56:295 – 001
Multivariate Statistics and Advanced Quality Control
Fall 05

HW1 Due: September 7 (Wednesday), 6:15pm

Note: Solution

Solve the following problems from the textbook (J&W)

1.3

$$\mathbf{x} = \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}, \ \mathbf{S}_n = \begin{bmatrix} 6 & 4 & -1.4 \\ 4 & 8 & 1.2 \\ -1.4 & 1.2 & 2 \end{bmatrix} \text{ (or } \mathbf{S} = \begin{bmatrix} 7.5 & 5 & -1.75 \\ 5 & 10 & 1.5 \\ -1.75 & 1.5 & 2.5 \end{bmatrix} \text{), and}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0.5774 & -0.4042 \\ 0.5774 & 1 & 0.3 \\ -0.4042 & 0.3 & 1 \end{bmatrix}$$

2.7

a) Eigenvalues: $\lambda_1 = 10, \ \lambda_2 = 5$.

Normalized eigenvectors: $\mathbf{e}_1' = [2/\sqrt{5}, -1/\sqrt{5}] = [.894, -.447]$

$\mathbf{e}_2' = [1/\sqrt{5}, 2/\sqrt{5}] = [.447, .894]$

b) $$A = \begin{bmatrix} 9 & -2 \\ -2 & 9 \end{bmatrix} = 10 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, \ -1/\sqrt{5}] + 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, \ 2/\sqrt{5}]$$

c) $$A^{-1} = \frac{1}{9(6)-(-2)(-2)} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} .12 & .04 \\ .04 & .18 \end{bmatrix}$$

d) Eigenvalues: $\lambda_1 = .2, \ \lambda_2 = .1$

Normalized eigenvectors: $\mathbf{e}_1' = [1/\sqrt{5}, \ 2/\sqrt{5}]$

$\mathbf{e}_2' = [2/\sqrt{5}, \ -1/\sqrt{5}]$

2.26

2.26 a) $\rho_{13} = \sigma_{13}/\sigma_{11}^{1/2}\sigma_{22}^{1/2} = 4/\sqrt{25}\,\sqrt{9} = 4/15 = .267$

b) Write $X_1 = 1\cdot X_1 + 0\cdot X_2 + 0\cdot X_3 = \mathbf{c}_1'\mathbf{X}$ with $\mathbf{c}_1' = [1,0,0]$

$$\frac{1}{2}X_2 + \frac{1}{2}X_3 = \mathbf{c}_2'\mathbf{X} \quad \text{with} \quad \mathbf{c}_2' = [0, \tfrac{1}{2}, \tfrac{1}{2}]$$

Then $\text{Var}(X_1) = \sigma_{11} = 25$. By (2-43),

$$\text{Var}(\tfrac{1}{2}X_2 + \tfrac{1}{2}X_3) = \mathbf{c}_2'\,\boldsymbol{\Sigma}\,\mathbf{c}_2 = \frac{1}{4}\sigma_{22} + \frac{2}{4}\sigma_{23} + \frac{1}{4}\sigma_{33} = 1 + \frac{1}{2} + \frac{9}{4}$$

$$= \frac{15}{4} = 3.75$$

By (2-45), (see also hint to Exercise 2.28),

$$\text{Cov}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) = \mathbf{c}_1'\,\boldsymbol{\Sigma}\,\mathbf{c}_2 = \frac{1}{2}\sigma_{12} + \frac{1}{2}\sigma_{13} = -1 + 2 = 1$$

$$\text{Corr}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2) = \frac{\text{Cov}(X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2)}{\sqrt{\text{Var}(X_1)}\,\sqrt{\text{Var}(\tfrac{1}{2}X_1 + \tfrac{1}{2}X_2)}} = \frac{1}{5\sqrt{3.75}} = .103$$

2.31

(a) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (b) 1 (c) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ (d) 4

4.1

4.1 (a) We are given $p = 2$, $\mu = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & -.8 \times \sqrt{2} \\ -.8 \times \sqrt{2} & 1 \end{bmatrix}$ so $|\Sigma| = .72$ and

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{.72} & \frac{\sqrt{2}}{.9} \\ \frac{\sqrt{2}}{.9} & \frac{2}{.72} \end{bmatrix}$$

$$f(\mathbf{x}) = \frac{1}{(2\pi)\sqrt{.72}} \exp\left(-\frac{1}{2} \left[\frac{1}{.72}(x_1 - 1)^2 + \frac{2\sqrt{2}}{.9}(x_1 - 1)(x_2 - 3) + \frac{2}{.72}(x_2 - 3)^2 \right] \right)$$

(b)

$$\frac{1}{.72}(x_1 - 1)^2 + \frac{2\sqrt{2}}{.9}(x_1 - 1)(x_2 - 3) + \frac{2}{.72}(x_2 - 3)^2$$

4.4

a) $3X_1 - 2X_2 + X_3$ is $N(13, 9)$

b) Require Cov $(X_2, X_2 - a_1X_1 - a_3X_3) = 3 - a_1 - 2a_3 = 0$. Thus any $\underset{\sim}{a}' = [a_1, a_3]$ of the form $\underset{\sim}{a}' = [3 - 2a_3, a_3]$ will meet the requirement. As an example, $\underset{\sim}{a}' = [1, 1]$.

4.5(b)

$X_2 | x_1, x_3$ is $N(-2x_1 - 5, 1)$

4.6

(a) X_1 and X_2 are independent since they have a bivariate normal distribution with covariance $\sigma_{12} = 0$.

(b) X_1 and X_3 are dependent since they have nonzero covariance $\sigma_{13} = -1$.

(c) X_2 and X_3 are independent since they have a bivariate normal distribution with covariance $\sigma_{23} = 0$.

(d) X_1, X_3 and X_2 are independent since they have a trivariate normal distribution where $\sigma_{12} = 0$ and $\sigma_{32} = 0$.

(e) X_1 and $X_1 + 3X_2 - 2X_3$ are dependent since they have nonzero covariance

$$\sigma_{11} + 3\sigma_{12} - 2\sigma_{13} = 4 + 3(0) - 2(-1) = 6$$