Generating Uniformly-Distributed Numbers

Beginning with a "seed" $X_0$, a sequence of random numbers $X_1, X_2, X_3, \ldots$ is generated by some operation

$$X_{i+1} = 6(X_i), \quad i=1,2,3,\ldots$$

Since the sequence is determined by the operator 6 and the initial "seed", the numbers are not, in fact, random, but if uniformly distributed in the interval [0,1] they will be called "pseudo-random".

Example: Congruential Method

Choose $A > 0$.

Determine $X_{i+1}$ by the operation:

$$X_{i+1} = [AX_i] \text{ Modulo } M$$

i.e., multiply $X_i$ by $A$ and divide the result by $M$. Keep the remainder of the division, and call it $X_{i+1}$.

Example: "Midsquare Technique"

Determine $X_{i+1}$ as follows:

Compute $X_i^2$, discard the final 2 digits, and take the last 4 digits of the remaining number:

$$\begin{array}{c|c|c|c}
  i & X_i & X_i^2 \\
  \hline
  0 & 1912 & 05655744 \\
  1 & 6557 & 42994249 \\
  2 & 9942 & 95843364 \\
  3 & 6433 & 71115489 \\
  4 & 1154 & 01351716 \\
\end{array}$$

Example: $X_{i+1} = [AX_i] \text{ Modulo } M$

Select $X_0 = 5$.

A = 5.

M = 17

$$\begin{array}{c|c|c|c|c|c|c}
  i & X_i & AX_i & [AX_i + 1] & \text{remainder} \\
  \hline
  0 & 5 & 25 & 1 & 6 \\
  1 & 6 & 40 & 2 & 6 \\
  2 & 6 & 30 & 1 & 13 \\
  3 & 13 & 65 & 3 & 14 \\
  4 & 14 & 70 & 4 & 2 \\
  5 & 2 & 10 & 0 & 10 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}$$

Usually, $M = 2^n - 1$, where $B = \# \text{ bits/word for the computer used}$

E.g., for IBM system 360, the choice was

$$M = 2^{32} - 1$$

$A = X_0 = 3^{19} = 65539$

(with these values, the sequence repeats after $2^{32}$ numbers!)
**Inverse Transformation Method**

F(x) = cumulative distribution function (CDF) of probability distribution to be simulated

R = random variable uniformly distributed in the interval [0, 1]

Randomly generate R (uniformly dist'd in [0, 1])

Find X such that F(X) = R,

i.e., \( X = F^{-1}(R) \)

> Inverse of CDF

X generated in this way has the desired distribution

**Example**

**Exponential Distribution**

\( F(x) = 1 - e^{-\lambda x} \) for \( x \geq 0 \)

\[
F(x) = 1 - e^{-\lambda x} = R
\]

\( e^{-\lambda x} = 1 - R = \overline{R} \)

\( -\lambda x = \ln \overline{R} \)

\( x = -\frac{\ln \overline{R}}{\lambda} \)

Suppose that we wish to simulate a Poisson process with \( \lambda = 2/\text{hr} \). For this purpose, we need to randomly generate the arrival times \( t_1, t_2, t_3, \ldots \)

The time between arrivals will have the exponential distribution with parameter \( \lambda = 2/\text{hr} \). We will randomly generate values having this distribution.

Randomly generate \( R \) (uniformly distributed random number table earlier in this stack).

\( R = 0.3821 \Rightarrow \)

\( X = -\frac{\ln 0.3821}{\lambda} - \frac{(-0.9621)}{2} = 0.4810 \)

**Generating the first 3 random values**

\[
X = \frac{-\ln \frac{0.4281}{2}}{2} = \frac{(-0.8621)}{2} = 0.4362
\]

\[
X = \frac{-\ln \frac{0.9216}{2}}{2} = \frac{(-3.8258)}{2} = 1.9129
\]

\[
X = \frac{-\ln \frac{0.5105}{2}}{2} = \frac{(-0.6724)}{2} = 0.3362
\]
### Rejection Method

Generates sample values for any random variable that
- assumes values only within a finite interval \([a,b]\)
- has a density function that is bounded by a finite value \(c\)

\[ y = f(x) \]

### Algorithm

1. Generate 2 random numbers \(R_1\) & \(R_2\)
   - uniformly distributed in \([0,1]\)
2. Let \(X = (b-a)R_1 + a\) and \(Y = cR_2\) to get a point \((X,Y)\) uniformly distributed in the rectangle
3. Accept \(X\) if \(Y \leq f(X)\), i.e., the point lies in the shaded region under the graph of \(y = f(x)\).
   - Otherwise, reject \(X\) and return to step 1.

### Example - Beta Distribution

PDF:

\[ f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \ 0 < x < 1 \]

Mode:

\[ \frac{\alpha-1}{\alpha+\beta-2} \]

Mean:

\[ \mu = \frac{\alpha}{\alpha+\beta} \]

Variance:

\[ \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \]

### Example

Suppose \(\alpha = 3, \beta = 2\), \(a = 0, b = 1\),

\[ C = f(0.5) = 1.77777778 \]

Select the second column from the table of uniformly-distributed random numbers:

| 0.0278 | 0.1630 | 0.3929 | 0.7423 | 0.5745 | 0.1220 | 0.8593 | 0.9665 | 0.4053 |

The first 2 uniformly-distributed random numbers are

\[ R_1 = 0.4875, \quad R_2 = 0.3519 \]
The next 2 uniformly-distributed random numbers are

\[ R_1 = 0.8147, \ R_2 = 0.1466 \]
\[ X = 0.8147, \ Y = R_2 C = 0.260978 \]
\[ f(X) = 1.47568 \ > \ Y \] \hspace{1cm} ACCEPT! \hspace{1cm} again, the point \((X,Y)\) is under the density curve

\[ f(X) = 1.73711 \ > \ Y \] \hspace{1cm} ACCEPT! \hspace{1cm} again, the point \((X,Y)\) is under the density curve

The next 2 uniformly-distributed random numbers are

\[ R_1 = 0.7233, \ R_2 = 0.5742 \]
\[ X = 0.7233, \ Y = R_2 C = 1.0208 \]
\[ f(X) = 1.47568 \ > \ Y \] \hspace{1cm} ACCEPT! \hspace{1cm} again, the point \((X,Y)\) is under the density curve

\[ f(X) = 1.73711 \ > \ Y \] \hspace{1cm} ACCEPT! \hspace{1cm} again, the point \((X,Y)\) is under the density curve

The next 2 uniformly-distributed random numbers are

\[ R_1 = 0.1129, \ R_2 = 0.6050 \]
\[ X = 0.1129, \ Y = R_2 C = 1.4311 \]
\[ f(X) = 0.13569 \ < \ Y \] \hspace{1cm} REJECT! \hspace{1cm} the point \((X,Y)\) is NOT under the curve, and is rejected!

\[ f(X) = 0.0719 \ < \ Y \] \hspace{1cm} REJECT! \hspace{1cm} the point \((X,Y)\) is NOT under the curve, and is rejected!

The first 4 random numbers having the desired BETA distribution are, therefore,

\[ 0.4876 \]
\[ 0.8147 \]
\[ 0.7233 \hspace{1cm} \text{rejected} \]
\[ 0.5063 \]