Lecture 15: Determination of Natural Frequencies and Mode Shapes (Optional)

1. Eigenvalue problems

The following type of equations often occur in practice,

\[ A\mathbf{x} = \lambda \mathbf{x} \quad (a) \]

For a matrix of order N, there are N vectors \( \mathbf{x}_i \) (i=1 to N). Every vector is associated with a value \( \lambda_i \)

\( \mathbf{x}_i \): Eigenvectors or Characteristic vectors

\( \lambda_i \): Eigenvalues

Theoretical analysis

Solving the following characteristic equation to obtain the eigenvalues

\[ \det(A - \lambda I) = 0 \]

Solving the following linear algebra equations to obtain the eigenvectors

\[ (A - \lambda_i I) \mathbf{x}_i = 0 \]

For vibrating system

Solving the following characteristic equation to obtain the natural frequencies

\[ \det(m^{-1}k - \lambda I) = 0 \]

or

\[ \det(k - \lambda m) = 0 \]
\[ \omega_i = \sqrt{\lambda_i} \]

Solving the following linear algebra equations to obtain the eigenvectors

\[ (m^{-1}k - \lambda_i I) \varphi_i = 0 \]

or

\[ (k - \lambda_i m) \varphi_i = 0 \]

2. Eigenvectors by gauss elimination

Example:
3. Vector iteration (Power method) for the largest eigenvalue

\[
\begin{bmatrix}
16 & -24 & 18 \\
3 & -2 & 0 \\
-9 & 18 & -17
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \lambda
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

a). Guess a solution:
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

b). Substitute the guessed solution into LHR of the equation. Then, normalizing the resulting vector

\[
\begin{bmatrix}
16 & -24 & 18 \\
3 & -2 & 0 \\
-9 & 18 & -17
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
10 \\
1 \\
-8
\end{bmatrix} = 10 \begin{bmatrix}
1.0 \\
0.1 \\
-0.8
\end{bmatrix}
\]

c). Do iteration till convergence
Illustrating convergence towards the eigenvalue –8 and eigenvector {−0.5, 0.25, 1}

4. Calculation of intermediate eigenvalues - deflation

Using orthogonality of eigenvectors, a modified matrix $A^*$ can be established if the largest eigenvalue $\lambda_1$ and its corresponding eigenvector $x_1$ are known.

$$A^* = A - \lambda_1 x_1 (x_1)^T$$

The power method can be employed to obtain the largest eigenvalue of $A^*$, which is the second largest eigenvalue of $A$.

Proof:
Example

Using iterative methods to find eigenvalues and eigenvectors of

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{bmatrix}
\]
5. Transformation methods

\[ A\mathbf{x} = \lambda \mathbf{x} \quad \Rightarrow \quad P^{-1}AP\mathbf{x} = \lambda \ P^{-1} \ P \ \mathbf{x} \]

For an orthogonal matrix \( P \), \( P^{-1} = P^T \)

\[ P^TAP\mathbf{x} = \lambda \ P^T \ P \ \mathbf{x} \Rightarrow A^*\mathbf{x} = \lambda \ \mathbf{x} \]

If \( A \) is symmetrical, \( A^* = P^TAP \) is also symmetrical.

In this course, we only consider \( A \) as a 2x2 matrix.

Select a ‘rotation matrix’ as \( P \),

\[ P = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \]

With a proper \( \alpha \), \( A^* \) can be written as a diagonal matrix. Then, eigenvalues can be determined.

Example:
6. Dunkerley’s formula

Dunkerley’s formula gives the approximate value of the fundamental frequency of a composite system in terms of the natural frequencies of its component parts.

Flexibility matrix

Dunkerley’s formula
Example

FIGURE 7.1 Beam carrying masses.
7. Rayleigh’s method

Rayleigh principle:
The frequency of vibration of a conservative system vibrating about an equilibrium position has a stationary value in the neighborhood of a natural mode. This stationary value, in fact, is a minimum value in the neighborhood of the fundamental natural mode.

Rayleigh’s method

The above equation can be used to find an approximate value of the first natural frequency of the system. For this, we select a trial vector \( X \) to represent the first natural mode \( X^{(1)} \) and substitute it on the right hand side of the above equation. This yields the approximate value of \( \omega_1^2 \). Because Rayleigh’s quotient is stationary, remarkably good estimates of \( \omega_1^2 \) can be obtained even if the trail vector \( X \) deviates greatly from the true natural mode \( X^{(1)} \). Obviously, the estimated value of the fundamental frequency \( \omega_1 \) is more accurate if the trail vector \( X \) chosen resembles the true natural mode \( X^{(1)} \) closely.
Example