Inferences about a Covariance Matrix

Testing the Hypothesis That a Covariance Matrix Is Equal to a Given Matrix

- Consider the hypothesis: H_0: $\mathbf{\Sigma}=\mathbf{\Sigma}_0$ vs. H_1: $\mathbf{\Sigma}\neq\mathbf{\Sigma}_0$. The hypothesized covariance matrix $\mathbf{\Sigma}_0$ is a target value for $\mathbf{\Sigma}$ or a nominal value from previous experience.
- Obtain a random sample of n observations, $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ from $N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ and calculate sample covariance matrix $\mathbf{S}$. To see if $\mathbf{S}$ is significantly different from $\mathbf{\Sigma}_0$, the likelihood ratio test statistic is:

$$W = -pn + pn\ln(n) - n\ln(|\mathbf{A}|/|\mathbf{\Sigma}_0|) + \mathrm{tr}(\mathbf{\Sigma}_0^{-1}\mathbf{A})$$

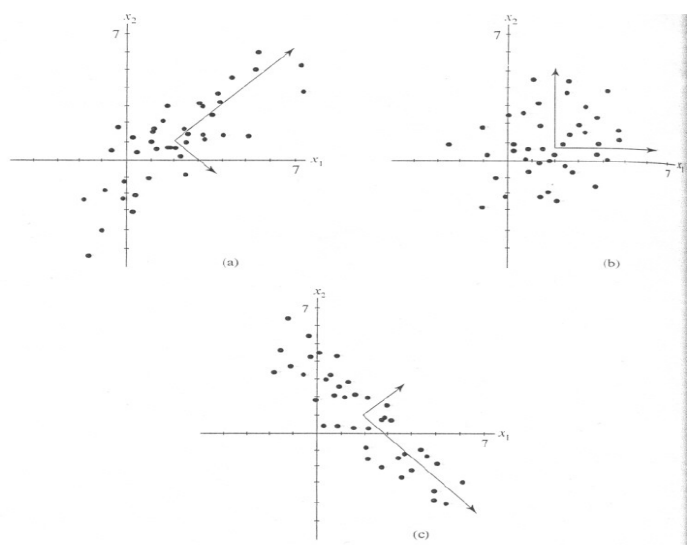
where $\mathbf{A}=(n\text{-}1)\mathbf{S}$.

- The likelihood ratio test is biased, a modified likelihood ratio test statistic W' can be obtained by replacing all n in W with n-1 (keep $\mathbf{A}$ the same). The modified likelihood ratio test is unbiased.
- W and W' are asymptotically (when n is large) distributed as $\chi^2_{p(p+1)/2}$. So we reject H_0 if W or W' is greater than $\chi^2_{p(p+1)/2}(\alpha)$.

Generalized Variance

- Sometimes it is desirable to assign a *single* numerical value for the variation/dispersion expressed by **S**. One choice for a value is the determinant of **S**.
- Generalized sample variance = |**S**|
- (Volume of ellipsoid with constant generalized distance)^2=(constant)(generalized sample variance)
- Example 3.8 of J&W

Scatter Plots for Ex 3.8



Distribution of |S|

- The distribution of |**S**| is the same as the distribution of $|\mathbf{A}|/(n\text{-}1)^p$, where **A** is as defined on pg. 43 of lecture notes.
- Asymptotic Distribution:

$$\sqrt{n}\left(|\mathbf{S}|/|\mathbf{\Sigma}|-1\right) \sim N(0,2p)$$

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