r_o u^*}{\nu} + B$$

Combine with

$$\frac{V}{u_{max}} = \left(1 + 1.3 \sqrt{f}\right)^{-1}$$

$$\kappa = \frac{3}{2\sqrt{8}}$$
TABLE 10.1  EXONENTS FOR POWER-LAW EQUATION AND RATIO OF MEAN TO MAXIMUM VELOCITY

<table>
<thead>
<tr>
<th>$\text{Re} \rightarrow$</th>
<th>$4 \times 10^3$</th>
<th>$2.3 \times 10^4$</th>
<th>$1.1 \times 10^5$</th>
<th>$1.1 \times 10^6$</th>
<th>$3.2 \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \rightarrow$</td>
<td>$\frac{1}{6.0}$</td>
<td>$\frac{1}{6.6}$</td>
<td>$\frac{1}{7.0}$</td>
<td>$\frac{1}{8.8}$</td>
<td>$\frac{1}{10.0}$</td>
</tr>
<tr>
<td>$\bar{V}/V_{\text{max}}$ -&gt;</td>
<td>0.791</td>
<td>0.807</td>
<td>0.817</td>
<td>0.850</td>
<td>0.865</td>
</tr>
</tbody>
</table>


Power law fit to velocity profile:

$$\frac{\bar{U}}{U_{\text{in}}^n} = \left(1 - \frac{r}{r_o}\right)^m$$

$m = m(\text{Re})$

**Figure 8.17**  Exponent, $n$, for power-law velocity profiles.

**Figure 8.18**  Typical laminar flow and turbulent flow velocity profiles.
Viscous Distribution and Resistance – Rough Pipes

For laminar flow, effect of roughness is small; however, for turbulent flow the effect is large. Both laminar sublayer and overlap layer are affected.

Inner layer:

$$\bar{u} = \bar{u}(y, k, \rho, \tau_w)$$

not function of $\mu$ as was case for smooth pipe (or wall)

$$u^+ = u^+(y/k)$$

Outer layer: unaffected

Overlap layer:

$$u_R^+ = \frac{1}{\kappa} \ln \frac{y}{k} + \text{constant}$$

rough

$$u_S^+ = \frac{1}{\kappa} \ln y^+ + B$$

smooth

$$u_S^+ - u_R^+ = \frac{1}{\kappa} \ln k^+ + \text{constant}$$

$\Delta B(k^+)$

i.e., rough-wall velocity profile shifts downward by $\Delta B(k^+)$, which increases with $k^+$.

Three regions of flow depending on $k^+$

1. $k^+ < 5$  hydraulically smooth (no effect of roughness)
2. $5 < k^+ < 70$  transitional roughness (Re dependence)
3. $k^+ > 70$  fully rough (independent Re)
For 3, \[ \Delta B = \frac{1}{\kappa} \ln k^+ - 3.5 \] from data

\[ u^+ = \frac{1}{\kappa} \ln \frac{y}{k} + 8.5 \neq f(\text{Re}) \]

\[ \frac{V}{u^*} = 2.44 \ln \frac{D}{k} + 3.2 \]

\[ \frac{1}{f^{1/2}} = -2 \log \frac{k/D}{3.7} \]

\[ \begin{align*}
\text{Composite Log-Law} \\
\text{Smooth wall log law} \\
\frac{u^+}{B^*} = \frac{1}{\kappa} \ln y^+ + B - \Delta B(k^+) \\
B^* = 5 - \frac{1}{\kappa} \ln (1 + 0.3k^+) \quad \text{from data}
\end{align*} \]

\[ \frac{1}{f^{1/2}} = -2 \log \left[ \frac{k/D}{3.7} + \frac{2.51}{\text{Ref}^{1/2}} \right] \quad \text{Moody Diagram} \]

\[ = 1.14 - 2 \log \left( \frac{k_s}{D} + \frac{9.35}{\text{Ref}^{1/2}} \right) \]

fully rough flow
Fig. 6.12 Effect of wall roughness on turbulent pipe-flow velocity profiles: (a) logarithm-law downshift; (b) correlation with roughness

Fig. 6.11 Experimental rough-pipe velocity profiles by Scholz (1955), showing the inward shift $\Delta B$ of the logarithmic overlap layer.

FIGURE 6.12 Composite plot of the profile-shift parameter $\Delta B(k^+)$ for various roughness geometries, as compiled by Clauser (1956).

Fig. 6.13 The Moody chart for pipe friction with smooth and rough walls. (From Ref. 8, by permission of the ASME.)
FIGURE 10.7
Resistance coefficient $f$ versus $Re$ for sand-roughened pipe. [After Nikuradse (30)]

$Re^{1/2} = \frac{D^{3/2}}{v} \left( \frac{2h}{k} \right)^{1/2}$

Complete turbulence, rough pipes

Smooth pipes

Boundary material | Equivalent sand roughness, $k_s$
--- | ---
Glass, plastic | Smooth
Cooper or brass tubing | $1.5 \times 10^{-5}$ ($5 \times 10^{-5}$)
Wrought iron, steel | $4.6 \times 10^{-5}$ ($1.5 \times 10^{-4}$)
Asphalted cast iron | 0.12 ($4 \times 10^{-4}$)
Galvanized iron | 0.15 ($5 \times 10^{-4}$)
Cast iron | 0.26 ($8.5 \times 10^{-4}$)
Concrete | 0.3 to 3.0 ($10^{-3}$ to $10^{-2}$)

FIGURE 10.8
Resistance coefficient $f$ versus $Re$. Reprinted with minor variations. [After Moody (29). Reprinted with permission from the A.S.M.E.]
FIGURE 10.9
Relative roughness for various kinds of pipe. [After Moody (29). Reprinted with permission from the A.S.M.E.]
There are basically three types of problems involved with uniform flow in a single pipe:

1. Determine the head loss, given the kind and size of pipe along with the flow rate, \( Q = A \times V \)
2. Determine the flow rate, given the head, kind, and size of pipe
3. Determine the pipe diameter, given the type of pipe, head, and flow rate

1. Determine the head loss
The first problem of head loss is solved readily by obtaining \( f \) from the Moody diagram, using values of \( \text{Re} \) and \( k_s/D \) computed from the given data. The head loss \( h_f \) is then computed from the Darcy-Weisbach equation.

\[
f = f(\text{Re}_D, k_s/D)\\
\]

\[
h_f = f \frac{L}{D} \frac{V^2}{2g} = \Delta h\\
\Delta h = \left( \frac{p_1}{\gamma} + z_1 \right) - \left( \frac{p_2}{\gamma} + z_2 \right) = -\Delta \left( \frac{p}{\gamma} + z \right)\\
\]

\[
\text{Re}_D = \text{Re}_D(V, D)\\
\]

2. Determine the flow rate
The second problem of flow rate is solved by trial, using a successive approximation procedure. This is because both \( \text{Re} \) and \( f(\text{Re}) \) depend on the unknown velocity, \( V \). The solution is as follows:
1) solve for $V$ using an assumed value for $f$ and the Darcy-Weisbach equation

$$V = \left[ \frac{2gh_f}{L/D} \right]^{1/2} \cdot f^{-1/2}$$

known from given data

2) using $V$ compute $Re$

3) obtain a new value for $f = f(Re, k_s/D)$ and repeat as above until convergence

Or can use

$$Re = f^{1/2} = \frac{D^{3/2}}{\nu} \left( \frac{2gh_f}{L} \right)^{1/2}$$

table on Moody Diagram

1) compute $Re f^{1/2}$ and $k_s/D$

2) read $f$

3) solve $V$ from $h_f = f \frac{L V^2}{D 2g}$

4) $Q = VA$

3. Determine the size of the pipe

The third problem of pipe size is solved by trial, using a successive approximation procedure. This is because $h_f$, $f$, and $Q$ all depend on the unknown diameter $D$. The solution procedure is as follows:
1) solve for \( D \) using an assumed value for \( f \) and the Darcy-Weisbach equation along with the definition of \( Q \)

\[
D = \left( \frac{8LQ^2}{\pi^2 gh_f} \right)^{1/5} \cdot f^{1/5}
\]

known from given data

2) using \( D \) compute \( \text{Re} \) and \( k_s/D \)

3) obtain a new value of \( f = f(\text{Re}, k_s/D) \) and reapeat as above until convergence

**Flows at Pipe Inlets and Losses From Fittings**

For real pipe systems in addition to friction head loss these are additional so called minor losses due to

1. entrance and exit effects  
2. expansions and contractions  
3. bends, elbows, tees, and other fittings  
4. valves (open or partially closed)  

\[
\text{can be large effect}
\]

For such complex geometries we must rely on experimental data to obtain a loss coefficient
$K = \frac{h_m}{V^2} \frac{1}{2g}$

head loss due to minor losses

In general,

$K = K(\text{geometry, Re, } \varepsilon/D)$

dependence usually not known

Loss coefficient data is supplied by manufacturers and also listed in handbooks. The data are for turbulent flow conditions but seldom given in terms of Re.

Modified Energy Equation to Include Minor Losses (where $V = \bar{V}$):

$$\frac{p_1}{\gamma} + z_1 + \frac{1}{2g} \alpha_1 V_1^2 + h_p = \frac{p_2}{\gamma} + z_2 + \frac{1}{2g} \alpha_2 V_2^2 + h_t + h_f + \sum h_m$$

$h_m = K \frac{V^2}{2g}$

Note: $\Sigma h_m$ does not include pipe friction and e.g. in elbows and tees, this must be added to $h_f$. 
1. Flow in a bend. 

\[ \frac{\partial p}{\partial r} > 0 \] which is an adverse pressure gradient in \( r \) direction. The slower moving fluid near wall responds first and a swirling flow pattern results. 

This swirling flow represents an energy loss which must be added to the \( h_L \). 

Also, flow separation can result due to adverse longitudinal pressure gradients which will result in additional losses.
This shows potential flow is not a good approximate in internal flows (except possibly near entrance)

2. Valves: enormous losses

3. Entrances: depends on rounding of entrance

4. Exit (to a large reservoir): $K = 1$
   i.e., all velocity head is lost

5. Contractions and Expansions
   sudden or gradual
   theory for expansion:
   \[ h_L = \frac{(V_1 - V_2)^2}{2g} \]
   from continuity, momentum, and energy
(assuming \( p = p_1 \) in separation pockets)

\[
K_{SE} = \left( 1 - \frac{d^2}{D^2} \right)^2 = \frac{h_m}{V_1^2} \frac{1}{2g}
\]

no theory for contraction:

\[
K_{SC} = 0.42 \left( 1 - \frac{d^2}{D^2} \right)
\]

from experiment

If the contraction or expansion is gradual the losses are quite different. A gradual expansion is called a diffuser. Diffusers are designed with the intent of raising the static pressure.

\[
C_p = \frac{p_2 - p_1}{\frac{1}{2} \rho V_1^2}
\]

\[
C_{p,\text{ideal}} = 1 - \left( \frac{A_1}{A_2} \right)^2
\]

Bernoulli and continuity equation

\[
K = \frac{h_m}{V^2} = C_{p,\text{ideal}} - C_p
\]

Energy equation
Actually very complex flow and

\[ C_p = C_p \ (\text{geometry, inlet flow conditions}) \]

i.e., fully developed (long pipe) reduces \( C_p \)
thin boundary layer (short pipe) high \( C_p \)
(more uniform inlet profile)

\[ K = .5 \]
See textbook Table 8.2 for a table of the loss coefficients for pipe components.
### Table 8.2 Loss Coefficients for Various Transitions and Fittings

<table>
<thead>
<tr>
<th>Description</th>
<th>Sketch</th>
<th>Additional Data</th>
<th>$K$</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pipe entrance</td>
<td><img src="image" alt="Pipe Entrance" /></td>
<td>$r/d$</td>
<td>$K_e$</td>
<td>(2)*</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$&gt;0.2$</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>Contraction</td>
<td><img src="image" alt="Contraction" /></td>
<td>$D_2/D_1$</td>
<td>$K_C$</td>
<td>(2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\theta = 60^\circ$</td>
<td>0.08</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.20</td>
<td>0.08</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.40</td>
<td>0.07</td>
<td>0.42</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.60</td>
<td>0.06</td>
<td>0.27</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.80</td>
<td>0.06</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.90</td>
<td>0.06</td>
<td>0.10</td>
</tr>
<tr>
<td>Expansion</td>
<td><img src="image" alt="Expansion" /></td>
<td>$D_1/D_2$</td>
<td>$K_E$</td>
<td>(2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\theta = 20^\circ$</td>
<td>0.0</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.20</td>
<td>0.30</td>
<td>0.87</td>
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<tr>
<td></td>
<td></td>
<td>0.40</td>
<td>0.25</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.60</td>
<td>0.15</td>
<td>0.41</td>
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<tr>
<td></td>
<td></td>
<td>0.80</td>
<td>0.10</td>
<td>0.15</td>
</tr>
<tr>
<td>$h_L = K_C V^2/2g$</td>
<td><img src="image" alt="Expansion" /></td>
<td>$K_E$</td>
<td>$K_E$</td>
<td>(2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\theta = 180^\circ$</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$90^\circ$ miter bend</td>
<td><img src="image" alt="Miter Bend" /></td>
<td>$r/d$</td>
<td>$K_b$</td>
<td>(37)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>0.35</td>
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</tr>
<tr>
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<td></td>
<td>2</td>
<td>0.19</td>
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<td>0.28</td>
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<tr>
<td>$90^\circ$ smooth bend</td>
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<td>$K_b$</td>
<td>(37)</td>
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<tr>
<td></td>
<td></td>
<td>1</td>
<td>0.35</td>
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</tr>
<tr>
<td></td>
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<td>2</td>
<td>0.19</td>
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<td>0.16</td>
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<td>0.21</td>
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<td>8</td>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>0.32</td>
<td></td>
</tr>
<tr>
<td>Threaded pipe fittings</td>
<td><img src="image" alt="Threaded Pipes" /></td>
<td>$K_e$</td>
<td>10.0</td>
<td>(37)</td>
</tr>
<tr>
<td>Globe valve—wide open</td>
<td></td>
<td>$K_v$</td>
<td>5.0</td>
<td></td>
</tr>
<tr>
<td>Angle valve—wide open</td>
<td></td>
<td>$K_v$</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>Gate valve—half open</td>
<td></td>
<td>$K_v$</td>
<td>5.6</td>
<td></td>
</tr>
<tr>
<td>Return bend</td>
<td></td>
<td>$K_h$</td>
<td>2.2</td>
<td></td>
</tr>
<tr>
<td>Tee</td>
<td><img src="image" alt="Tee" /></td>
<td>$K_t$</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>straight-through flow</td>
<td></td>
<td>$K_t$</td>
<td>1.8</td>
<td></td>
</tr>
<tr>
<td>side-outlet flow</td>
<td></td>
<td>$K_s$</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>$90^\circ$ elbow</td>
<td></td>
<td>$K_s$</td>
<td>0.4</td>
<td></td>
</tr>
</tbody>
</table>

FIGURE 10.14
EGL and HGL at a sharp-edged pipe entrance.

FIGURE 10.15
Head losses in a pipe.