From the Central Limit Theorem we know that, for “large” n,

\[ Y = \sum_{i=1}^{n} X_i \] has approximately a Normal distribution

\[ f_Y(y) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2 \right\} \]

\[ Y = \prod_{i=1}^{n} X_i \] has approximately a Lognormal distribution

\[ f_Y(y) = \frac{1}{y \sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{\ln(y/\mu)}{\sigma} \right)^2 \right\} \]

When the minimum value of X is 0, i.e., X is nonnegative, the limiting distribution for \( T = \min \{ X_i \} \) as \( n \to \infty \) is the Weibull distribution:

- **CDF**
  \[ F_T(t) = 1 - e^{-\left(\frac{t}{\mu}\right)^k} \]

- **Mean value**
  \[ \mu_T = u \Gamma \left( 1 + \frac{1}{k} \right) \]

- **Standard deviation**
  \[ \sigma_T = u \sqrt{\Gamma \left( 1 + \frac{2}{k} \right) - \Gamma^2 \left( 1 + \frac{1}{k} \right) } \]

The “Gamma” function \( \Gamma \) is a generalization of the factorial function, defined for all \( x \geq 0 \), (not necessarily integer) and satisfies

\[ \Gamma(1+x) = x! \]

when \( x \) is a nonnegative integer.
### Table of values of $\Gamma\left(1 + \frac{1}{k}\right)$ for $k = 0.1$ through 9.9

<table>
<thead>
<tr>
<th>$k$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>$3.63 \times 10^6$</td>
<td>120.000</td>
<td>9.2610</td>
<td>3.3230</td>
<td>2.0000</td>
<td>1.5050</td>
<td>1.2660</td>
<td>1.1330</td>
<td>1.0520</td>
<td>1.0230</td>
</tr>
<tr>
<td>For example, $\Gamma\left(1 + \frac{1}{2.5}\right) = \Gamma(1.4) = 0.8873$</td>
<td></td>
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<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table of values of $\Gamma\left(1 + \frac{2}{k}\right)$ for $k = 0.1$ through 9.9

<table>
<thead>
<tr>
<th>$k$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values</td>
<td>$2.43 \times 10^{18}$</td>
<td>3628800</td>
<td>2594</td>
<td>120</td>
<td>24.0000</td>
<td>9.2610</td>
<td>5.0290</td>
<td>3.3230</td>
<td>2.4790</td>
<td>1.9670</td>
</tr>
<tr>
<td>For example, $\Gamma\left(1 + \frac{2}{2.5}\right) = \Gamma(1.8) = 0.9314$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Computing Weibull Parameters**

Given $\mu_Y$ and $\sigma_Y$, we wish to solve for the parameters $u$ & $k$:

\[
\begin{align*}
\mu_T &= u \Gamma\left(1 + \frac{1}{k}\right) \\
\sigma_T &= u \sqrt{\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)}
\end{align*}
\]

This is a system of two nonlinear equations in two unknowns $u$ & $k$, which might be solved by (for example) the **Newton-Raphson** method.

Solve for $u$ in terms of the mean $\mu_T$:

\[
\mu_T = u \Gamma\left(1 + \frac{1}{k}\right) \Rightarrow u = \frac{\mu_T}{\Gamma\left(1 + \frac{1}{k}\right)}
\]

Use this to eliminate $u$ from the second equation:

\[
\sigma_T = u \sqrt{\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)} \Rightarrow \sigma_T = \frac{\mu_T}{\Gamma\left(1 + \frac{1}{k}\right)} \sqrt{\Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)}
\]

This gives us a **single** (nonlinear) equation in a **single** variable $k$. 

\[
\sigma_T = \frac{\mu_T}{\Gamma\left(1+\frac{1}{k}\right)} \sqrt{\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)}
\]

\[
\Rightarrow \frac{\sigma_T}{\mu_T} = \frac{1}{\Gamma\left(1+\frac{1}{k}\right)} \sqrt{\frac{\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)}} - 1
\]

Thus, the coefficient of variation of the Weibull distribution is determined by \(k\) alone.

Several methods might be used to estimate the parameters \(u\) & \(k\) of the Weibull distribution:

(a) method of moments, i.e., matching the mean and standard deviation
(b) linear regression (after transforming to linear form)
(c) maximum likelihood method

Method (a) requires that we have sufficient data to compute the mean & standard deviation.
Methods (b) & (c) require a sample of observations of \(T\).
Given the coefficient of variation \( \frac{\sigma}{\mu} \), we can either

- approximate \( k \) through the use of the table (or graph), or
- use a numerical method, e.g., the Newton-Raphson or Secant method, to solve the nonlinear equation

\[
\frac{\sigma_r}{\mu_r} = \sqrt{\frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)}} - 1
\]

(The Newton-Raphson method requires derivatives, while the secant method does not!)

Let \( X_i = \) lifetime of link \( i \) of a chain \((X_i \geq 0)\)

\( T = \min \{X_1, X_2, \ldots X_n\} = \) lifetime of the chain

For large \( n \) (long chains) the distribution of \( T \) should be approximately Weibull.

Suppose that we have estimates for the mean \( \mu_T = 150 \) hours and standard deviation \( \sigma_T = 50 \) hours.

What is the probability that...

- the chain fails during its first 100 hours of use?
- the chain has not yet failed after 200 hours?

Since the lifetime of the chain is the minimum of the lifetimes (times until failure) of the individual links of the chain, and these lifetimes are each bounded below by zero, we will assume the Weibull distribution.

The coefficient of variation of the lifetime \( T \) is \( \frac{\sigma}{\mu} = \frac{50}{150} = \frac{1}{3} \)

We will use the **Secant Method**

(which, unlike the Newton-Raphson method, does not require derivatives)

\[
\frac{\sigma_r}{\mu_r} = \sqrt{\frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)}} - 1
\]

for \( k \).
Using the secant method, we get the following approximations to the value of $k$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.250000</td>
<td>7.973291</td>
</tr>
<tr>
<td>2.000000</td>
<td>0.189390</td>
</tr>
<tr>
<td>2.042579</td>
<td>0.179577</td>
</tr>
<tr>
<td>2.821777</td>
<td>0.050615</td>
</tr>
<tr>
<td>3.127600</td>
<td>0.016787</td>
</tr>
<tr>
<td>3.279356</td>
<td>0.002206</td>
</tr>
<tr>
<td>3.302318</td>
<td>0.000109</td>
</tr>
<tr>
<td>3.303517</td>
<td>0.000001</td>
</tr>
<tr>
<td>3.303525</td>
<td>0.000000</td>
</tr>
</tbody>
</table>
Given \( k=3.3035 \) and mean \( \mu_T = 150 \), we can now solve for the parameter \( u \):

\[
u = \frac{\mu_T}{\Gamma\left(1+\frac{1}{k}\right)} = \frac{150}{\Gamma(1.3027)} = \frac{150}{0.8971} = 167.21
\]

What is the probability that chain fails during its first 100 hours of use? i.e., \( P\{T \leq 100\} = F_T(100) = ? \)

What is the probability that it has not yet failed after 200 hours of use? i.e., \( P\{T \geq 200\} = 1 - F_T(200) = ? \)

\[
F_T(t) = 1 - e^{-\left(\frac{t}{u}\right)^k} = 1 - e^{-\left(\frac{t}{167.21}\right)^{3.3025}}
\]

Therefore,

\[
P\{T \leq 100\} = F_T(100) = 1 - \exp\left(\frac{100}{167.21}\right)^{3.3025}
\]

\[
= 1 - \exp(0.60529)^{3.3025} = 1 - \exp(0.19052) = 1 - 0.82653 = 0.17347
\]

That is, the chain is about 17% likely to fail during its first 100 hours of use.

\[
P\{T \geq 200\} = 1 - F_T(200) = e^{-\left(\frac{200}{167.21}\right)^{3.3025}} = 0.16423
\]

That is, the chain is about equally likely (16%) to survive its first 200 hours of use.