Weibull Distribution:

pdf: \[ f(t) = \frac{k}{u} \left( \frac{t}{u} \right)^{k-1} \exp \left\{ -\left( \frac{t}{u} \right)^k \right\} \]

Suppose \( t_1, t_2, \ldots, t_n \) are times to failure of a group of \( n \) mechanisms.

The likelihood function is

\[
L(t; k, u) = \prod_{i=1}^{n} \frac{k}{u} \left( \frac{t_i}{u} \right)^{k-1} \exp \left\{ -\left( \frac{t_i}{u} \right)^k \right\} = \frac{k^n}{u^n} \left( \prod_{i=1}^{n} t_i^{k-1} \right) \exp \left\{ -u^{-k} \sum_{i=1}^{n} t_i^k \right\}
\]

We wish to choose values of \( u \) & \( k \) which maximize \( L \) (or equivalently, the logarithm of \( L \)), i.e., which make the observed values of \( t \) as large as possible!

The log-likelihood function is

\[
\ln L(t; k, u) = n \ln k - nk \ln u + (k - 1) \sum_{i=1}^{n} \ln t_i - u^{-k} \sum_{i=1}^{n} t_i^k
\]

The optimality conditions for the maximum of the log-likelihood function are

\[
\frac{\partial}{\partial u} \ln L(t; u, k) = 0
\]
\[
\frac{\partial}{\partial k} \ln L(t; u, k) = 0
\]

This gives us a pair of nonlinear equations in two unknowns (\( u \) & \( k \)):
But the left side of the first equation can be factored:
\[-\frac{n\hat{k}}{\hat{u}} + \hat{k}\hat{u}^{-i-1}\sum_{i=1}^{n} t_i^k = 0 \Rightarrow \hat{k}\hat{u}^{-1}\left[-n+\hat{u}^{-i}\sum_{i=1}^{n} t_i^k\right] = 0\]

Since the first factor cannot be zero, we set the second factor equal to zero and solve for \(\hat{u}\) in terms of \(\hat{k}\):
\[\hat{u} = \left(\frac{1}{n} \sum_{i=1}^{n} t_i^k\right)^{\frac{1}{k}}\]

Eliminating \(\hat{u}\) in the second equation by substituting the first, we get the following nonlinear equation in \(\hat{k}\) alone:
\[\frac{1}{k} - \frac{1}{n} \sum_{i=1}^{n} t_i^k \ln t_i + \frac{1}{n} \sum_{i=1}^{n} \ln t_i = 0\]

This can now be solved by, for example, the secant method.

**Example: MLE of Weibull parameters, given censored data**

The CDF of the Weibull distribution is
\[F(t;k,u) = 1 - \exp\left\{-\left(\frac{t}{u}\right)^k\right\}\]

and so the likelihood function is
\[L(t;k,u) = \left[\exp\left\{-\left(\frac{t}{u}\right)^k\right\}\right]^{n-r} \times \prod_{i=1}^{r} \frac{k}{u} \left(\frac{t_i}{u}\right)^{k-1} \exp\left\{-\left(\frac{t_i}{u}\right)^k\right\} = \frac{k^r}{u^{nk}} \left[\prod_{i=1}^{r} t_i^{k-1}\right] \exp\left\{-u^{-k} \left[\sum_{i=1}^{r} t_i^k + (n-r)\tau^k\right]\right\}\]

The log-likelihood function is
\[\ln L(t;k,u) = r \ln k - nk \ln u + (k-1) \sum_{i=1}^{r} \ln t_i - u^{-k} \left[\sum_{i=1}^{r} t_i^k + (n-r)\tau^k\right]\]

The optimality conditions for a maximum of the log-likelihood at \((\hat{k},\hat{u})\) are
\[\frac{\partial}{\partial u} \ln L(t;\hat{u},\hat{k}) = 0\]
\[\frac{\partial}{\partial k} \ln L(t;\hat{u},\hat{k}) = 0\]
A result similar to the uncensored case can be derived:

\[ \hat{u} = \left( \frac{\sum_{i=1}^{r} t_i^k + (n-r) \tau^k}{n} \right)^{\frac{1}{k}} \]

and

\[ \frac{1}{k} \sum_{i=1}^{r} t_i^k \ln t_i + (n-r) \tau^k \ln \tau \frac{1}{k} \sum_{i=1}^{r} t_i^k + (n-r) \tau^k = 0 \]

This second equation can be solved for \( \hat{k} \) by the secant method, and then \( \hat{k} \) can be used to calculate \( \hat{u} \) by the first equation.

**Example:** Twenty devices are tested simultaneously until 500 days have passed, at which time the following failure times (in days) have been recorded:

\[
\begin{align*}
31.5 \\
74.0 \\
87.5 \\
100.1 \\
103.3 \\
181.9 \\
279.9 \\
297.1 \\
462.5 \\
465.4
\end{align*}
\]

Estimate the lifetime for which the device is 90% reliable.

A plot of \( Y \) vs \( X \), obtained by the transformations:

\[ Y = \log \log \frac{1}{R(t)} \]

where \( R(t) \) is the observed fraction of the devices which have survived until time \( t \), and

\[ X = \log t \]

should be a line if the Weibull model were to fit the data perfectly.

**Least Squares Regression Results:**

\[
\begin{align*}
u \text{ (scale parameter)} &= 653.504 \\
k \text{ (shape parameter)} &= 0.908313
\end{align*}
\]

so that

\[
\text{mean} = 630.396 \\
\text{standard deviation} = 754.336
\]

Note: this is determined by minimizing the sum of the squared errors in the linearized version of \( F(t) = 1 - e^{-\left(\frac{t}{\hat{u}}\right)^k} \), namely

\[ y = kx - k \ln u \]

where \( x = \ln t \) & \( y = \ln \ln \frac{1}{R(t)} \),

rather than in the original equation!
If we use these parameters found by linear regression, the reliability function would have the values:

<table>
<thead>
<tr>
<th>t</th>
<th>F(t)</th>
<th>1-F(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.12824</td>
<td>0.01</td>
<td>0.99</td>
</tr>
<tr>
<td>8.90435</td>
<td>0.02</td>
<td>0.98</td>
</tr>
<tr>
<td>13.993</td>
<td>0.03</td>
<td>0.97</td>
</tr>
<tr>
<td>19.3163</td>
<td>0.04</td>
<td>0.96</td>
</tr>
<tr>
<td>24.837</td>
<td>0.05</td>
<td>0.95</td>
</tr>
<tr>
<td>30.5337</td>
<td>0.06</td>
<td>0.94</td>
</tr>
<tr>
<td>36.3925</td>
<td>0.07</td>
<td>0.93</td>
</tr>
<tr>
<td>42.4042</td>
<td>0.08</td>
<td>0.92</td>
</tr>
<tr>
<td>48.5623</td>
<td>0.09</td>
<td>0.91</td>
</tr>
<tr>
<td>54.8622</td>
<td>0.10</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Hence, according to this model, 90% of the devices should be operating at approximately 55 days.

**SECANT METHOD**

If our first two “guesses” at the value of $k$ are 0.5 and 2.0, then we determine that $g(0.5) = 1.13739$ & and $g(2.0) = -0.618085$.

The secant joining the two points on the graph of $g$ cross the $k$ axis at 1.47187.

We then repeat, with the 2 improved “guesses” $k=0.5$ and $k=1.47187$. The Maximum Likelihood result:

Solving the nonlinear equation for $k$:

$$g(k) = \frac{1}{k} - \frac{\sum_{i=1}^{r} t_i^k \ln t_i + (n-r) \tau^k \ln \tau}{\sum_{i=1}^{r} t_i^k + (n-r) \tau^k} + \frac{1}{r} \sum_{i=1}^{r} \ln t_i = 0$$
**SECANT METHOD RESULTS:**

<table>
<thead>
<tr>
<th>k</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.13739</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.618085</td>
</tr>
<tr>
<td>1.47187</td>
<td>-0.397478</td>
</tr>
<tr>
<td>0.5203</td>
<td>1.05148</td>
</tr>
<tr>
<td>1.21083</td>
<td>-0.217582</td>
</tr>
<tr>
<td>1.09244</td>
<td>-0.108608</td>
</tr>
<tr>
<td>0.974445</td>
<td>0.025006</td>
</tr>
<tr>
<td>0.996528</td>
<td>0.00227829</td>
</tr>
<tr>
<td>0.994684</td>
<td>0.0000438242</td>
</tr>
<tr>
<td>0.994648</td>
<td>7.83302E-8</td>
</tr>
<tr>
<td>0.994648</td>
<td>-2.68896E-12</td>
</tr>
</tbody>
</table>

Once we determine the value of \( \hat{k} \) which maximizes the likelihood function, then the corresponding value of the parameter \( \hat{u} \) is found by

\[
\hat{u} = \left( \frac{\sum_{i=1}^{r} t_i^\hat{k} + (n-r) \tau^\hat{k}}{n} \right)^{\frac{1}{\hat{k}}}
\]

**Maximum Likelihood Result:**

\( u \) (scale parameter) = 710.339,
\( k \) (shape parameter) = 0.994648

<table>
<thead>
<tr>
<th>t</th>
<th>F(t)</th>
<th>1-F(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.9646</td>
<td>0.01</td>
<td>0.99</td>
</tr>
<tr>
<td>14.0526</td>
<td>0.02</td>
<td>0.98</td>
</tr>
<tr>
<td>21.2337</td>
<td>0.03</td>
<td>0.97</td>
</tr>
<tr>
<td>28.5026</td>
<td>0.04</td>
<td>0.96</td>
</tr>
<tr>
<td>35.8579</td>
<td>0.05</td>
<td>0.95</td>
</tr>
<tr>
<td>43.2993</td>
<td>0.06</td>
<td>0.94</td>
</tr>
<tr>
<td>50.8272</td>
<td>0.07</td>
<td>0.93</td>
</tr>
<tr>
<td>58.4427</td>
<td>0.08</td>
<td>0.92</td>
</tr>
<tr>
<td>66.1467</td>
<td>0.09</td>
<td>0.91</td>
</tr>
<tr>
<td>73.9408</td>
<td>0.10</td>
<td>0.90</td>
</tr>
</tbody>
</table>

According to this model, then, 90% of the devices should be operating at 73.94 (approximately 74) days.