EXAMPLE

Minimize $x_1^2 - 2x_1 - x_2$
subject to $2x_1^2 + 3x_2^2 \leq 6$
$x_1 \geq 0, \ x_2 \geq 0$

$f(x) = -2.213$

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Having determined that the constraints imply that $0 \leq x_1 \leq 2$, we select 9 grid points, in this case evenly distributed.

<table>
<thead>
<tr>
<th>i</th>
<th>$\xi_{1i}$</th>
<th>$f_{1i}$</th>
<th>$g_{1i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>-0.4375</td>
<td>0.125</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>-0.75</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.75</td>
<td>-0.9375</td>
<td>1.125</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>-1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>-0.9375</td>
<td>3.125</td>
</tr>
<tr>
<td>6</td>
<td>1.5</td>
<td>-0.75</td>
<td>4.5</td>
</tr>
<tr>
<td>7</td>
<td>1.75</td>
<td>-0.4375</td>
<td>6.125</td>
</tr>
<tr>
<td>8</td>
<td>2.0</td>
<td>0</td>
<td>8.0</td>
</tr>
</tbody>
</table>

Likewise, we determine that feasibility requires that $0 \leq x_2 \leq 2$, and select 9 grid points for $x_2$.  

<table>
<thead>
<tr>
<th>i</th>
<th>$\xi_{2i}$</th>
<th>$f_{2i}$</th>
<th>$g_{2i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>-0.25</td>
<td>0.1875</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>-0.5</td>
<td>0.75</td>
</tr>
<tr>
<td>3</td>
<td>0.75</td>
<td>-0.75</td>
<td>1.6875</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>-1.0</td>
<td>3.0</td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>-1.25</td>
<td>4.6875</td>
</tr>
<tr>
<td>6</td>
<td>1.5</td>
<td>-1.5</td>
<td>6.75</td>
</tr>
<tr>
<td>7</td>
<td>1.75</td>
<td>-1.75</td>
<td>9.1875</td>
</tr>
<tr>
<td>8</td>
<td>2.0</td>
<td>-2.0</td>
<td>12.0</td>
</tr>
</tbody>
</table>
The piecewise-linear approximation has the LP formulation:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{2} \sum_{j=0}^{8} \lambda_{ij} f_{ij} \\
\text{subject to} & \quad \sum_{i=1}^{2} \sum_{j=0}^{8} \lambda_{ij} g_{ij} \leq 6 \\
& \quad \sum_{j=0}^{8} \lambda_{ij} = 1, \quad \forall \ i \\
& \quad \lambda_{ij} \geq 0, \quad \forall \ i \ & & j
\end{align*}
\]

That is, the LP problem:

\[
\begin{align*}
\text{Min} & \quad -0.4375\lambda_{11} - 0.75\lambda_{12} - 0.9375\lambda_{13} - \lambda_{14} - 0.9375\lambda_{15} - 0.75\lambda_{16} - 0.4375\lambda_{17} \\
& \quad -0.25\lambda_{21} - 0.5\lambda_{22} - 0.75\lambda_{23} - \lambda_{24} - 1.25\lambda_{25} - 1.5\lambda_{26} - 1.75\lambda_{27} - 2\lambda_{28} \\
\text{subject to} & \quad 1.25\lambda_{11} + 0.5\lambda_{12} + 1.125\lambda_{13} + 2\lambda_{14} \\
& \quad + 3.125\lambda_{15} + 4.5\lambda_{16} + 6.125\lambda_{17} + 8\lambda_{18} \\
& \quad + 0.1875\lambda_{21} + 0.75\lambda_{22} + 1.6875\lambda_{23} + 3\lambda_{24} \\
& \quad + 4.6875\lambda_{25} + 6.75\lambda_{26} + 9.1875\lambda_{27} + 12\lambda_{28} \leq 6 \\
\text{"convexity" constraints} & \quad \begin{cases} 
\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} + \lambda_{17} + \lambda_{18} = 1 \\
\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} + \lambda_{25} + \lambda_{26} + \lambda_{27} + \lambda_{28} = 1 \\
\lambda_{ij} \geq 0, \quad \forall \ i \ & & j
\end{cases}
\end{align*}
\]
...at most TWO $\lambda_i$'s will be positive, and these will be weights of adjacent grid points!

$\lambda_{10} = 0 \quad \lambda_{20} = 0
\lambda_{11} = 0 \quad \lambda_{21} = 0
\lambda_{12} = 0 \quad \lambda_{22} = 0
\lambda_{13} = 1 \quad \lambda_{23} = 0
\lambda_{14} = 0 \quad \lambda_{24} = 0
\lambda_{15} = 0 \quad \lambda_{25} = 0.9090909
\lambda_{16} = 0 \quad \lambda_{26} = 0.0909090
\lambda_{17} = 0 \quad \lambda_{27} = 0
\lambda_{18} = 0 \quad \lambda_{28} = 0

The solution obtained from the piecewise-linear approximation is reasonably close to the "true" optimum:

$X_1 = 0.75 \lambda_{13}$
$= (0.75)(1) = 0.75$

$X_2 = 1.25 \lambda_{25} + 1.5 \lambda_{26}$
$= (1.25)(0.9090909) + (1.5)(0.0909090)$
$= 1.2727$

LP objective = 2.21023

\[ X_1^* = 0.7906 \]
\[ X_2^* = 1.258 \]
\[ f(X^*) = 2.213 \]
Given a set of \( p_j \) grid points \( \{ \gamma_{jk} \}_{k=1}^{p_j} \) for \( X_j \), and assuming that, for each \( j \), \( f_j \) & \( g_{ij} \) are convex functions, we obtain a piecewise-linear approximation:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{j=1}^{n} \sum_{k=1}^{p_j} f_j(\gamma_{jk}) \lambda_{jk} \\
\text{subject to} & \quad \sum_{j=1}^{n} \sum_{k=1}^{p_j} g_{ij}(\gamma_{jk}) \lambda_{jk} \leq b_i, \quad i=1, \ldots, m \\
& \quad \sum_{k=1}^{p_j} \lambda_{jk} = 1, \quad j=1, \ldots, n \\
& \quad \lambda_{ik} \geq 0, \quad j=1, \ldots, n
\end{align*}
\]
The "finer" the mesh of the grid, i.e., the nearer the grid points, the more accurate is the piecewise-linear approximation, generally.... But the greater the computational burden!

What is needed is a "fine" mesh only in the vicinity of the optimal solution, with a coarse mesh elsewhere. The "grid refinement" method to be introduced next will iteratively select additional grid points to improve the approximation in the vicinity of the optimum!

\[
\begin{align*}
\text{Minimize} & \quad x_1^2 - 2x_1 - x_2 \\
\text{subject to} & \quad 2x_1^2 + 3x_2^2 \leq 6 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0
\end{align*}
\]

**EXAMPLE**

Instead of initially using 9 grid points for each variable, as before, we will use an initial "rough" grid, \{0, 1, 2\} for both \(x_1\) & \(x_2\)

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_1^2 - 2x_1)</th>
<th>(2x_1^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x_2)</th>
<th>(-x_2)</th>
<th>(3x_2^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>12</td>
</tr>
</tbody>
</table>

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Piecewise-Linear Approximation

Minimize \(-\lambda_{12} - \lambda_{22} - 2\lambda_{23}\)
subject to
\[2\lambda_{12} + 8\lambda_{13} + 3\lambda_{22} + 12\lambda_{23} \leq 6\]
\[\lambda_{11} + \lambda_{12} + \lambda_{13} = 1\]
\[\lambda_{21} + \lambda_{22} + \lambda_{23} = 1\]
\[\lambda_{jk} \geq 0, \ \forall \ j & k\]

LP Solution:

\[z = -\frac{19}{9}\]
\[\lambda_{12}^* = 1\]
\[\lambda_{22}^* = \frac{8}{9}, \ \lambda_{23}^* = \frac{1}{9}\]

Optimal Simplex Multipliers (dual variables):
\[\pi = \left[-\frac{1}{9}, -\frac{7}{9}, -\frac{2}{3}\right]\]

How can we "refine the grid, i.e., add additional grid points, so as to get a better approximation and a better solution?"
If \( \gamma_1 \) were a new grid point for \( x_1 \), then we would generate a new column for the tableau:

\[
\begin{bmatrix}
2\gamma_1^2 \\
1 \\
0
\end{bmatrix}
\]

with cost coefficient \( \gamma_1^2 - 2\gamma_1 \)

and reduced cost

\[
(\gamma_1^2 - 2\gamma_1) - \left[ -\frac{1}{9}, -\frac{7}{9}, -\frac{2}{3} \right] \begin{bmatrix}
2\gamma_1^2 \\
1 \\
0
\end{bmatrix}
\]

cost of simplex multipliers column of coefficients

\[
\text{cost of variable}
\]

\[
(\gamma_1^2 - 2\gamma_1) - \left[ -\frac{1}{9}, -\frac{7}{9}, -\frac{2}{3} \right] \begin{bmatrix}
2\gamma_1^2 \\
1 \\
0
\end{bmatrix}
\]

\[
= \gamma_1^2 - 2\gamma_1 + \left( \frac{1}{9} \right) (2\gamma_1^2) + \left( \frac{7}{9} \right) (1) + \left( \frac{2}{3} \right) (0)
\]

\[
= \frac{11}{9} \gamma_1^2 - 2\gamma_1 + \frac{7}{9}
\]
Given a choice of grid points to choose from, let's select that grid point whose column, when added to the LP tableau, has the smallest (i.e., "most negative") reduced cost.

*Note that this rule does not necessarily give us the grid point which will yield the most improvement in the approximation or the objective function.*

To identify this grid point, we will minimize the reduced cost, $\frac{11}{9} \gamma_1^2 - 2 \gamma_1 + \frac{7}{9}$, which is a function of $\gamma_1$.

Differentiating the reduced cost function

$$\frac{11}{9} \gamma_1^2 - 2 \gamma_1 + \frac{7}{9}$$

and equating the derivative to zero yields (in this example) a linear equation which is easily solved for the grid point $\gamma_1$:

$$2 \left( \frac{11}{9} \right) \gamma_1 - 2 = 0 \quad \Rightarrow \gamma_1 = \frac{9}{11}$$

with reduced cost $-0.0404 < 0$
The column which we therefore generate for the LP tableau, corresponding to this new grid point, is

\[
\begin{bmatrix}
2\gamma_1^2 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
1.3388 \\
1 \\
0
\end{bmatrix}
\]

with objective coefficient

\[\gamma_1^2 - 2\gamma_1 = -0.96694\]

Likewise, selection of a new grid point for \(x_2\) is done by choosing \(\gamma_2\) in order to minimize the reduced cost of the generated column

\[-\gamma_2 - \begin{bmatrix}
-\frac{1}{9} & -\frac{7}{9} & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
3\gamma_2^2 \\
0 \\
1
\end{bmatrix} = \frac{1}{3}\gamma_2^2 - \gamma_2 + \frac{1}{3}\]

whose derivative, \(\frac{2}{3}\gamma_2 - 1\), is zero at \(\gamma_2 = \frac{3}{2}\)
Thus, we refine the grids:
\[
\{0, \frac{9}{11}, 1, 2\} \quad \text{for } x_1
\]
\[
\{0, 1, \frac{3}{2}, 2\} \quad \text{for } x_2
\]
generate the new columns for the LP tableau, and re-optimize the LP:

\[
\begin{align*}
\text{Minimize} & \quad -\lambda_{12} - \lambda_{22} - 2\lambda_{23} - 0.96694\lambda_{14} - 1.5\lambda_{24} \\
\text{subject to} & \quad 2\lambda_{12} + 8\lambda_{13} + 3\lambda_{22} + 12\lambda_{23} + 1.3388\lambda_{14} + 6.75\lambda_{24} \leq 6 \\
& \quad \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 1 \\
& \quad \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1 \\
& \quad \lambda_{jk} \geq 0, \quad \forall j, k
\end{align*}
\]

\[\text{New LP optimum}\]
\[
z = -2.1884
\]
\[
\lambda_{14} = 1, \lambda_{11} = \lambda_{12} = \lambda_{13} = 0
\]
\[
\lambda_{22} = 0.5570, \lambda_{24} = 0.4430, \lambda_{21} = \lambda_{23} = 0
\]
\[
\Rightarrow \quad \begin{align*}
x_1 &= 0.8182 \\
x_2 &= 1.2215
\end{align*}
\]

with Simplex multiplier vector
\[
\pi = [ -0.1333, -0.7884, -0.6 ]
\]
Let's further refine the grid

- Reduced cost for grid point $\gamma_1$'s column is
  \[
  (\gamma_1^2 - 2\gamma_1 ) + 0.1333 (2\gamma_1^2 ) + 0.7884
  \]
  which is minimized (with value -0.0010) at $\gamma_1 = 0.7895$

- Reduced cost for grid point $\gamma_2$'s column is
  \[- \gamma_2 + 0.1333 (3\gamma_2^2 ) + 0.6
  \]
  which is minimized (with value -0.025) at $\gamma_2 = 1.25$

---

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\[
\text{Minimize } -\lambda_{12} - \lambda_{22} - 2\lambda_{23} - 0.96694\lambda_{14} - 1.5\lambda_{24}
\]
\[
- 0.95569\lambda_{15} - 1.25\lambda_{25}
\]

subject to
\[
2\lambda_{12} + 8\lambda_{13} + 3\lambda_{22} + 12\lambda_{23} + 1.3388\lambda_{14} + 6.75\lambda_{24}
\]
\[
+ 1.2466 \lambda_{15} + 4.6875\lambda_{25} \leq 6
\]
\[
\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} = 1
\]
\[
\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} + \lambda_{25} = 1
\]
\[
\lambda_{jk} \geq 0, \forall j & k
\]

which has optimum $-2.2137$ at
\[
\lambda_{14} = 0.714751, \lambda_{15} = 0.285249
\]
\[
\lambda_{25} = 1
\]

$\Rightarrow$ $x_1 = 0.8100$
\[
x_2 = 1.25
\]

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Suppose that the simplex multipliers are

\[ \pi = [\pi_1, \pi_2, \ldots, \pi_m | \pi_{m+1}, \pi_{m+2}, \ldots, \pi_{m+n}] \]

These simplex multipliers are used by the revised simplex method to compute the reduced cost of a nonbasic variable.

Corresponding to a new grid point \( \gamma_j \) for \( x_j \) is the LP column

\[
\begin{bmatrix}
g_{1j}(\gamma_j) \\
g_{2j}(\gamma_j) \\
\vdots \\
g_{mj}(\gamma_j) \\
0 \\
0 \\
\vdots \\
1 \\
0
\end{bmatrix}
\]

with objective coefficient \( f_j(\gamma_j) \)

and reduced cost function

\[ f_j(\gamma_j) - \sum_{i=1}^{m} \pi_i g_{ij}(\gamma_j) - \pi_{m+j} \]
For each $j=1, 2, \ldots n$:

- find the grid point $\gamma_j$ which minimizes the reduced cost function

$$f_j(\gamma_j) - \sum_{i=1}^{m} \pi_i g_{ij}(\gamma_j) - \pi_{m+j}$$

- if the value of the reduced cost function exceeds some tolerance $\varepsilon > 0$ in absolute value, generate the LP column and add to the tableau

If no new column was added to the LP tableau, then terminate.

Otherwise, re-optimize the LP, and repeat the procedure.

*Note that in our example, we were able to minimize the reduced cost function analytically; more generally, it is necessary to use a one-dimensional search technique (e.g., golden section search, fibonacci search, quadratic interpolation, etc.).*