Oscillating flow of a viscoelastic fluid in a pipe with the fractional Maxwell model

Youbing Yin *, Ke-Qin Zhu

Department of Engineering Mechanics, Tsinghua University, Beijing, 100084,
People’s Republic of China

Abstract

The unidirectional oscillating flow of a viscoelastic fluid with the fractional Maxwell model is studied. The flow is produced by a periodic pressure gradient in an infinite straight pipe. Exact solutions are obtained in the time and frequency domains by using Fourier transform. The fractional Maxwell model exhibits resonance phenomena similar to that of the Maxwell model. However, the number of the resonance peaks, the amplitudes of the velocity enhancement and the frequencies where resonance occurs are totally different. The amplitudes of resonance peaks decay rapidly with frequency. In addition, a critical dimensionless radius is found for the fractional Maxwell model. Below the critical radius, the resonance behavior is weakened, but above this value, the enhancement is strengthened.

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* Corresponding author.
E-mail address: yinyb03@mails.tsinghua.edu.cn (Y. Yin).
1. Introduction

Investigation of oscillating flow in a pipe has many applications in various branches of science. In the field of bio-engineering, this type of investigation is of particular interest since blood in veins is forced by a periodic pressure gradient. In the petroleum and chemical industries, there are also many problems which involve the dynamic response of the fluid to the frequency of the periodic pressure gradient. In these fields, however, the fluids, such as blood, oil, and polymer solution, not only have the viscosity of the fluid, but also exhibit the elasticity of the solid. These kinds of fluids are often treated as viscoelastic fluids. Because of the difficulty to suggest a single model which exhibits all characteristic of viscoelastic fluids, there exist many viscoelastic models and constitutive equations. Among those the Maxwell model is the simplest one. It is constructed by the series of the spring and dashpot and can qualitatively reflect some properties of the viscoelastic fluid, such as stress relaxation.

During the past few years, attention has been given to the study on oscillating flow of viscoelastic fluids in a pipe [1–4]. In those studies, the Maxwell model was adopted to describe the viscoelastic fluid. Rahaman et al. [1] analyzed some kinds of basic unsteady pipe flows of viscoelastic fluid with Maxwell model, including periodic oscillation flow. They obtained the exact solutions and analyzed the velocity profile. Andrienko et al. [2] also studied the unidirectional oscillating flow of viscoelastic fluid with the Maxwell model in a tube, and they found that the instantaneous velocities drastically increase at certain frequencies of the oscillating pressure gradient and called this phenomenon as the resonance-like behavior of viscoelastic fluids. In addition, Rio et al. [3] studied the flow of a Maxwell fluid in a tube under the oscillating pressure gradient in order to analyze the effects of elasticity on the dynamics of fluids in porous media. They gave an analytical expression for the dynamic permeability and showed that a dramatic enhancement at certain frequencies occurred. Based on Rio’s research [3], Tsiklauri et al. [4] studied the similar problem while considering the longitudinal vibration of the wall of the tube. The transition from a dissipative regime to an elastic regime was found in their research.

However, the Maxwell model is a linear model for viscoelastic fluids and it is proper only under the condition that non-linear effects are negligible, such as very low strain and stress. The model also leads to an exponential stress relaxation modulus; for real materials, however, the stress relaxation obeys an algebraic decay [5]. Recently, fractional calculus has encountered much success in the description of viscoelasticity [6–9]. The fractional derivative models with the algebraic stress decay can be easily constructed. And the fractional Maxwell model is one among these fractional models. Experimental research has shown that a better agreement of the experimental data could be achieved with the fractional Maxwell model than the ordinary Maxwell model [6]. With this mind, Tan et al. [10] studied four unsteady flows of a viscoelastic fluid with the
fractional Maxwell model between two infinite parallel plates. Hayat et al. [11] also took the fractional Maxwell model into account and discussed three types of unidirectional flows which were induced by general periodic oscillations of a plate.

The purpose of this paper is to discuss the unidirectional oscillating flow of a viscoelastic fluid. The flow is produced by a period pressure gradient in an infinite straight pipe. Although there are many literatures on this problem [1–4], here we consider the fractional Maxwell model to describe the viscoelastic fluid. First, we introduce the construction of the fraction model for the viscoelastic fluid. Then we give the governing equations on this problem. By using the Fourier transform, we obtain the exact solutions of the flow in the time and frequency domains. Based on the solution in the time domain, we define the velocity enhancement as the ratio of the amplitude of oscillating velocity on the axis of the tube to the velocity when the pressure is constant. We discuss further the velocity enhancement of the factional Maxwell model using two pairs of fractional parameters, which are obtained from a experimental research in [6].

2. Construction of the fractional model for the viscoelastic fluid

There are two methods to construct the fractional model [5]. The direct one is to replace the time derivatives of an integer order by the Riemann–Liouville fractional calculus operators. For example, the fractional Maxwell model can be constructed through the replacement

\[ \sigma + \dot{\lambda}^\alpha \frac{d^\alpha \sigma}{dt^\alpha} = E \dot{\lambda}^\beta \frac{d^\beta E}{dt^\beta}, \]  

(1)

where \( \sigma \) is the shear stress, \( \varepsilon \) is the shear strain, \( E \) is a shear modulus, \( \lambda = \mu/E \) is a relaxation time, \( \mu \) is the constant viscosity coefficient, and \( \alpha \) and \( \beta \) are fractional parameters which satisfy \( 0 < \alpha < 1, 0 < \beta < 1 \). This approach is very simple, but its physical meaning is ambiguous. The models must be determined after further analysis. For example, the restriction of \( \alpha \leq \beta \) must be given in Eq. (1) to satisfy the physical meaning.

In the following, we give the second method which is able to guarantee that the fractional models obtained through it are physically correct.

First, we introduce the stress–strain relation with fractional-order derivative [12]

\[ \sigma(t) = E \dot{\lambda}^\gamma \frac{d^\gamma E}{dt^\gamma}, \quad (0 < \gamma < 1) \]  

(2)

\[ G(t) = \frac{E}{\Gamma(1 - \gamma)} \left( \frac{t}{\lambda} \right)^{-\gamma}, \]  

(3)
where $G(t)$ is the relaxation modulus which has the algebraic form and $\Gamma(\cdot)$ is the Gamma function. This model is also known as Scott–Blair’s model, and it can be realized physically through hierarchical arrangements of springs and dashpots, such as ladders, trees and so on. We can also interpret the model as an interpolation between Hooke’s law ($\gamma = 0$) and Newton’s law ($\gamma = 1$).

In order to construct the fractional model, we introduce the fractional element defined as the mechanical elements which obey Eq. (2). A fractional element is determined by three parameters $(\gamma, E, \lambda)$ and is symbolized by a triangle, as shown in Fig. 1(c). In the following, we will treat a fractional element as the same status as a spring and a dashpot which are shown in Fig. 1(a) and (b) to construct the viscoelastic models.

We carry out a proper generalization of the ordinary Maxwell model, which is depicted in Fig. 2(a). We simply replace the two classical elements by fractional elements with the parameters $(\gamma_1, E_1, \dot{\lambda}_1)$ and $(\gamma_2, E_2, \dot{\lambda}_2)$ in Fig. 2(b).

![Fig. 1. Single elements: (a) elastic element; (b) viscous element and (c) fractional element.](image)

![Fig. 2. (a) The classical Maxwell model and (b) the fractional Maxwell model.](image)
The fractional elements obey the stress–strain relations: \( \sigma_1(t) = E_1 \lambda_1^{\gamma_1} \frac{d^{\gamma_1} \varepsilon_1(t)}{dt^{\gamma_1}} \) and \( \sigma_2(t) = E_2 \lambda_2^{\gamma_2} \frac{d^{\gamma_2} \varepsilon_2(t)}{dt^{\gamma_2}} \). Due to the series connection of the elements, it is easy to obtain
\[
\sigma(t) + \frac{E_1 \lambda_1^{\gamma_1}}{E_2 \lambda_2^{\gamma_2}} \frac{d^{\gamma_1-\gamma_2} \sigma(t)}{dt^{\gamma_1-\gamma_2}} = E_1 \lambda_1^{\gamma_1} \frac{d^{\gamma_1} \varepsilon(t)}{dt^{\gamma_1}}. \tag{4}
\]
Without loss of generality, we can assume \( \gamma_1 > \gamma_2 \). Eq. (4) can be further simplified by introducing new variables
\[
\lambda = \left( \frac{E_1 \lambda_1^{\gamma_1}}{E_2 \lambda_2^{\gamma_2}} \right)^{1/\gamma_1-\gamma_2}, \quad E = E_1 \left( \frac{\lambda}{\lambda} \right)^{\gamma_1}
\]
and this leads to
\[
\sigma(t) + \lambda^{\gamma_1-\gamma_2} \frac{d^{\gamma_1-\gamma_2} \sigma(t)}{dt^{\gamma_1-\gamma_2}} = E \lambda^{\gamma_1} \frac{d^{\gamma_1} \varepsilon(t)}{dt^{\gamma_1}}, \tag{5}
\]
which is the constitutive equation of the fractional Maxwell model. If we replace \( \gamma_1-\gamma_2 \) and \( \gamma_1 \) with \( \alpha \) and \( \beta \), respectively, we can get the same equation as Eq. (1)
\[
\sigma + \lambda^\alpha \frac{d^\alpha \sigma}{dt^\alpha} = E \lambda^\beta \frac{d^\beta \dot{\varepsilon}}{dt^\beta}. \tag{6}
\]
where \( \dot{\varepsilon} \) is the shear rate.

**3. Governing equations and analytical solutions**

The unidirectional oscillating flow of a viscoelastic fluid in an infinite straight pipe with the radius of \( a \) under a periodic pressure gradient is studied. The viscoelastic fluid is described by the incompressible fractional Maxwell model.

The constitutive equation for the fractional Maxwell model is given in Eq. (7). It can be seen that this model includes the classical Maxwell model as a special case for \( \alpha = \beta = 1 \) and the Newtonian fluid model when \( \alpha = 0, \beta = 1 \).

Following the rules of fractional calculus [13], Eq. (7) can be rewritten as follows:
\[
\sigma + \lambda^\alpha \frac{d^\alpha \sigma}{dt^\alpha} = E \lambda^\beta \frac{d^{\beta-1} \dot{\varepsilon}}{dt^{\beta}}, \tag{8}
\]
For the incompressible flow, the continuity equation is
\[ \text{div} \bar{u} = 0 \] (9)
and the momentum equation, in the absence of body forces, is
\[ \rho \frac{D\bar{u}}{Dt} = -\nabla p + \text{div} \tau, \] (10)
where \( \bar{u}, \rho, p, \tau \) refer to the velocity vector, density, pressure and stress tensor, respectively. \( D/Dt \) denotes the material derivative.

We construct the cylindrical coordinate system \((r, \theta, z)\) and make the \(z\)-axis direct along the axis of the tube. For the problem under consideration here, the velocity field has only \(z\) directional component, i.e.
\[ \bar{u} = u_z(r, t) \hat{e}_z, \] (11)
where \( \hat{e}_z \) is the unit vector in the \(z\) direction. And only the \(rz\) component of the stress tensor is non-zero. Then the constitutive equation (8) becomes
\[ \sigma_{rz} + \lambda^z \frac{d^z \sigma_{rz}}{dr^z} = E \lambda^\rho \frac{d^{\rho-1} \partial u_z}{\partial r^\rho} \] (12)
and meanwhile, the momentum equation (10) and no-slip boundary condition read
\[ \rho \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\partial (r \sigma_{rz})}{r \partial r} \] (13)
\[ u_z(a, t) = 0. \] (14)

From Eqs. (12) and (13), we can obtain the governing equation for \(u_z\) as follows:
\[ \rho \frac{\partial u_z}{\partial t} + \frac{\partial p}{\partial z} + \lambda^z \frac{d^z}{dr^z} \left( \rho \frac{\partial u_z}{\partial t} + \frac{\partial p}{\partial z} \right) = E \lambda^\rho \frac{d^{\rho-1}}{dr^\rho} \frac{\partial}{\partial r} \left( r \frac{\partial u_z}{\partial r} \right). \] (15)

Let the pressure gradient oscillate with frequency \(\omega\) and amplitude \(P_0\), i.e.
\[ \frac{\partial p}{\partial z} = P_0 e^{i\omega t} \] (16)
and we assume that all physical quantities vary in time as \(e^{i\omega t}\). In order to solve Eq. (15), we define the Fourier transform of the velocity \(u_z\) and pressure \(p\)
\[ U_z(r, \omega) = \int_{-\infty}^{\infty} u_z(r, t) e^{-i\omega t} dt, \] (17)
\[ P(z, \omega) = \int_{-\infty}^{\infty} p(z, t) e^{-i\omega t} dt \] (18)
and the Fourier transform for the fractional derivative is given by [13]
\[
\int_{-\infty}^{\infty} \frac{d}{dr} [u_z(r, t)] e^{-i\omega t} dt = (i\omega)^{\beta} U_z(r, t),
\]
where \(\omega\) is the frequency.

Transforming Eqs. (14) and (15), we get
\[
U_z(a, \omega) = 0,
\]
and
\[
\rho(i\omega)(1 + \lambda^2(i\omega)^2) U_z + (1 + \lambda^2(i\omega)^2) \frac{\partial P}{\partial z} = E\lambda^\beta(i\omega)^{\beta-1} \frac{\partial}{\partial r} \left( r \frac{\partial U_z}{\partial r} \right).
\]

Solving Eq. (22) under the boundary condition (21), we can get the velocity solution in the frequency domain
\[
U_z(r, \omega) = \frac{i}{\rho \omega} \frac{\partial P}{\partial z} \left( 1 - \frac{J_0(\zeta r)}{J_0(\zeta a)} \right),
\]
where the parameter \(\zeta\) is determined from
\[
\zeta^2 = -\frac{\rho(1 + \lambda^2(i\omega)^2)}{E\lambda^\beta(i\omega)^{\beta-2}} = -\frac{\rho + \rho\lambda^2|\omega|^2(\cos \frac{\beta\pi}{2} + i \text{sign} \omega \sin \frac{\beta\pi}{2})}{E\lambda^\beta|\omega|^{\beta-2}(\cos \frac{(\beta-2)\pi}{2} + i \text{sign} \omega \sin \frac{(\beta-2)\pi}{2})}
\]
and \(J_0\) is the zeroth-order Bessel function. If we take \(\alpha = \beta = 1\) and replace \(\omega\) with \(-\omega\), the same solution as in [3] can be obtained.

Using the inverse Fourier transform to Eq. (23), we can get the solution of the velocity in the time domain
\[
u_z(r, t) = \frac{i\rho_0}{\rho \omega} \frac{\partial P}{\partial z} \left( 1 - \frac{J_0(\zeta r)}{J_0(\zeta a)} \right) e^{i\omega t}.
\]

We introduce the dimensionless parameters
\[
\tilde{u} = u_z/u_0, \quad \tilde{r} = r/\sqrt{\nu \lambda}, \quad \tilde{\omega} = \omega \lambda, \quad \tilde{t} = t/\lambda
\]
and
\[
\sigma = \frac{1 + (i\omega \lambda)^2}{(i\omega \lambda)^{\beta-2}} = -\frac{1 + |\tilde{\omega}|^2(\cos \frac{\beta\pi}{2} + i \text{sign} \tilde{\omega} \sin \frac{\beta\pi}{2})}{|\tilde{\omega}|^{\beta-2}(\cos \frac{(\beta-2)\pi}{2} + i \text{sign} \tilde{\omega} \sin \frac{(\beta-2)\pi}{2})},
\]
where \(u_0 = \rho_0 a^2/(4\mu)\) is the velocity at \(\tilde{r} = 0\) when the pressure gradient is constant and equal to \(-P_0\), i.e. \(-dp/dx = P_0\). Besides, it should be noted that the
physical condition of $\lambda > 0$ is used in Eq. (28). Thus, we can get the dimensionless form of Eq. (26)

$$\tilde{u} = \frac{4i}{\tilde{a}^2 \tilde{\omega}} \left( 1 - \frac{J_0(\sqrt{\tilde{\sigma} \tilde{\alpha}})}{J_0(\sqrt{\tilde{\sigma} \tilde{a}})} \right) e^{i\tilde{\omega}t}. \tag{29}$$

We also introduce enhancement $A_u$ defined as the amplitude of dimensionless velocity on the axis of the tube in Eq. (29). It can be obtained as follows:

$$A_u = \left| \frac{4i}{\tilde{a}^2 \tilde{\omega}} \left( 1 - \frac{1}{J_0(\sqrt{\tilde{\sigma} \tilde{a}})} \right) \right|. \tag{30}$$

Note that the enhancement $A_u$ denotes the ratio of the amplitude of the velocity on the center axis under oscillating pressure gradient $P_0 e^{i\omega t}$ to that under the constant pressure gradient $-P_0$.

4. Results and discussion

It has been mentioned that the fractional Maxwell model will be simplified to the ordinary Maxwell model for $\alpha = \beta = 1$. The variations of the velocity enhancement $A_u$ for the Maxwell model are given at four dimensionless radii of $\tilde{a} = 0.05$, $\tilde{a} = 0.1$, $\tilde{a} = 0.5$ and $\tilde{a} = 5$ in Fig. 3(a)–(d), respectively. It can

![Fig. 3. Velocity enhancement $A_u$ (y-axis) versus the dimensionless frequency $\tilde{\omega}$ (x-axis) for the Maxwell model when: (a) $\tilde{a} = 0.05$; (b) $\tilde{a} = 0.1$; (c) $\tilde{a} = 0.5$ and (d) $\tilde{a} = 5$.](image-url)
be seen from these figures that drastic enhancements of the velocity amplitude occur at certain frequencies, which are the so-called resonance phenomena. For a fixed dimensionless radius \( \tilde{a} \) in any figure of Fig. 3(a)–(d), we can observe that the amplitude of the enhancement decreases as the frequency increases and the largest enhancement occurs at the smallest resonance frequency. But if the dimensionless radius grows, the number of the resonance peaks in the same frequency region increases, meanwhile the magnitude of the peak and the resonance frequency with the same order decreases. With further increase in the dimensionless radius, the curve even becomes monotonous and has no peaks, as shown in Fig. 3(d). A detailed discussion about this part can be obtained from [2].

Real viscoelastic fluids can be exactly described by the fractional Maxwell models. Here we will take two polymers [6], i.e. PTFE (Polytetrafluorethylene) and PMMA (Methylmethacrylate) for example to discuss the results of the fractional model. The fractional parameters of these polymers are \( \alpha = 0.0357, \beta = 0.0520 \) (PTFE) and \( \alpha = 0.5865, \beta = 0.6921 \) (PMMA), respectively. In Figs. 4 and 5, we give the curves of velocity enhancement \( A_v \) versus dimensionless frequency \( \tilde{\omega} \) at four dimensionless radii ((a) \( \tilde{a} = 0.05 \), (b) \( \tilde{a} = 0.1 \), (c) \( \tilde{a} = 0.5 \), (d) \( \tilde{a} = 5 \)) for these two polymers, respectively.

Figs. 4 and 5 show that the resonance phenomena similar to the Maxwell model also exist in the case of the fractional Maxwell model. The similar laws

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![Fig. 4. Velocity enhancement \( A_v \) (y-axis) versus the dimensionless frequency \( \tilde{\omega} \) (x-axis) for the fractional Maxwell model with \( \alpha = 0.0357, \beta = 0.0520 \) when: (a) \( \tilde{a} = 0.05 \); (b) \( \tilde{a} = 0.1 \); (c) \( \tilde{a} = 0.5 \) and (d) \( \tilde{a} = 5 \).]
which are mentioned for the Maxwell model also can be observed. First, if the
dimensionless radius is fixed, the largest enhancement occurs at the smallest
resonance frequency and the amplitude of the enhancement decreases as the
frequency increases. Second, as the dimensionless radius grows, the number
of the resonance peaks in the same frequency region increases, but the magni-
tude of the peak and the resonance frequency with the same order decrease.
The curve can even become monotonous once the dimensionless radius is large
enough.

However, some differences between the classical Maxwell model and the
fractional Maxwell model can also be observed from Figs. 3–5.

First, it can be seen that the number of the resonance peaks in the same fre-
quency region, the amplitudes of the enhancement and the frequencies where
resonance occurs are totally different for these two models through the compar-
ison between the figures with the same radius in Figs. 3–5.

Furthermore, we can see that the amplitudes of resonance peaks for the frac-
tional models given by Figs. 4 and 5 decay more quickly as the frequency grows
than those of the Maxwell model in Fig. 3. For the fractional Maxwell model,
the peaks except the first one is so low that they can be neglected, namely, the
first resonance peak is the most important. Take the cases of the dimensionless
radius $\tilde{a} = 0.05$ in Figs. 3(a) and 4(a) for an example. For the classical Maxwell
model, the first peak value is about 2500 and the second one is above 1500 yet.

![](fig5.png)

Fig. 5. Velocity enhancement $A_u$ (y-axis) versus the dimensionless frequency $\tilde{\omega}$ (x-axis) for the
fractional Maxwell model with $x = 0.5865$, $\beta = 0.6921$ when: (a) $\tilde{a} = 0.05$; (b) $\tilde{a} = 0.1$; (c) $\tilde{a} = 0.5$
and (d) $\tilde{a} = 5$. 
However, for the fractional model with the parameters $\alpha = 0.0357$, $\beta = 0.0520$, the first peak value is near 2300 but the second one is only slightly larger than 500.

In addition, we can find that a critical dimensionless radius exists for the fractional models. Below the critical radius, both the amplitudes of resonance peaks and the number of pikes in the same frequency region are less than those of the classical Maxwell model, namely, the resonance behavior for the fractional Maxwell model is weakened. However, above this value, the first peak value for the fractional model is much higher and the number of the peaks increases compared with the case of the classical Maxwell model, namely, the enhancement is strengthened. The critical value is about $\tilde{a}_c = 0.095$ for the fractional Maxwell model with parameters $\alpha = 0.0357$, $\beta = 0.0520$ and about $\tilde{a}_c = 1.3$ for parameters $\alpha = 0.5865$, $\beta = 0.6921$.

5. Conclusions

In this paper, we analyze the unidirectional oscillating flow of a viscoelastic fluid in an infinite straight pipe. A fractional Maxwell model is considered. Exact solutions are obtained in the time and frequency domains by using Fourier transform. Based on the exact solutions, we discuss the velocity enhancement of the fractional Maxwell model in detail through the comparison between the fractional Maxwell model and the classical Maxwell model.

We observe that the resonance phenomenon similar to the Maxwell model also exists in the case of the fractional Maxwell model. There exists some similar laws as follows: (1) For a fixed dimensionless radius, the amplitude of the enhancement decreases as the frequency increases, and the largest enhancement occurs at the smallest resonance frequency. (2) If the dimensionless radius grows, the number of the resonance peaks in the same frequency region increases, but the magnitude of the peak and the resonance frequency with the same order decrease. (3) The curve will become monotonous once the dimensionless radius is large enough.

However, there are also some differences between these two models. (1) The number of the resonance peaks in the same frequency region, the amplitudes of the enhancement and the frequencies where resonance occurs are totally different. (2) For the fractional, the amplitudes of resonance peaks models decay rapidly with frequency and only the first resonance peak is primary. (3) A critical dimensionless radius exists for the fractional models. Below the critical radius, the resonance behavior is weakened, but above this value, the enhancement is strengthened.

Through the analysis above, we find that although there are some similarities between the fractional Maxwell model and the classical Maxwell model in describing the pipe oscillating flow, there are also distinct differences. Recently,
the research has shown that the fractional models can describe the real viscoelastic fluids better than the classical linear models. So we can expect that the results presented in this paper will be of importance to the fundamental research in this area and its practical applications.

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References