Farkas' Lemma

Let \( A \in \mathbb{R}^{m \times n} \), i.e., \( A \) is an \( m \times n \) matrix, \( b \in \mathbb{R}^m \), \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \).

The following statements are equivalent:

1. \( y^T A \leq 0 \Rightarrow y^T b \leq 0 \)
2. \( \exists x \) such that \( Ax = b \), \( x \geq 0 \)

If \( y = 0 \) is optimal for \( \text{D} \), then by LP duality theory, \( \text{P} \) is feasible (with LP value 0), proving that \( 1 \Rightarrow 2 \).

Suppose that \( Ax = b \) for some \( x \geq 0 \), and \( y^T A \leq 0 \) for some \( y \).

Then \( y^T A \leq 0 = y^T A x \leq 0 = y^T b \leq 0 \)

proving that \( 2 \Rightarrow 1 \).

\[ \text{QED} \]

Proof

Consider the primal/dual LP pair:

\( \text{P} \) Minimize \( 0x \)
subject to \( A x = b \)
\( x \geq 0 \)

\( \text{D} \) Maximize \( y^T b \)
subject to \( A^T y \leq 0 \)
\[ i.e., \ y^T A \leq 0 \]

Problem \( \text{D} \) is feasible (e.g., let \( y = 0 \), for which the objective \( y^T b \) is zero.)

If statement \( 1 \) is true, i.e., \( y^T A \leq 0 \Rightarrow y^T b \leq 0 \)
then \( y = 0 \) must be optimal for problem \( \text{D} \).

GEOMETRIC ILLUSTRATION OF FARKAS' LEMMA

Let \( A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 1 \end{bmatrix} \), \( b = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \)

The columns of \( A \) are points (vectors) in \( \mathbb{R}^2 \)

\[ A^1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} , A^2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} , A^3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} , b = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \]

\[ \text{(requirements space)} \]

For example,

\[ b = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 1 A^1 + 0 A^2 + 1 A^3 \]
\[ = \frac{4}{7} A^1 + \frac{4}{7} A^2 + 0 A^3 \]
\[ = \frac{11}{14} A^1 + \frac{4}{7} A^2 + \frac{1}{2} A^3 \]

.... etc.

The system \( Ax = b \) has a solution if & only if \( b \) lies in the cone generated by \( A^1, A^2, \) and \( A^3 \)

Let \( H_j \) be the hyperplane (a line in \( \mathbb{R}^2 \)) through the origin, orthogonal to \( A_j \), and let \( H_i \) be the closed halfspace on the side of \( H_j \) not containing \( A_i \)
Farkas' Lemma

\[ y^T A^j = 0 \iff y \perp A^j \]
\[ \iff y \in H_j \]

Also,
\[ y^T A^j \leq 0 \iff y \in H^*_j \]
Therefore,
\[ y^T A \leq 0 \iff y \in \cap H_j \]
(the intersection of the half-spaces, shaded above)

\[ \text{EXAMPLE 1} \]
\[ b = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \]

Note that \( b \) is in the cone generated by \( A^1, A^2, \text{and } A^3 \)
and that \( \cap H^*_j \subseteq H^*_5 \)

\[ \text{APPLICATION TO NONLINEAR PROGRAMMING} \]

Consider the problem
\[
\begin{align*}
\text{Minimize } & f(x) \\
\text{subject to } & g_i(x) \leq 0, i = 1, 2, \ldots, m
\end{align*}
\]

Denote
\[ b = -\nabla f(x^*) \]
\[ A^1 = \nabla g_i(x^*) \]
\[ y = d \quad \text{(direction vector)} \]
\[ x_i = \lambda_i \text{ for } i \in I \equiv \{ i \mid g_i(x^*) = 0 \} \]
(Lagrange multiplier)
\( \cap \) (index set of tight constraints)

Likewise, for a given \( b \), let \( H_b \) - hyperplane through the origin, orthogonal to \( b \), and \( H^*_b \) - closed halfspace on side of \( H_b \), not containing \( b \).

Then the statement
\[ y^T A \leq 0 \iff y^T b \leq 0 \]
simply says that
\[ \cap H^*_j \subseteq H^*_5 \]

\[ \text{EXAMPLE 2} \]
\[ b = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \]

In this case, the vector \( b \) does not lie in the cone generated by \( A \), nor does \( \cap H^*_j \) lie entirely in the closed half-space \( H^*_5 \)

\[ \text{points in } \cap H^*_j \text{ which don't lie in } H^*_5 \]

\[ \text{Farkas' Lemma} \]

1. \[ y^T A \leq 0 \iff y^T b \leq 0 \]
2. \[ \exists x \text{ such that } A x = b, x \geq 0 \]

are equivalent statements

That is,
1. \[ d^T \nabla g_i(x^*) \leq 0 \forall i \in I \Rightarrow -d^T \nabla f(x^*) \leq 0 \]
2. \[ \exists \lambda_i \geq 0 \text{ such that } \sum_{i \in I} \lambda_i \nabla g_i(x^*) = -\nabla f(x^*) \]

are equivalent statements

If a constraint is not tight, then any direction is feasible with respect to that constraint!
1. $d^T V g_i(x^*) \leq 0 \forall i \in I \Rightarrow -d^T V f(x^*) \leq 0$

Directional derivatives satisfying $d^T V g_i(x^*) \leq 0 \forall i \in I$ are feasible directions.

Directional derivatives satisfying $d^T V f(x^*) \geq 0$ are directions of ascent.

Every feasible direction is non-improving.

2. $\exists \lambda_i \geq 0$ such that $\sum_{i \in I} \lambda_i V g_i(x^*) = -V f(x^*)$

The steepest descent direction at $x^*$ is in the cone generated by the gradients of the tight constraints at $x^*$.

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K-K-T "Necessary" Condition for Optimality

If $x^*$ is an optimal solution to

Minimize $f(x)$

subject to $g_i(x) \leq 0, i=1,2,\ldots,m$

then

The directional derivative of $f(x)$ is nonnegative in every feasible direction at $x^*$.

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K-K-T "Necessary" Condition for Optimality

If $x^*$ is an optimal solution to

Minimize $f(x)$

subject to $g_i(x) \leq 0, i=1,2,\ldots,m$

then

The steepest descent direction at $x^*$ is in the cone generated by the gradients of the tight constraints at $x^*$.

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(Equivalent condition, according to Farkas' Lemma)