the Poisson process
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The Poisson Process as a limiting case of the Bernoulli Process
Consider the following situation:

\[ \cdot \cdot \cdot 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ \cdots \ t \]

A time interval of length \( t \) seconds is divided into one-second intervals, with the probability of a vehicle arriving at an intersection during a one-second interval being a small number \( p \). (Assume that the probability that more than one vehicle arrives is negligible.)
Consider the Bernoulli process \( \{X_k; k=1,2,...\} \) where \( X_k = 1 \) if a vehicle arrives during the \( k^{\text{th}} \) second, and the associated counting process \( \{N_t\} \) which counts the number of arrivals during the interval \([0,t]\).

Then \( N_t \) has the binomial distribution:

\[
P(N_t = x) = \binom{t}{x} p^x (1-p)^{n-x}
\]

with expected value \( \nu = tp \).
Consider what happens as we divide \([0,t]\) into \(n\) smaller time intervals, but in such a way that the expected number of arrivals in \([0,t]\) remains constant, \(\nu\).

That is, the probability of an arrival in each of these small intervals must be \(\frac{\nu}{n}\), and

\[
P(N_t = x) = \binom{n}{x} \left(\frac{\nu}{n}\right)^x (1 - \frac{\nu}{n})^{n-x}
\]
\[ P(N_t = x) = \binom{n}{x} \left( \frac{\nu}{n} \right)^x \left( 1 - \frac{\nu}{n} \right)^{n-x} \]

\[ = \frac{n!}{x!(n-x)!} \left( \frac{\nu}{n} \right)^x \left( 1 - \frac{\nu}{n} \right)^n \left( 1 - \frac{\nu}{n} \right)^{-x} \]

\[ = \frac{\nu^x}{x!} \left( 1 - \frac{\nu}{n} \right)^n \left( \frac{n!}{(n-x)!} \right) \frac{1}{n^x \left( 1 - \frac{\nu}{n} \right)^x} \]
Consider the limit of this distribution as $n \rightarrow +\infty$

$$P(N_t = x) = \frac{v^x}{x!} \left(1 - \frac{v}{n}\right)^n \left(\frac{n!}{(n-x)!}\right) \frac{1}{n^x (1 - \frac{v}{n})^x}$$

$$\downarrow$$

$$e^{-v} \frac{n(n-1)(n-2) \cdots (n-x+1)}{\left[n \left(1 - \frac{v}{n}\right)\right]^x}$$

$$\rightarrow \frac{n^x}{n^x} = 1$$
\[ P(N_t = x) = \frac{\nu^x}{x!} e^{-\nu} \]

If the arrival rate is \( \lambda \) /second, then \( \nu = \lambda t \) and

\[ P(N_t = x) = \frac{\(\lambda t\)^x}{x!} e^{-\lambda t} \]

for \( x = 0, 1, 2, 3, \ldots \)

\[ \text{Poisson Distribution} \]
Poisson Distribution

\[ P(N_t = x) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \]

for \( x = 0, 1, 2, 3, \ldots \)

Mean Value

\[ E(N_t) = \lambda t \]

Variance

\[ \text{Var}(N_t) = \lambda t \]

mean and variance are equal!
Example A left-turn lane at an intersection has a capacity of 3 autos. 30% of autos arriving at the intersection wish to turn left. The expected number of autos arriving during a red signal is 6.

What is the probability that the capacity of the left-turn lane is exceeded during a red signal?
Given that $N$ autos arrive, the number $X$ of left-turning autos has the \textit{binomial} distribution.

The number $N$ of autos arriving during the red signal has the \textit{Poisson} distribution.

\[
P\{X>3\} = \sum_{N=4}^{\infty} P\{X>3 \mid N \text{ arrivals}\} \ P\{N \text{ arrivals}\}
\]

- \text{computed using binomial distn.}
- \text{computed using Poisson distn.}

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\[ P\{X > 3 \mid N \text{ arrivals}\} = \sum_{x=4}^{N} \binom{N}{x} (0.3)^x (0.7)^{N-x} \]

\[ = 1 - \sum_{x=0}^{3} \binom{N}{x} (0.3)^x (0.7)^{N-x} \]

\[ P\{N \text{ arrivals}\} = \frac{6^N}{N!} e^{-6} \]
Poisson Distribution

Probability that \( N \) autos arrive during the red signal

\[
\frac{6^N}{N!} e^{-6}
\]

\(0.0022\)

\(0.0099\)

\(0.0113\)

\(0.0052\)

\(0.0225\)

\(0.0413\)

\(0.0688\)

\(0.1033\)

\(0.1377\)

\(0.1606\)

\(0.1606\)

\(0.1339\)

\(0.0892\)

\(0.0446\)

\(0.0149\)

\(0.0025\)

\(N\)

\(0\)

\(1\)

\(2\)

\(3\)

\(4\)

\(5\)

\(6\)

\(7\)

\(8\)

\(9\)

\(10\)

\(11\)

\(12\)

\(13\)

\(14\)

\(15\)

\(\cdots\)

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The probability that the capacity of the left-turn lane is exceeded during each red signal is about 11%.

| \(N\) | \(P(N)\)  | \(P(X|N)\) | \(P(X|N)P(N)\) |
|-------|----------|------------|-----------------|
| 0     | 0.00247875 | 0.00000000 | 0.00000000     |
| 1     | 0.01487251  | 0.00000000 | 0.00000000     |
| 2     | 0.04481754  | 0.00000000 | 0.00000000     |
| 3     | 0.08923508  | 0.00000000 | 0.00000000     |
| 4     | 0.13386262  | 0.00810000 | 0.0010421      |
| 5     | 0.18082314  | 0.03078000 | 0.00494398     |
| 6     | 0.18062314  | 0.07047000 | 0.01131911     |
| 7     | 0.13787898  | 0.12503600 | 0.01735226     |
| 8     | 0.10325773  | 0.19410435 | 0.02004378     |
| 9     | 0.06693849  | 0.27034090 | 0.01850986     |
| 10    | 0.04130309  | 0.35038928 | 0.01447216     |
| 11    | 0.02252896  | 0.43043786 | 0.00969731     |
| 12    | 0.01126448  | 0.50740423 | 0.00571655     |
| 13    | 0.00519899  | 0.57939435 | 0.00301227     |
| 14    | 0.00228144  | 0.64483257 | 0.00143678     |
| 15    | 0.00089128  | 0.70313207 | 0.00062687     |

\[\sum P(X|N)P(N) = 0.1083\]
Time between arrivals

Suppose that the number of arrivals in an interval has the Poisson distribution with arrival rate $\lambda$/second.

Let $T_1 =$ time of the first arrival.
What is the distribution of $T_1$?
Poisson processes- Intro

\[ \begin{align*}
P(T_1 > t) &= \text{P( NO arrivals occur in interval } [0,t]) \\
&= \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} \\
\text{CDF: } P(T_1 \leq t) &= F(t) = 1 - e^{-\lambda t} \\
\text{Density function: } f(t) &= \frac{d}{dt} F(t) = \lambda e^{-\lambda t}
\end{align*} \]
Exponential Distribution

\[ F(t) = 1 - e^{-\lambda t} \]

Mean Value

\[ E(T_1) = \frac{1}{\lambda} \]

Variance

\[ \text{Var}(T_1) = \frac{1}{\lambda^2} \]
Example

Suppose that the arrival rate for northbound autos is 6 per 30 second red signal, i.e., 0.2/second

What is the distribution of the arrival time of the first auto? (This will also be the distribution of the time between arrivals!)

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\[ f(t) = \lambda e^{-\lambda t} \]
\[ \lambda = 0.2/\text{sec.} \]

\[ F(t) = 1 - e^{-\lambda t} \]

Exponential Distribution

\[ P[T_1 \leq t] \]

\begin{tabular}{|c|c|}
  \hline
  t & \( F(t) \) \\
  \hline
  1 & 0.18127 \\
  2 & 0.32968 \\
  3 & 0.45119 \\
  4 & 0.55087 \\
  5 & 0.63212 \\
  6 & 0.69981 \\
  7 & 0.75340 \\
  8 & 0.79810 \\
  9 & 0.83470 \\
 10 & 0.86486 \\
 11 & 0.88920 \\
 12 & 0.90928 \\
 13 & 0.92573 \\
 14 & 0.93919 \\
 15 & 0.95021 \\
 16 & 0.95924 \\
 17 & 0.96663 \\
 18 & 0.97268 \\
 19 & 0.97783 \\
 20 & 0.98168 \\
  \hline
\end{tabular}

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Memoryless Property

Exponential Distribution

Suppose that it is known that, at time $t_0$, the first arrival has not yet occurred, i.e., $T_1 > t_0$.

What is the conditional distribution of $T_1$? That is, what is $P(T_1 \leq t \mid T_1 > t_0)$ for $t > t_0$?
Memoryless Property

\[ P(T_1 \leq t \mid T_1 > t_0) = \frac{P(T_1 \leq t \cap T_1 > t_0)}{P(T_1 > t_0)} = \frac{P(t_0 \leq T_1 \leq t)}{P(T_1 > t_0)} \]

\[ = \frac{F(t) - F(t_0)}{1 - F(t_0)} = \frac{1 - e^{-\lambda t}}{e^{-\lambda t_0}} - \frac{1 - e^{-\lambda t_0}}{e^{-\lambda t_0}} \]

\[ = \frac{e^{-\lambda t_0} - e^{-\lambda t}}{e^{-\lambda t_0}} = 1 - e^{-\lambda(t - t_0)} \]
Memoryless Property

\[ P\{T_1 \leq t \mid T_1 > t_0 \} = 1 - e^{-\lambda(t-t_0)} = P\{T_1 \leq t - t_0\} \]

If the time \( \tau \) is reckoned from time \( t_0 \), i.e., \( \tau = t - t_0 \), then

\[ P\{T_1 \leq t \mid T_1 > t_0 \} = P\{T_1 \leq t - t_0\} = P\{T_1 \leq \tau\} \]

In other words, the failure of an arrival to occur before time \( t_0 \) does not alter one's prediction of the length of time (from \( t_0 \)) before the next arrival.
Time of \( k^{th} \) Arrival

Let \( T_k \) = time of \( k^{th} \) arrival,
\[ \tau_k = T_k - T_{k-1} = \text{time between arrivals } k-1 \text{ and } k. \]

Suppose that \( \tau_k \) (\( k=1,2,3,\ldots \)) have identical and independent exponential distributions with rate \( \lambda \).

Then \( T_k \) is the \textit{sum} of \( k \) random variables with exponential distributions.

It is said to have a \textit{k-Erlang} distribution.

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Erlang Distribution

time of $k^{th}$ arrival in a Poisson process

Density function

$$f(t) = \frac{\lambda (\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}$$

$$= \frac{\lambda (\lambda t)^{k-1} e^{-\lambda t}}{\Gamma(k)}$$

where the Gamma function is defined by

$$\Gamma(k) = \int_0^\infty e^{-u} u^{k-1} du$$

$$= (k-1)! \text{ if } k \text{ integer}$$
Erlang Distribution

CDF

\[ F(t) = \frac{\Gamma(k, \lambda t)}{\Gamma(k)} \]

where \( \Gamma(k, x) \) is the "incomplete Gamma function" defined by

\[ \Gamma(k, x) = \int_0^x e^{-u} u^{k-1} \, du \]

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Alternate computation, when $k$ is integer

CDF

$$F(t) = P\{T_k \leq t\} = P\{N_t \geq k\}$$

$$= 1 - P\{N_t < k\}$$

$$= 1 - P\{N_t \leq k-1\}$$

where $N_t =$ # arrivals at time $t$

has the Poisson distribution:

$$F(t) = 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

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Erlang Distribution

Mean Value \[ \mu = \frac{k}{\lambda} \]

Variance \[ \sigma^2 = \frac{k}{\lambda^2} \]

(These expressions result from the fact that the random variable is the sum of \( k \) i.i.d. random variables.)

More generally, when \( k \) is not an integer, the probability distribution is called the Gamma distribution.

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Erlang Distribution

example:
\[ \lambda = 0.2 \]

As \( k \) increases, the distribution becomes less skewed, and more "normal."

In the limit, as \( k \to \infty \), the \( k \)-Erlang distribution converges to the Normal distribution.