56:295 – 001
Multivariate Statistics and Advanced Quality Control
Fall 05

HW4 Due: October 12 (Wednesday), 6:15pm

Solution

1. Read the paper: George C. Runger "Projections and the U2 Multivariate Control Chart", Journal of Quality Technology, 28(3), 1996, which can be downloaded from homework page of course website.

2. In a process 10 variables are being monitored. However, the assignable cause anticipated only shifts the mean of the first variable and the last variable.
(a) Find the orthonormal matrix **U** corresponding to the mean shift subspace.
(b) If the control limit is selected such that the in-control ARL is equal to 200, find a mean shift $\boldsymbol{\mu}$ (from **0**) such that the ARL under this mean shift for the χ^2 chart is 31. What is the ARL under this mean shift for the U^2 chart? Assume the covariance matrix $\boldsymbol{\Sigma}$ of the process variables is a 10-by-10 identity matrix.

Ans. (a) $\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$
(b) From the table in Runger (1996) (page 62 of lecture notes), the out-of-control ARL is 31 for chi-square chart when the noncentrality parameter λ=3. For the same mean shift, the U^2 chart will have ARL equal to 11 based on the table. From $\lambda = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} = \boldsymbol{\mu}^T \boldsymbol{\mu}$, μ_1 and μ_{10}, the first and last component of $\boldsymbol{\mu}$, should satisfy $\mu_1^2 + \mu_{10}^2 = 3$. One example of $\boldsymbol{\mu}$ is $\boldsymbol{\mu} = [\sqrt{3/2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \sqrt{3/2}]^T$. Note that $\boldsymbol{\mu}$ should be in the subspace **U** of the possible mean shifts.

3. This question uses the data in the file Data_hw4.mat on homework page of the course website. It is a MAT format Matlab data file. You can use command "load" in Matlab to read the data. There is one variable, "Y", in the file. Y is a 150 by 20 matrix. The data are measurement data from the auto-body assembly process shown on page 65 of the lecture notes. Mean shifts in the subspace spanned by the columns of $\boldsymbol{\Gamma}$ on page 66 of lecture notes are present in the data. Matrix $\boldsymbol{\Gamma}$ is stored in the file Gamma_20by3.mat, which is downloadable on the homework page. Each row of the matrix Y represents a 20-dimension observation (10 points, 20 measurements for two directions of each point) on one product (car body). So Y contains a sample of 150 observations. Please refer to the Matlab help for more details on reading and using the data. Please use Matlab for computation.

(i) Let $\mathbf{Y}_1$ denote the vector of measurements from the first observation (first row of matrix Y). Fit a linear model as $\mathbf{Y}_1 = \boldsymbol{\Gamma\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\Gamma}$ is the one given on page 66 of lecture notes and $\boldsymbol{\varepsilon}$ is a zero mean random vector with covariance matrix equal to $\boldsymbol{\Sigma}$, which is also given on slide 66. Please find the generalized least squares estimation of $\boldsymbol{\beta}$. Note that you should only use the data in the first row of Y here.

Ans. $\hat{\boldsymbol{\beta}} = (\boldsymbol{\Gamma}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\Gamma})^{-1} \boldsymbol{\Gamma}^T \boldsymbol{\Sigma}^{-1} \mathbf{Y}_1 = [0.1393 \quad 0.0982 \quad 0.0176]^T$

(ii) Calculate the U^2 statistics for each of the 150 observations to detect mean shifts in the subspace spanned by the columns of $\boldsymbol{\Gamma}$ on page 66 of lecture notes. Please evaluate $\boldsymbol{\Sigma}$ as shown on page 66 of lecture notes. Use α=0.005 to obtain the control limit for the U^2 chart. List all out-of-control observations when the U^2 chart is used.

Ans. The upper control limit for the U^2 chart is $\chi_3^2(0.005)$ =12.838. There are 16 out-of-control signals on U^2 chart:

Obs#	U^2
1	14.1903
9	13.8421
12	13.0559
13	18.2756
15	14.4649
82	14.5103
102	15.4406
104	15.7937
107	15.6334
109	19.9197
112	24.1446
123	15.6455
133	19.2553
138	13.1833
142	13.2615
143	25.3248

(iii) Calculate the χ^2 statistics for each of the 150 observations. Use α=0.005 to obtain the control limit for the χ^2 chart. List all out-of-control observations when the χ^2 chart is used. Compare with part (ii) to comment on the advantage of U^2 chart to the χ^2 chart in detecting the mean shifts in this data set.

Ans. The upper control limit for the χ^2 chart is $\chi_{20}^2(0.005) = 40.00$. There are 3 out-of-control signals on the χ^2 chart:

#109	44.7049
#133	40.0352
#143	56.7644

The U^2 chart and χ^2 chart have the same type I error ($\alpha = 0.005$) and hence the same in-control ARL ($= 1/\alpha = 200$). But the U^2 chart is much more sensitive to the mean shifts in this data set than the χ^2 chart. It can detect the mean shift much quicker than the χ^2 chart.