Consider the nonlinear programming problem

\[ \begin{align*}
\text{Minimize } & \quad f(x_1, x_2, \ldots, x_n) \\
\text{subject to } & \quad h_i(x_1, x_2, \ldots, x_n) = 0, \ i = 1, 2, \ldots, m \\
& \quad a_j \leq x_j \leq b_j, \ j = 1, 2, \ldots, n
\end{align*} \]

There are, however, several differences between the two algorithms:

In **GRG**, unlike the simplex method,

- nonbasic (independent) variables need not be at their bound (lower or upper)
- at each iteration, several nonbasic (independent) variables may have their values changed (increased or decreased)
- the basis need not change at each iteration

At the beginning of each iteration, the \( n \) variables are partitioned into two sets:

- Dependent variables (one per equation)
- Independent variables

(after re-ordering the variables):

\[
\begin{align*}
x = \begin{bmatrix} x_D \  x_I \end{bmatrix}
\end{align*}
\]

where

\[
\begin{align*}
x_D = \text{vector of } m \text{ dependent variables} \\
x_I = \text{vector of } (n-m) \text{ independent variables}
\end{align*}
\]

Suppose that we are given an initial point \( x^0 \) which satisfies:

1) \( h_i(x^0) = 0 \ \forall \ i \)

2) \( a_D < x_D < b_D \) \hspace{1cm} (nondegeneracy)

3) \( J_D(x^0) \) is nonsingular, i.e., \( [J_D(x^0)]^{-1} \) exists

4) \( a_I \leq x_I \leq b_I \)

Denote the change in \( x \) by

\[
\delta = \begin{bmatrix} \delta_D \\
\delta_I \end{bmatrix}
\]

For “small” \( \delta \), the change in the objective is

\[
\Delta f = \begin{bmatrix} V_{x^0} f(x^0) \end{bmatrix} \cdot \delta
\]

i.e.,

\[
\Delta f = \begin{bmatrix} \begin{align*}
V_{x^0} f(x^0) & \cdot \delta_D \\
V_{x^0} f(x^0) & \cdot \delta_I
\end{align*} \end{bmatrix} = V_{x^0} f(x^0) \cdot \delta_D + V_{f} f(x^0) \cdot \delta_I
\]
We want to choose \( \delta \) so that we maintain feasibility:
\[
(h_l(X^0 + \delta) - h_u(X^0)) = \Delta h = [\nabla h(X^0)]^T \cdot \delta = 0 \quad \forall \ i
\]
i.e., \[\Delta h = \nabla h(X^0)^T \cdot \delta_d + \nabla h(X^0) \cdot \delta_i = 0 \quad \forall \ i\]
This system of equations (linear in \( \delta \)) may be written:
\[
\Delta h = J(X^0) \cdot \delta = J_d(X^0) \cdot \delta_d + J_i(X^0) \cdot \delta_i = 0
\]
Since we assume that \( J_d(X^0) \) is nonsingular,
\[
J_d(X^0) \cdot \delta_d + J_i(X^0) \cdot \delta_i = 0
\]
\[
\Rightarrow \delta_d = -[J_d(X^0)^{-1} J_i(X^0)] \cdot \delta_i
\]
This equation tells us the required changes in the dependent variables which are required to maintain feasibility when the independent variables are changed by the amount \( \delta_i \).

We now make the substitution
\[
\delta_d = -[J_d(X^0)^{-1} J_i(X^0)] \cdot \delta_i
\]
into the estimate of change in the objective function:
\[
\Delta f = \nabla f(X^0)^T \cdot \delta_d + \nabla f(X^0) \cdot \delta_i
\]
\[
\Delta f \approx \nabla f(X^0)^T \left[ -[J_d(X^0)^{-1} J_i(X^0)] \cdot \delta_i \right] + \nabla f(X^0) \cdot \delta_i
\]
\[
\Delta f \approx \left[ \nabla f(X^0) - \nabla f(X^0)^T \left[ J_d(X^0)^{-1} J_i(X^0) \right] \right] \cdot \delta_i \equiv \Gamma \cdot \delta_i
\]
That is,
\[
\Delta f \approx \Gamma_1 \cdot \delta_i
\]
where the "reduced gradient" \( \Gamma_1 \) is defined as
\[
\Gamma_1 \equiv \nabla f(X^0) - \nabla f(X^0)^T \left[ J_d(X^0)^{-1} J_i(X^0) \right]
\]
This gives us an estimate of the change in the objective when we change the independent variables \( X_i \) by the amount \( \delta_i \) and change the dependent variables \( X_d \) by the amount required to maintain feasibility!

Since the objective is to be minimized, we choose to move each independent variable in the negative of the direction given by the reduced gradient, taking into account the upper & lower bounds on \( X_i \):
\[
\delta_i = \begin{cases} 
0 & \text{if } \Gamma_1 > 0 \text{ and } x^0_i = a_i \\
0 & \text{if } \Gamma_1 < 0 \text{ and } x^0_i = b_i \\
- \Gamma_1 & \text{otherwise}
\end{cases}
\]
for each \( i \in I \).

Once the step direction \( \delta_i \) (for the independent variables) is chosen, then the step direction for the dependent variables is determined by
\[
\delta_d = -[J_d(X^0)^{-1} J_i(X^0)] \cdot \delta_i
\]
(By the nondegeneracy assumption, i.e., \( a_d < X_d^0 < b_d \), some positive step can always be made in the dependent variables.)

Note that, unlike the Simplex LP method, which chooses a single nonbasic (\( \approx \) independent) variable to be changed, GRG simultaneously changes many of the independent variables!
Having found the direction $\delta$ in which to move, we next do a one-dimensional search along this direction in order to

Minimize $f(x^0 + \lambda \delta)$

subject to

\[ a \leq x^0 + \lambda \delta \leq b \]

i.e.,

\[ a - x^0 \leq \lambda \delta \leq b - x^0 \]

This can be done by any of several one-dimensional search methods, e.g., golden section search, cubic interpolation, etc.

In general, when the constraints are nonlinear, for the optimal stepsize $\lambda^*$, $h(x^0 + \lambda^* \delta) = 0$

Then we need to move back onto the feasible surface by solving $h(x)=0$, using $x^0 + \lambda^* \delta$ as an initial "guess" (e.g., using the Newton-Raphson method).

**Example:**

Minimize $f(x) = x_1^2 - x_1 - x_2$

subject to

\[ g_1(x) = 2x_1 + x_2 \leq 1 \]
\[ g_2(x) = x_1 + 2x_2 \leq 1 \]

$x_j \geq 0, j=1,2$

We first write the inequality constraints as equations:

\[ h_1(x) = 2x_1 + x_2 + x_3 - 1 = 0 \]
\[ h_2(x) = x_1 + 2x_2 + x_4 - 1 = 0 \]

For standard GRG form, we need both upper & lower bounds on the variables, which we deduce:

$2x_1 + x_2 \leq 1 \implies x_2 \leq 1$

$x_1 + 2x_2 \leq 1 \implies x_1 \leq 1$

\[ x_3 = 1 - (2x_1 + x_2) \]
\[ \frac{2x_1 + x_2 \geq 0}{2x_1 + x_2 \leq 0} \implies x_3 \leq 1 \]

\[ x_4 = 1 - (x_1 + 2x_2) \]
\[ \frac{x_1 + 2x_2 \geq 0}{x_1 + 2x_2 \leq 0} \implies x_4 \leq 1 \]

$X^0 = \left[ \begin{array}{c} 1/4 \\ 0 \\ 1/2 \\ 3/4 \end{array} \right]$

at lower bound

To avoid degeneracy in the initial partition, we cannot allow $x_2$ to be dependent ("basic"), and so our choice of two dependent variables is limited to $x_1, x_3, \text{ and } x_4$. For the starting partition of the variables, let's define (arbitrarily)

\[ \begin{align*}
X_D &= \left[ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right], \\
D &= \{3,4\} \quad \text{and} \quad I = \{1,2\}
\end{align*} \]
\[ J_D(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_0(x) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

\[
\Gamma_1 = \nabla f(x^0) - \nabla f(x^0) J_D \quad \text{reduced gradient}
\]

\[
\Rightarrow \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix}
\]

\[
\begin{align*}
\delta_1 &= \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \\
\delta_D &= -[J_D(x^0)]^{-1} J_0(x^0) \delta_1
\end{align*}
\]

\[
\Rightarrow \delta_D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}
\]

\[
\begin{align*}
\Gamma_1 &= \begin{bmatrix} -1/2 \\ -1 \end{bmatrix} \\
\Gamma_2 &= \begin{bmatrix} \end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow 0 < x^0_1 < 1 \\
&\Rightarrow \delta_1 = -\Gamma_1 = 1/2 \quad \delta_2 = -\Gamma_2 = 1
\end{align*}
\]

\[
\begin{align*}
&\text{(Neither independent variable is at its upper bound, and so } \delta_2 = -\Gamma_2) \\
&\end{align*}
\]

**objective function**

\[
Z = c' \times \delta
\]

\[
\begin{align*}
&\text{Constraint functions for GRG example problem 3} \\
&\text{(3 linear equality constraints)}
\end{align*}
\]

\[
\begin{align*}
&X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \\
&\text{COP} \begin{bmatrix} 4 & 2 & 1 & 1 \end{bmatrix}
\end{align*}
\]

**Equality Constraints**

\[
\begin{align*}
&V=\begin{bmatrix} \delta \end{bmatrix} \\
&\text{A constraint function for GRG example problem 3}
\end{align*}
\]

\[
\begin{align*}
&X = \begin{bmatrix} x_1 \end{bmatrix} \\
&\text{COP} = \begin{bmatrix} 1 & 0 \end{bmatrix}
\end{align*}
\]

**Gradient for objective function**

\[
\begin{align*}
&\text{Gradient for objective function of GRG Example 2} \\
&\text{Max. 2 1 0 0 0 0 = C= \begin{bmatrix} 1 \end{bmatrix}}
\end{align*}
\]

**Jacobian of Equality Constraints**

\[
\begin{align*}
&\text{Jacobian matrix of linear equality constraints} \\
&\text{for GRG example problem 3}
\end{align*}
\]

**Computing Maximum Step Size**

Based upon the lower & upper bounds:

\[
\begin{align*}
0 \leq x_1 + \delta_1 &= \frac{1}{4} + \frac{3}{4} \lambda \leq 1 \\
0 \leq x_2 + \delta_2 &= 0 + \delta_2 \leq 1 \\
0 \leq x_3 + \delta_3 &= \frac{1}{4} + \frac{3}{4} \lambda \leq 1 \\
0 \leq x_4 + \delta_4 &= \frac{3}{4} - \lambda \leq 1
\end{align*}
\]

\[
\text{lowest upper bound !}
\]

\[
\begin{align*}
&\delta \text{ was normalized by scaling so that } \max |\delta| = 1
\end{align*}
\]
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Max Step Size = 0.625
Optimal Step Size = 0.625
x = 0.375 0.15 0.125
h(x) = 0, f(x) = -0.494375

Note that x (which contributed the maximum stepsize) has reached its lower bound!

X_3 is the slack in the inequality constraint, and so GRG6 has moved to the boundary of that constraint as X_3 decreases to 0.

Dependent variables cannot be at either lower or upper bound, and so X_4 must become independent, and replaced by either X_1 or X_2.

(X_4 is already dependent.)

Variable(s) 3 has reached a bound and must be removed from D
Variables 1 2 are candidates to enter D
Try entering variable 1
Determinant of J(1,D) = 2 < checking that the Jacobian submatrix is nonsingular

3 is replaced by 1 in Independent Variable Set.
h(x) = 0, f(x) = -0.494375

Iteration 2

x = 0.375 0.25 0.125
f(x) = -0.494375
Dependent Index Set: 1 4
Independent Index Set: 3 2
h(x) = 0 0
Gradient = -0.25 -1 0 0
Negative of Reduced Gradient = -0.125 0.275
Search Direction = -0.4375 0.875 0 1.3125
Normalized Search Direction = -0.333333 0.666667 0 0
Max Step Size = 0.125
Optimal Step Size = 0.125
x = 0.3125 0.3125 0 0
h(x) = 0, f(x) = -0.555556

Variable(s) 4 has reached a bound and must be removed from D
Variables 2 are candidates to enter D
Try entering variable 2
Determinant of J(1,D) = 3

4 is replaced by 2 in Independent Variable Set.
h(x) = 0, f(x) = -0.555556

Iteration 3

x = 0.333333 0.333333 0 0
f(x) = -0.555556
Dependent Index Set: 1 4
Independent Index Set: 3 2
h(x) = 0 0
Gradient = -0.333333 -1 0 0
Negative of Reduced Gradient = -0.111111 -0.555556
Search Direction = -0.070741 0.070737 0.111111 0
Normalized Search Direction = -0.666667 0.333333 1 0
Max Step Size = 0.5
Optimal Step Size = 0.125
x = 0.25 0.375 0.125 0
h(x) = -1.11022E-16 -1.11022E-16, f(x) = -0.5625

Iteration 4

x = 0.25 0.375 0.125 0
f(x) = -0.5625
Dependent Index Set: 1 2
Independent Index Set: 3 4
h(x) = -1.11022E-16 -1.11022E-16

Generalized Reduced Gradient

*** GRG HAS CONVERGED ***

Variable(s) 4 has reached a bound and must be removed from D
Variables 2 are candidates to enter D
Try entering variable 2
Determinant of J(1,D) = 3

4 is replaced by 2 in Independent Variable Set.
h(x) = 0, f(x) = -0.555556

Iteration 5

x = 0.333333 0.333333 0 0
f(x) = -0.555556
Dependent Index Set: 1 4
Independent Index Set: 3 2
h(x) = 0 0
Gradient = -0.333333 -1 0 0
Negative of Reduced Gradient = -0.111111 -0.555556
Search Direction = -0.070741 0.070737 0.111111 0
Normalized Search Direction = -0.666667 0.333333 1 0
Max Step Size = 0.5
Optimal Step Size = 0.125
x = 0.25 0.375 0.125 0
h(x) = -1.11022E-16 -1.11022E-16, f(x) = -0.5625

Optimal Solution
Iteration 1

\[ \begin{align*}
x &= 0.25 \\
F(x) &= 0.4575 \\
\text{Independent Index Set: } &2 4 \\
\text{Gradient} &= -0.5 -1 0 0 \\
\text{Negative of Reduced Gradient} &= 0.51 \\
\text{Search Direction} &= 0.51 72 -2.5 \\
\text{(Normalized Search Direction)} &= 0.2 0.4 -0.8 -1 \\
\text{Max Step Size} &= 0.35 \\
\text{Optimal Step Size} &= 0.25 \\
2 &= 0.3 0.35 0.05 0 \\
h(x) &= 0 0, F(x) = -0.56 \\
\end{align*} \]

Variable(s) in bound are candidates to enter D

Try entering variable 2

Determinant of \( {J_f}^T \): 2

4 is replaced by 2 in dependent variable set.

\[ h(x) = 0 0, F(x) = -0.56 \]

Iteration 2

\[ \begin{align*}
x &= 0.1 0.35 0.05 0 \\
F(x) &= 0.56 \\
\text{Independent Index Set: } &3 4 \\
\text{Gradient} &= -0.4 -1 0 0 \\
\text{Negative of Reduced Gradient} &= -0.1 -0.5 \\
\text{Search Direction} &= -0.1 0.05 0.15 0 \\
\text{(Normalized Search Direction)} &= -0.666667 0.333333 1 0 \\
\text{Max Step Size} &= 0.45 \\
\text{Optimal Step Size} &= 0.075 \\
\text{Max Step Size} &= 0.25 0.375 0.125 0 \\
h(x) &= 0 0, F(x) = -0.5655 \\
\end{align*} \]

Iteration 3

\[ \begin{align*}
x &= 0.25 0.375 0.125 0 \\
F(x) &= 0.5625 \\
\text{Independent Index Set: } &3 4 \\
\text{Gradient} &= -0.8 -1 0 0 \\
\text{Negative of Reduced Gradient} &= -2.238655 \times 10^{-16} -0.5 \\
\text{*** GRG HAS CONVERGED ***} \\
\end{align*} \]

Generalized Reduced Gradient Solution