Generating Uniformly-Distributed Numbers

Inverse Transformation Method

Rejection Method
Generating Uniformly-Distributed Numbers

Beginning with a "seed" $X_0$, a sequence of random numbers $X_1, X_2, X_3, ...$ is generated by some operation

$$X_{i+1} = G(X_i), \ i=1,2,3,...$$

Since the sequence is determined by the operator $G$ and the initial "seed", the numbers are not, in fact, random, but if uniformly distributed in the interval $[0,1]$ they will be called "pseudo-random".

Example: "Midsquare Technique"

Determine $X_{i+1}$ as follows:

Compute $X_i^2$, discard the final 2 digits, and take the last 4 digits of the remaining number:

<table>
<thead>
<tr>
<th>i</th>
<th>$X_i$</th>
<th>$X_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1912</td>
<td>03655744</td>
</tr>
<tr>
<td>1</td>
<td>6557</td>
<td>42994249</td>
</tr>
<tr>
<td>2</td>
<td>9942</td>
<td>98843364</td>
</tr>
<tr>
<td>3</td>
<td>8433</td>
<td>71115489</td>
</tr>
<tr>
<td>4</td>
<td>1154</td>
<td>01331716</td>
</tr>
</tbody>
</table>
Example

**Congruential Method**

Choose $A > 0$.
Determin $X_{i+1}$ by the operation:

$$X_{i+1} = [AX_i] \text{ Modulo } M$$

i.e., multiply $X_i$ by $A$ and divide the result by $M$. Keep the remainder of the division, and call it $X_{i+1}$.

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Example

$$X_{i+1} = [AX_i] \text{ Modulo } M$$

Select $X_0 = 5$,
$A = 5$,
$M = 17$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$X_i$</th>
<th>$AX_i$</th>
<th>$[AX_i + M]$ remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>40</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>65</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>70</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Usually, \( M = 2^B - 1 \), where \( B = \# \) bits/word for the computer used.
E.g., for IBM system 360, the choice was

\[
M = 2^{32} - 1 \\
A = X_0 = 3^{19} = 65539
\]

(with these values, the sequence repeats after \( 2^{39} \) numbers!)

Uniformly-distributed
Random Number Table

<table>
<thead>
<tr>
<th>0218</th>
<th>3519</th>
<th>0707</th>
<th>3695</th>
<th>6478</th>
<th>3977</th>
<th>2017</th>
<th>3644</th>
<th>7993</th>
<th>5547</th>
</tr>
</thead>
<tbody>
<tr>
<td>5105</td>
<td>8147</td>
<td>7365</td>
<td>2901</td>
<td>7228</td>
<td>2307</td>
<td>7241</td>
<td>4225</td>
<td>6078</td>
<td>9344</td>
</tr>
<tr>
<td>4549</td>
<td>1468</td>
<td>4395</td>
<td>3808</td>
<td>9446</td>
<td>5954</td>
<td>6851</td>
<td>2930</td>
<td>9217</td>
<td>5668</td>
</tr>
<tr>
<td>6758</td>
<td>7233</td>
<td>0303</td>
<td>0981</td>
<td>5955</td>
<td>4881</td>
<td>5916</td>
<td>3197</td>
<td>8532</td>
<td>9810</td>
</tr>
<tr>
<td>8431</td>
<td>5742</td>
<td>0744</td>
<td>3115</td>
<td>4411</td>
<td>5132</td>
<td>2175</td>
<td>8044</td>
<td>5668</td>
<td>3463</td>
</tr>
<tr>
<td>5072</td>
<td>1129</td>
<td>0723</td>
<td>1390</td>
<td>0722</td>
<td>6669</td>
<td>8144</td>
<td>0434</td>
<td>3014</td>
<td>9675</td>
</tr>
<tr>
<td>1797</td>
<td>8050</td>
<td>3603</td>
<td>9301</td>
<td>2162</td>
<td>8267</td>
<td>6733</td>
<td>5878</td>
<td>9918</td>
<td>3984</td>
</tr>
<tr>
<td>5280</td>
<td>5063</td>
<td>6863</td>
<td>6449</td>
<td>6400</td>
<td>0863</td>
<td>2414</td>
<td>4309</td>
<td>0851</td>
<td>3393</td>
</tr>
<tr>
<td>7223</td>
<td>4603</td>
<td>1542</td>
<td>9279</td>
<td>7217</td>
<td>2279</td>
<td>4575</td>
<td>5332</td>
<td>0000</td>
<td>6645</td>
</tr>
</tbody>
</table>
Inverse Transformation Method

\[ F(x) = \text{cumulative distribution function (CDF)} \]
\[ = P\{X \leq x\} \]
\[ R = \text{random variable uniformly distributed in the interval } [0, 1] \]

Randomly generate \( R \) (uniformly dist'd in \([0, 1]\))

Find \( X \) such that \( F(X) = R \)
\[ \text{i.e., } \quad X = F^{-1}(R) \]

\( X \) generated in this way has the desired distribution
**Example**

**Exponential Distribution**

\[ F(x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0 \]

\[ F(x) = 1 - e^{-\lambda x} = R \]

\[ e^{-\lambda x} = 1 - R = R \]

\[ -\lambda x = \ln R \]

\[ x = -\frac{\ln R}{\lambda} \]

Both \( R \) & \( 1-R \) are uniformly distributed in \([0, 1]\).

\[ x = -\frac{\ln (1-R)}{\lambda} \]

\[ F(x) = 1 - e^{-\lambda x} \]
Suppose that we wish to simulate a Poisson process with $\lambda = 2$/hr. For this purpose, we need to randomly generate the arrival times $T_1, T_2, T_3, \ldots$

The time between arrivals will have the exponential distribution with parameter $\lambda = 2$/hr. We will randomly generate values having this distribution.

Inverse Transformation Method for Exponential Distribution

Let's use the first column of the uniformly distributed random number table earlier in this stack.

$R = 0.3821 \Rightarrow$

$$\chi = -\frac{\ln 0.3821}{2} = -\frac{(-0.9621)}{2} = 0.4810$$

$$\chi = -\frac{\ln R}{\lambda}$$
Generating the first 3 random values

\[ X = - \frac{\ln 0.3821}{2} = - \frac{(-0.9621)}{2} = 0.4810 \]

\[ X = - \frac{\ln 0.0218}{2} = - \frac{(-3.8258)}{2} = 1.9129 \]

\[ X = - \frac{\ln 0.5105}{2} = - \frac{(-0.6724)}{2} = 0.3362 \]

\[ X = - \frac{\ln \bar{R}}{\lambda} \]
Rejection Method

Generates sample values for any random variable that

- assumes values only within a finite interval \([a,b]\)
- has a density function that is bounded by a finite value \(c\)

\[ y = f(x) \]
Algorithm

1) Generate 2 random numbers \( R_1 \) & \( R_2 \) uniformly distributed in \([0, 1]\)

2) Let \( X = (b-a)R_1 + a \) and \( Y = cR_2 \) to get a point \((X,Y)\) uniformly distributed in the rectangle

3) Accept \( X \) if \( Y \leq f(X) \), i.e., the point lies in the shaded region under the graph of \( y=f(x) \). Otherwise, reject \( X \) and return to step 1.
Example

Beta Distribution

\[ f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1 \]

Mode = \[ \frac{\alpha-1}{\alpha+\beta-2} \]

\[ \mu = \frac{\alpha}{\alpha + \beta} \]

\[ \sigma^2 = \frac{\alpha \beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \]

Suppose \( \alpha = 3, \beta = 2 \). \( a=0, b=1 \)

\[ C = f\left(\frac{2}{3}\right) = 1.77777778 \]
Select the second column from the table of uniformly-distributed random numbers:

<table>
<thead>
<tr>
<th>4876</th>
</tr>
</thead>
<tbody>
<tr>
<td>3519</td>
</tr>
<tr>
<td>9147</td>
</tr>
<tr>
<td>1468</td>
</tr>
<tr>
<td>7233</td>
</tr>
<tr>
<td>5742</td>
</tr>
<tr>
<td>1129</td>
</tr>
<tr>
<td>8050</td>
</tr>
<tr>
<td>5063</td>
</tr>
<tr>
<td>4603</td>
</tr>
</tbody>
</table>

The first 2 uniformly-distributed random numbers are

\[ R_1 = 0.4875, \]
\[ R_2 = 0.3519 \]

\[ X = R_1 = 0.4875, \]
\[ Y = R_2 C = 0.3519 \times 1.7777778 = 0.6256 \]

\[ f(X) = 1.4619 > Y \quad \text{ACCEPT!} \]

the point \((x, y)\) is under the density curve!
The next 2 uniformly-distributed random numbers are 
\[ R_1 = 0.8147, \ R_2 = 0.1468 \]
\[ X = 0.8147, \ Y = R_2 \cdot C = 0.260978 \]
\[ f(X) = 1.47588 \quad > Y \quad \text{ACCEPT!} \]

Again, the point \((x, y)\) is under the density curve!

The next 2 uniformly-distributed random numbers are 
\[ R_1 = 0.7233, \ R_2 = 0.5742 \]
\[ X = 0.7233, \ Y = R_2 \cdot C = 1.0208 \]
\[ f(X) = 1.73711 \quad > Y \quad \text{ACCEPT!} \]

Again, the point \((x, y)\) is under the density curve!
The next 2 uniformly-distributed random numbers are:

\[ R_1 = 0.1129, \quad R_2 = 0.8050 \]

\[ X = 0.1129, \quad Y = R_2 C = 1.4311 \]

\[ f(X) = 0.13569 < Y \quad \text{REJECT!} \]

The point \((x, y)\) is NOT under the curve, and is rejected.

The next 2 uniformly-distributed random numbers are:

\[ R_1 = 0.5063, \quad R_2 = 0.4603 \]

\[ X = 0.5063, \quad Y = R_2 C = 0.8183 \]

\[ f(X) = 1.5187 > Y \quad \text{ACCEPT!} \]

again, the point \((x, y)\) is under the density curve!
The first 4 random numbers having the desired BETA distribution are, therefore,

\begin{align*}
0.4876 \\
0.8147 \\
0.7233 \\
0.4129 \quad \text{rejected} \\
0.5063
\end{align*}