Midterm Exam Solutions

October 19, 1994

Possible Score
A. True/False & Multiple Choice  30 _____
B. Sensitivity analysis (LINDO)  20 _____
C.1. Transportation  15 _____
C.2. Decision Tree  15 _____
C.3. Simplex LP Method  15  _____
total possible:  80 _____

Part A

True/False:

True 1. A "pivot" in a nonbasic column of an LP simplex tableau will make it a basic column.
True 2. If you increase the right-hand-side of a "greater-than-or-equal" constraint in a minimization LP, the optimal objective value will either increase or stay the same.
False 3. The "reduced cost" in LP provides an estimate of the change in the objective value when a right-hand-side of a constraint changes.
False 4. In the two-phase simplex method, Phase One computes the optimal dual variables, followed by Phase Two in which the optimal primal variables are computed.
True 5. At the completion of the revised simplex method applied to an LP, the simplex multipliers give the optimal solution to the dual of the LP.
True 6. In the LP formulation of the project scheduling problem, the constraints include \( Y_B - Y_A = d_A \) if activity A must precede activity B, where \( d_A \) = duration of activity A.
False 7. In CPM, the "forward pass" is used to determine the latest time (LT) for each event (node).
True 8. The "minimum ratio test" is used to determine the pivot row in the simplex method.
True 9. The A-O-N project network does not require any "dummy" activities, except for the "begin" and "end" activities.
True 10. At the end of the first phase of the two-phase simplex method, the phase-one objective function must be zero if the LP is feasible.
False 11. In a transportation problem, if the current dual variables \( U_2 = 3 \) and \( V_4 = 1 \), and \( C_{24} = 5 \), then the current basic solution cannot be optimal.
True 12. In a transportation problem, if the current dual variables \( U_2 = 3 \) and \( V_4 = 1 \), and \( C_{24} = 5 \), then \( X_{24} \) cannot be basic.
True 13. All tasks on the critical path of a project schedule have their latest finish time equal to their earliest finish time.
True 14. A "dummy" activity in an A-O-A project network always has duration zero, but can be on the critical path.
False 15. If a zero appears on the right-hand-side of row i of an LP tableau, then at the next simplex iteration you cannot pivot in row i.
False 16. When maximizing in the simplex method, the value of the objective function will not improve at the next pivot if the current tableau is degenerate.
False 17. When minimizing in the simplex method, you must select the column which has the smallest (i.e., the most negative) reduced cost as the next pivot column.
True 18. If the "float" ("slack") of an activity of a project is positive, then the activity cannot be "critical" in the schedule.
False 19. If a primal minimization LP problem has a cost which is unbounded below, then the dual maximization problem has an objective which is unbounded above.
False 20. If $X_{ij}=0$ in the transportation problem, then dual variables $U$ and $V$ must satisfy $C_{ij}=U_i+V_j$.

**Multiple Choice:** Write the appropriate letter (a, b, c, d, or e) : (NOTA = None of the above).

- **e** 21. If, in the optimal primal solution of an LP problem (min $cx$ st $Ax\geq b$, $x\geq 0$), there is zero slack in constraint #1, then in the optimal dual solution,
  (a) dual variable #1 must be zero
  (b) dual variable #1 must be positive
  (c) slack variable for dual constraint #1 must be zero
  (d) dual constraint #1 must be slack
  (e) NOTA

- **c** 22. If, in the optimal dual solution of an LP problem (min $cx$ st $Ax=b$, $x\geq 0$), variable #2 is positive, then in the optimal primal solution,
  (a) variable #2 must be zero
  (b) variable #2 must be positive
  (c) slack variable for constraint #2 must be zero
  (d) constraint #2 must be slack
  (e) NOTA

- **b** 23. If you make a mistake in choosing the pivot row in the simplex method, the solution in the next tableau will
  (a) will be nonbasic
  (b) will be nonfeasible
  (c) will have a worse objective value
  (d) will be degenerate
  (e) NOTA

- **c** 24. If you make a mistake in choosing the pivot column in the simplex method, the solution in the next tableau will
  (a) be nonbasic
  (b) be nonfeasible
  (c) have a worse objective value
  (d) be degenerate
  (e) NOTA

- **d** 25. If there is a tie in the "minimum-ratio test" of the simplex method, the solution in the next tableau will
  (a) will be nonbasic
  (b) will be nonfeasible
  (c) will have a worse objective value
  (d) will be degenerate
  (e) NOTA

The problems (26)-(30) below refer to the following LP: (with inequalities converted to equations:)

Maximize $3X_1 + 2X_2$
subject to $2X_1 + X_2 \leq 100$
$X_1 + X_2 \leq 80$
$X_1 \leq 40$
$X_1 \geq 0, X_2 \geq 0$

The feasible region includes points A, B, C, D, E, F, G, H, I.
27. At point G, the basic variables include the variables
   (a) \( X_2 \) & \( X_3 \)  
   (b) \( X_3 \) & \( X_4 \)  
   (c) \( X_1 \) & \( X_5 \)  
   (d) \( X_1 \) & \( X_4 \)  
   (e) NOTA

28. Which point is degenerate in this problem?
   (a) point B  
   (b) point G  
   (c) point I  
   (d) point H  
   (e) NOTA

29. If point G is optimal, then which dual variables must be zero, according to the Complementary Slackness Theorem?
   (a) both \( Y_1 \) and \( Y_2 \)  
   (b) both \( Y_1 \) and \( Y_3 \)  
   (c) both \( Y_2 \) and \( Y_3 \)  
   (d) \( Y_1 \) only  
   (e) \( Y_2 \) only  
   (f) \( Y_3 \) only

30. For each alternative pair in parentheses, check the appropriate choice to obtain the dual LP of the above primal problem (with the inequality constraints):
   \[ \text{Max} \quad 100Y_1 + 80Y_2 + 40Y_3 \]
   subject to
   \[ 2Y_1 + Y_2 + Y_3 \leq X \]
   \[ Y_1 + Y_2 \leq X \]
   \[ AD + AR \leq 40 \]
   \[ BD + BR \leq 30 \]
   \[ -0.25 D + AD \geq 0 \]
   \[ -0.5 D + BD \geq 0 \]
   \[ -0.75 R + AR \leq 0 \]

LINDO analysis

**Problem Statement:** McNaughton Inc. produces two steak sauces, spicy Diablo and mild Red Baron. These sauces are both made by blending two ingredients A and B. A certain level of flexibility is permitted in the formulas for these products. Indeed, the restrictions are that:
   i) Red Baron must contain no more than 75% of A.
   ii) Diablo must contain no less than 25% of A and no less than 50% of B

Up to 40 quarts of A and 30 quarts of B could be purchased. McNaughton can sell as much of these sauces as it produces at a price per quart of $3.35 for Diablo and $2.85 for Red Baron. A and B cost $1.60 and $2.05 per quart, respectively. McNaughton wishes to maximize its net revenue from the sale of these sauces.

**Define**
   \( D = \text{quarts of Diablo to be produced} \)
   \( R = \text{quarts of Red Baron to be produced} \)
   \( AD = \text{quarts of A used to make Diablo} \)
   \( AR = \text{quarts of A used to make Red Baron} \)
   \( BD = \text{quarts of B used to make Diablo} \)
   \( BR = \text{quarts of B used to make Red Baron} \)

**The LINDO output for solving this problem follows:**

\[
\text{MAX} \quad 3.35 D + 2.85 R - 1.6 AD - 1.6 AR - 2.05 BD - 2.05 BR \\
\text{SUBJECT TO} \\
\quad 2) \quad - D + AD + BD = 0 \\
\quad 3) \quad - R + AR + BR = 0 \\
\quad 4) \quad AD + AR <= 40 \\
\quad 5) \quad BD + BR <= 30 \\
\quad 6) \quad -0.25 D + AD >= 0 \\
\quad 7) \quad -0.5 D + BD >= 0 \\
\quad 8) \quad -0.75 R + AR <= 0 \\
\text{END} \\
\text{OBJECTIVE FUNCTION VALUE} \\
\quad 1) \quad 99.0000000 \\
\text{VARIABLE} \quad \text{VALUE} \quad \text{REDUCED COST}
\]

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1. How many quarts of "Red Baron" are produced in the optimal solution?  __20 quarts__
2. How much total profit does the firm make on these two products?  $_ 99.00 ___
3. What additional amount should the firm be willing to pay to have another quart of ingredient B available?  $ _2.30 _____
What is the total amount the firm should be willing to pay for another quart of ingredient B?
$\_4.35\_ $

How many quarts should they be willing to buy at this price? 10 quarts

4. Using one more quart of ingredient B (i.e., a total of 31 quarts) would
   (___increase/ X decrease) the "slack variable" in row #5 by ____ unit (quart).
5. Using one more quart of ingredient B (i.e., a total of 31 quarts) would result the following changes in the variables:
   • the "slack variable" in row #6? (X_increase/decrease) by 0.75
   • the quantity of A used in producing "Red Baron"? (X_increase/decrease) by 1.5
   • the quantity of A used in producing "Diablo"? (X_increase/decrease) by 1.5
   • the total quantity of "Diablo" produced? (X_increase/decrease) by 3.0
   • the total quantity of "Red Baron" produced? (X_increase/decrease) by 2.0

6. How much can the price of "Red Baron" increase before the composition of the current optimal product mix changes? $0.50

7. What is the optimal value of the objective function of the LP which is the dual of this problem? $99.00 This dual LP objective is minimized/

(C.1) Transportation Problem. Consider the problem with the initial solution specified on the left below:

1. Is this a basic solution? (circle: Yes / No )
2. Is this a degenerate solution? (circle: Yes / No )
3. If the dual variable $u_1$ (for supply constraint #1) were assigned the value zero, the values of the other dual variables are: $u_2 = -2$, $u_3 = -3$, $v_1 = +5$, $v_2 = +7$, $v_3 = +9$, $v_4 = +2$.
4. If $X_{33}$ were to be increased, what would be the change in the cost function per unit increase in $X_{33}$? -1
5. If $X_{33}$ were to enter the basis, what would be its new value? _1
6. If $X_{33}$ were to enter the basis, what variable(s) would leave the basis? Circle as many as apply: $X_{13}$, $X_{14}$, $X_{21}$, $X_{22}$, $X_{32}$, $X_{33}$, $X_{34}$
7. If $X_{33}$ were to enter the basis, would the new basic solution be degenerate? (circle: Yes / No )
8. Which other variable instead of $X_{33}$ would result in a degenerate basic solution if it were to enter the basis? Circle as many as apply: $X_{11}$, $X_{12}$, $X_{24}$, $X_{32}$, $X_{33}$, $X_{34}$
9. What is the cost of the initial solution shown above? _81

C.2. Decision Tree. The decision sciences department is trying to determine which of two copying machines to purchase. Both machines will satisfy the department's needs for the next ten years.

Machine 1 costs $3000 and has a maintenance agreement, which, for an annual fee of $100, covers all repairs. Machine 2 costs $2500, and its annual maintenance cost is a random variable.
At present, the decision sciences department believes there is a 40% chance that the annual cost for machine 2 will be $0, a 40% chance it will be $100, and a 20% chance it will be $200.

For a fee of $100 the department can have Joe, a trained repairman, evaluate the quality of machine 2 before the purchase decision is made. If Joe believes that machine 2 is *satisfactory*, there is a 70% chance that its annual maintenance cost will be $0 and a 30% chance that it will be $100. If Joe believes that machine 2 is *unsatisfactory*, there is a 10% chance that the annual maintenance cost will be $0, a 40% chance it will be $100, and a 50% chance it will be $200. The department believes that there is a 70% probability that Joe will give a satisfactory report after evaluating machine 2. Based upon this data, the decision tree below is drawn:

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Fill the blank boxes in the decision tree above and answer the following questions:

1. If Joe's evaluation of machine 2 is "unsatisfactory" and they were to select machine 2, what would be their expected cost (including the cost of Joe's evaluation)?
   a. $ 2600  c. $ 3300  e. $ 4500
   b. $ 2900  d. $ 4000  f. NOTA

2. This computation is referred to as "folding back" which node in the tree above?
   a. Node # 4  c. Node #6  e. Node #8
   b. Node # 5  d. Node #7  f. NOTA

3. What is EVSI (i.e., the expected value of Joe's evaluation)?
   a. negative $ 30  c. $ 40  e. $ 120
   b. negative $ 20  d. $ 70  f. NOTA

4. Which machine should the department buy?
   a. Machine #1  c. a "toss-up"
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Suppose that the department were able to get "perfect information", i.e., a perfect prediction of the annual maintenance costs. Based upon this supposition, the department head drew the decision tree below:

Fill the blank boxes in the decision tree above and answer the following questions:

5. What is the minimum expected cost to the department if they were able to obtain this perfect prediction?
   a. $  2600   b. $  2900   c. $  3200   d. $  4000   e. $  4500   f. NOTA

6. What is EVPI (expected value of perfect information)?
   a. $   70   b. $   100   c. $   150   d. $   200   e. $   300   f. NOTA

C.3. Simplex Algorithm for LP: At an intermediate step of the simplex algorithm, the tableau is:

<table>
<thead>
<tr>
<th>z</th>
<th>x_1</th>
<th>x_2</th>
<th>x_3</th>
<th>x_4</th>
<th>x_5</th>
<th>x_6</th>
<th>x_7</th>
<th>x_8</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3</td>
<td>1</td>
<td>-2</td>
<td>-15</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-3</td>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

1. What are the basic variables for this tableau? (circle: X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8 RHS )
2. What are the current values of the variables?
   \[ X_1 = 0, \quad X_2 = 0, \quad X_3 = 4, \quad X_4 = 2, \quad X_5 = 6, \quad X_6 = 0, \quad X_7 = 0, \quad X_8 = 0 \]

3. Increasing \( X_1 \) would (circle: increase / decrease) the objective function.

4. Increasing \( X_6 \) would (circle: increase / decrease) the objective function.

5. What is the substitution rate of \( X_1 \) for \( X_3 \)? \(-4\)
   
   That is, if \( X_1 \) is increased by 1 unit, \( X_3 \) (circle: increases / decreases) by a quantity \( 4 \).

6. Suppose that \( X_6 \) and \( X_7 \) are slack variables in the first 2 constraints, and \( X_8 \) a surplus variable in the last constraint. (That is, the first two constraints were originally constraints, and the third was originally a constraint, all converted to equations.) What are the values of the simplex multipliers for this tableau? \( \Pi_1 = +3, \quad \Pi_2 = -1, \quad \Pi_3 = -2 \).

7. If the objective is to (circle: maximize / minimize) the objective \( z \), the optimal solution is unbounded.

8. If the objective is not that which you specified in (7), perform a pivot to improve the objective function, and write the new tableau below. (You need only complete the unshaded portion.)

If the objective is the maximize, then either \( X_1 \) or \( X_7 \) could be selected to enter the basis.

\[
\begin{array}{cccccccc}
-2 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\
0 & -2 & 0 & & & & & & -19 \\
1 & 4 & 0 & & & & & & \\
0 & 1 & 0 & & & & & & \\
0 & -2 & 1 & & & & & & \\
\end{array}
\]

If \( X_1 \) is entered into the basis:

\[
\begin{array}{cccccccc}
-2 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\
-1 & 0 & 0 & & & & & & -19 \\
1 & 0 & 0 & & & & & & \\
0 & 1 & 0 & & & & & & \\
-3 & 0 & 1 & & & & & & \\
\end{array}
\]

If \( X_7 \) is entered into the basis: