Lecture 2: Spring-Mass Systems

Reading materials: Sections 1.7, 1.8

1. Introduction

- All systems possessing mass and elasticity are capable of free vibration, or vibration that takes place in the absence of external excitation. Of primary interest for such a system is its natural frequency of vibration.

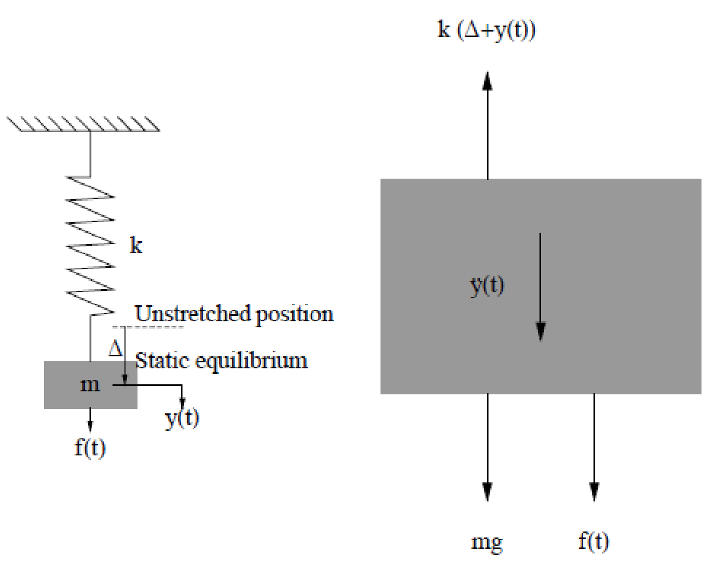
- The basic vibration model of a simple oscillatory system consists of a mass, a massless spring, and a damper.

- If damping in moderate amounts has little influence on the natural frequency, it may be neglected. The system can then be considered to be conservative.

- An undamped spring-mass system is the simplest free vibration system. It has one DOF.

2. Equation of Motion

- Natural frequency



3. Free vibration solution

Example 1: A ¼ kg mass is suspended by a spring having a stiffness of 0.1533 N/mm. determine its natural frequency in cycles per second. Determine its statistical deflection

Example 2:

A weight W=80lb suspended by a spring with k = 100 lb/in. Determine the vibration response, if the system is given an initial displacement of 2 inches and then released suddenly.

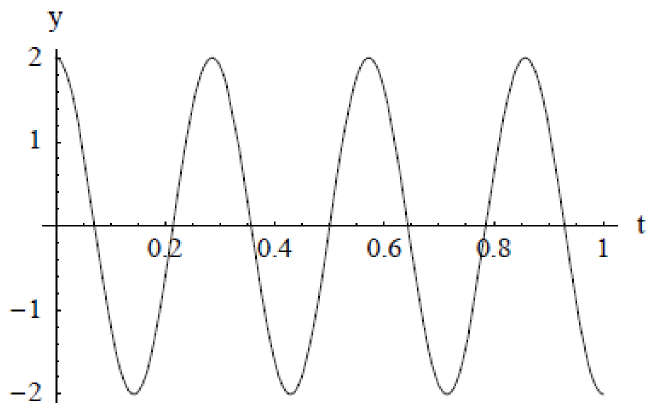
$$W = 80\ \text{lb}; \qquad g = 386.4\ \text{in}/s^2; \qquad m = W/g = 0.207039\ \text{lb}-s^2/\text{in}; \qquad k = 100\ \text{lb/in}$$

$$\omega = \sqrt{k/m} = 21.9773\ \text{rad/s}; \qquad f = 3.49779\ \text{Hz}; \qquad T = 0.285895\ \text{s}$$

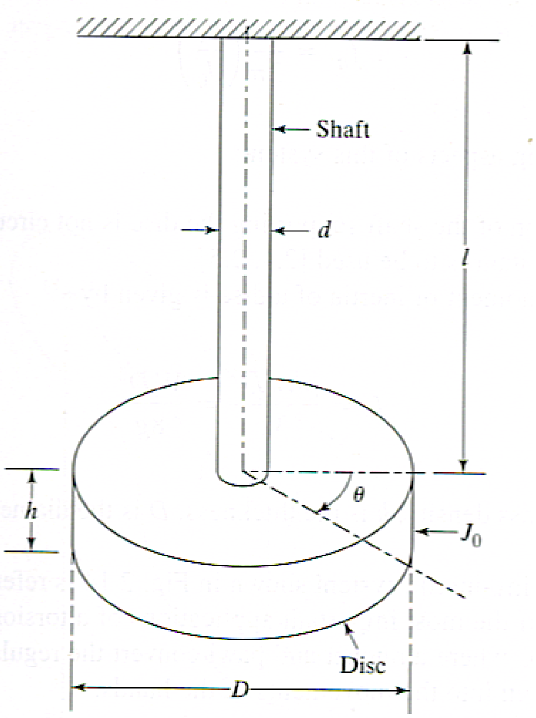
$$u_0 = 2\ \text{in}; \qquad v_0 = 0; \qquad v_0/\omega = 0$$

Then

$$y(t) = u_o \cos \omega t + \frac{v_0}{\omega} \sin \omega t = 2 \cos(21.9773\ t)$$



4. Free Vibration of an undamped torsional system



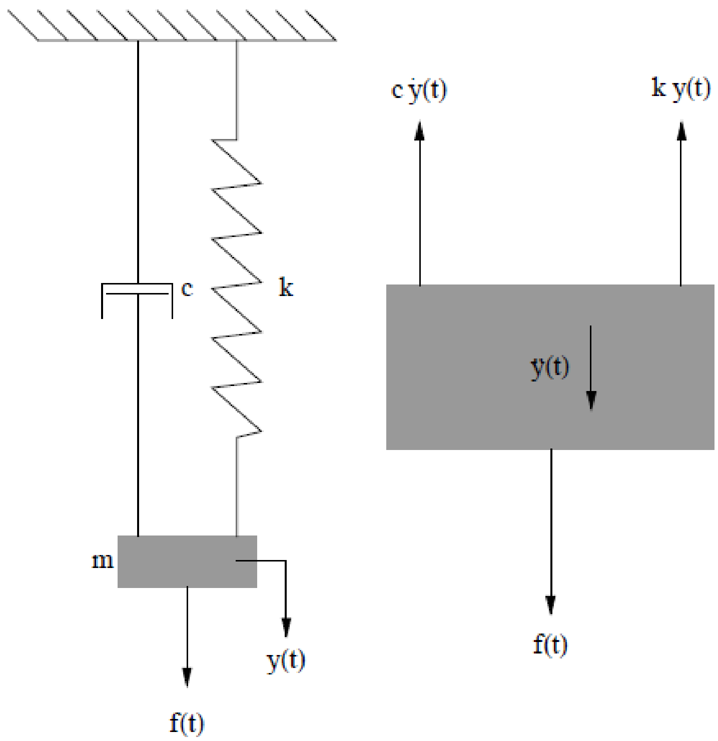
(a)

Example 3:

An automobile wheel and tire are suspended by a steel rod 0.50 cm in diameter and 2 m long. When the wheel is given an angular displacement and released, it makes 10 oscillations in 30.2 second. Determine polar moment of inertia of the wheel and tire.

Damping dynamic systems

- Spring-mass systems with viscous damping:



$$y(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} = e^{-(c/2m)t}\left(A_1 e^{\left(\sqrt{(c/2m)^2 - \omega^2}\right)t} + A_2 e^{-\left(\sqrt{(c/2m)^2 - \omega^2}\right)t} \right)$$

(a) Critical damping: $(c/2m)^2 = \omega^2 \quad \Longrightarrow \quad c_c = 2m\omega$

(b) Overdamped system: $(c/2m)^2 > \omega^2$

(c) Underdamped or lightly damped system: $(c/2m)^2 < \omega^2$

Introducing the damping ratio,

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega}$$

Therefore,

$$p_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} = -\xi\omega \pm \sqrt{(\xi\omega)^2 - \omega^2} = \omega\left(-\xi \pm \sqrt{\xi^2 - 1}\right)$$

$$y(t) = e^{-\xi\omega t}\left(A_1 e^{\left(\omega\sqrt{\xi^2-1}\right)t} + A_2 e^{-\left(\omega\sqrt{\xi^2-1}\right)t} \right)$$

Finally, we have

a) Critical damping: $\xi = 1$

b) Overdamped system: $\xi > 1$

c) Underdamped or lightly damped system: $0 < \xi < 1$

The above can be classified as critically damped motion; nonoscillatory motion; and oscillatory motion.

Underdamped or lightly-damped motion: $0 < \xi < 1$

$$y(t) = e^{-\xi \omega t}\left(u_0 \cos \omega_d t + \frac{\xi \omega u_0 + v_0}{\omega_d} \sin \omega_d t\right)$$

$$y(t) = e^{-\xi \omega t}(Y \sin \theta \cos \omega_d t + Y \cos \theta \sin \omega_d t) \equiv e^{-\xi \omega t}\, Y \sin(\omega_d t + \theta)$$

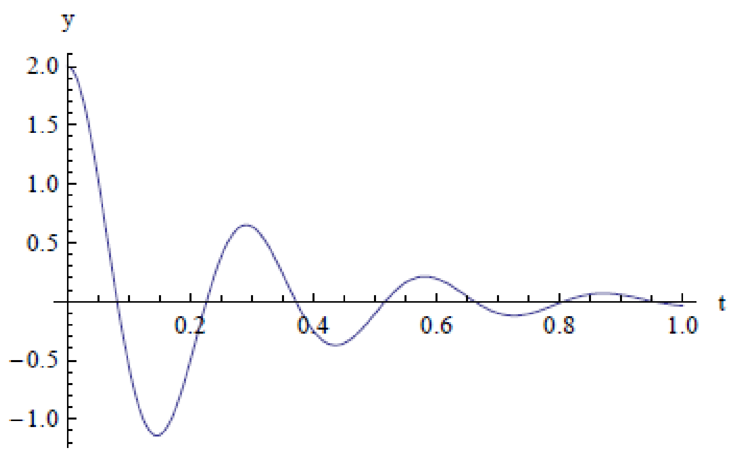
where

$$Y = \sqrt{u_0^2 + \left(\frac{\xi \omega u_0 + v_0}{\omega_d}\right)^2}$$

$$\theta = \tan^{-1}\left(u_0 \Big/ \left(\frac{\xi \omega u_0 + v_0}{\omega_d}\right)\right)$$

Example: A weight w = 80lb suspended by a spring with k=100lb/in. The damping coefficient is c=1.6 lb-s/in. Determine the vibration response. Initial displacement is 2 inches.

y(t)= $2.03165\, e^{-3.864\, t} \sin(21.6349\, t + 1.39406)$



Logarithmic decrement: If there are the displacements at two consecutive peaks at t_1 and t_1+T_d

$$y(t_1) \equiv y_1 = e^{-\xi\,\omega\, t_1}\, Y \sin(\omega_d\, t_1 + \theta)$$

$$y(t_2) \equiv y_2 = e^{-\xi\,\omega\,(t_1+T_d)}\, Y \sin(\omega_d\,(t_1 + T_d) + \theta)$$

The *logarithmic decrement* is defined as

$$\delta = \ln\left(\frac{y_1}{y_2}\right) = \ln\left(\frac{e^{-\xi\,\omega\, t_1}\, Y \sin(\omega_d\, t_1 + \phi)}{e^{-\xi\,\omega\,(t_1+T_d)}\, Y \sin(\omega_d\,(t_1+T_d)+\phi)}\right)$$

$$\delta = \ln\left(\frac{e^{-\xi\,\omega\, t_1}}{e^{-\xi\,\omega\,(t_1+T_d)}}\right) = \ln\left(\frac{1}{e^{-\xi\,\omega\, T_d}}\right) = \ln\left(e^{\xi\,\omega\, T_d}\right) \equiv \xi\,\omega\, T_d$$

$$\delta = \xi\,\omega\left(\frac{2\pi}{\omega\sqrt{1-\xi^2}}\right) = \frac{2\pi\,\xi}{\sqrt{1-\xi^2}}$$

The relationship between the logarithmic decrement and the damping ratio

$$\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

For lightly damped systems, the difference between two successive peaks may be too small to measure accurately. Since the logarithmic decrement between any two successive peaks is constant, we can determine the decrement from the first peak and the peak n cycles later.

$$\delta = \frac{1}{n}\ln\left(\frac{y_0}{y_n}\right)$$

Example: The following data are given for a vibrating system with viscous damping: w = 10lb, k = 30 lb/in, c = 0.12 lb in/s. Determine the logarithmic decrement and the ratio of any two successive amplitudes.

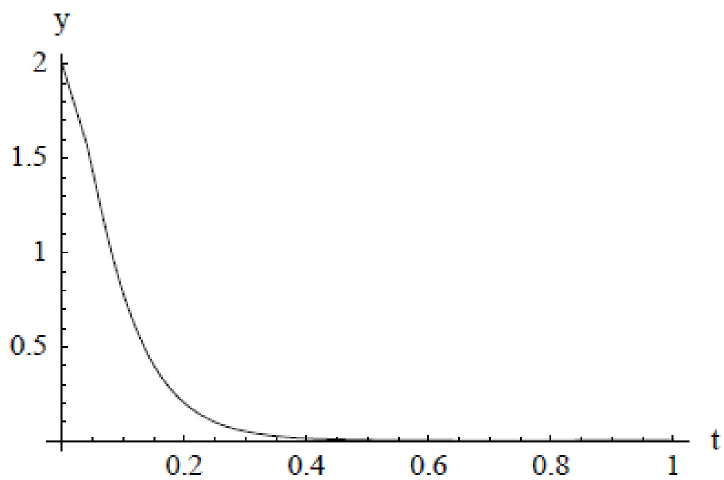
The dissipated energy in the first cycle,

$$\Delta U = \oint c\,\dot{y}\,dy = \int_0^{T_d} c\,\dot{y}\,\frac{dy}{dt}\,dt = \int_0^{2\pi/\omega_d} c\,\dot{y}^2\,dt$$

Overdamped (Nonoscillatory) motion: $\xi > 1$

$$y(t) = e^{-\xi\omega t}\left(\frac{\xi\omega u_0+\sqrt{\xi^2-1}\;\omega u_0+v_0}{2\sqrt{\xi^2-1}\;\omega}\,e^{\left(\omega\sqrt{\xi^2-1}\right)t} - \frac{\xi\omega u_0-\sqrt{\xi^2-1}\;\omega u_0+v_0}{2\sqrt{\xi^2-1}\;\omega}\,e^{-\left(\omega\sqrt{\xi^2-1}\right)t}\right)$$

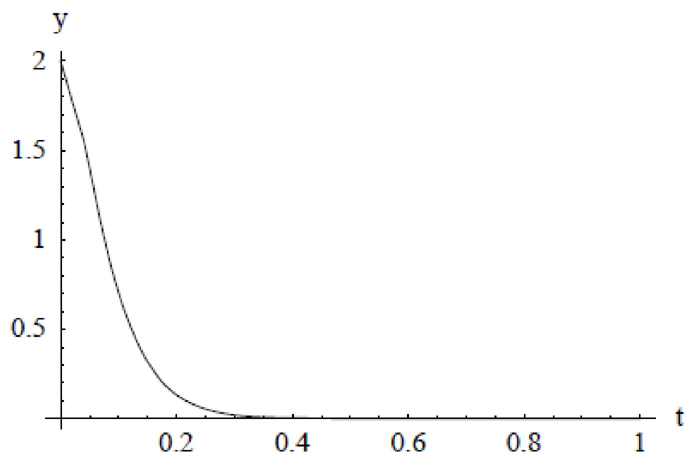
Example: A weight w = 80lb suspended by a spring with k=100lb/in. The damping coefficient is c=10 lb-s/in. Determine the vibration response. Initial displacement is 2 inches.



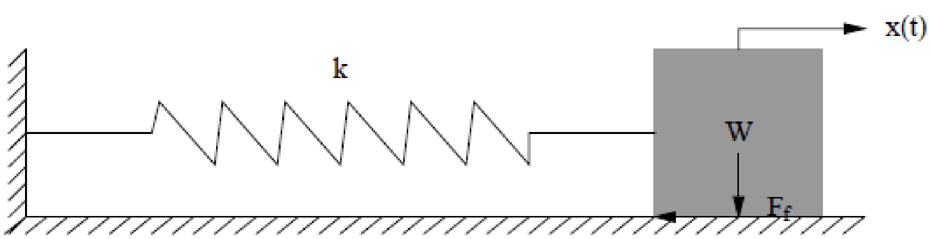
Critically damped motion: $\xi = 1$

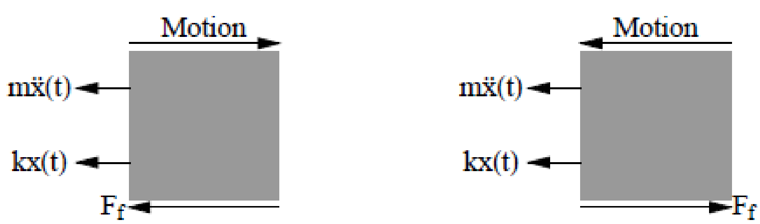
Example: A weight w = 80lb suspended by a spring with k=100lb/in. Determine the critically damped vibration response. Initial displacement is 2 inches.

$$y(t) = e^{-21.9773\,t}(43.9545\,t + 2)$$



6. Spring-Mass systems with Coulomb Damping (Optional)





The first half cycle

$$m\,\ddot{x}(t) + k\,x(t) = F_f; \quad x(0) = x_0; \quad \dot{x}(0) = 0$$

Assume

$$x(t) = A\sin\omega t + B\cos\omega t + \frac{F_f}{k}$$

Apply the I.C.

$$x(t) = \left(x_0 - \frac{F_f}{k}\right)\cos\omega t + \frac{F_f}{k}; \quad 0 \le \omega t < \pi$$

The second half cycle

$$m\,\ddot{x}(t) + k\,x(t) = -F_f; \quad \pi \le \omega t < 2\pi$$

$$x\left(\frac{\pi}{\omega}\right) = \left(x_0 - \frac{F_f}{k}\right)\cos\pi + \frac{F_f}{k} = -x_0 + \frac{2F_f}{k}$$

$$\dot{x}\left(\frac{\pi}{\omega}\right) = 0$$

Assume

$$x(t) = A\sin\omega t + B\cos\omega t - \frac{F_f}{k}$$

Apply the I.C.

$$x(t) = \left(x_0 - \frac{3F_f}{k}\right)\cos\omega t - \frac{F_f}{k}; \quad \pi \le \omega t < 2\pi$$

And so on…

$$x(t) = \left(x_0 - \frac{5F_f}{k}\right)\cos\omega t + \frac{F_f}{k}; \quad 2\pi \le \omega t < 3\pi$$

$$x(t) = \left(x_0 - \frac{7F_f}{k}\right)\cos\omega t - \frac{F_f}{k}; \quad 3\pi \le \omega t < 4\pi$$