Lecture 13: Frequency Domain Solution

Reading materials: Sections 4.6 and 4.7

1. Steady-state solution for complex forcing function

   - Equation of motion (Single degree of freedom)
     \[ m \ddot{u}(t) + c \dot{u}(t) + k u(t) = F e^{i \Omega t}; \quad u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0 \]

   - For steady-state solution, assume the solution
     \[ u(t) = U e^{i \Omega t} \]

   - Get the solution
     \[ -\Omega^2 m U e^{i \Omega t} + i \Omega c U e^{i \Omega t} + k U e^{i \Omega t} = F e^{i \Omega t} \]
     \[ U = \left[ -\Omega^2 m + i \Omega c + k \right]^{-1} F \]
     \[ \ddot{u}(t) = i \Omega U e^{i \Omega t} = \dot{U} e^{i \Omega t} \quad \Rightarrow \quad \dot{U} = i \Omega U \]
     \[ \dddot{u}(t) = -\Omega^2 U e^{i \Omega t} = \ddot{U} e^{i \Omega t} \quad \Rightarrow \quad \ddot{U} = -\Omega^2 U \]

     Not time derivatives.

   - For Periodic loading, using Fourier transform
     \[ f(t) = \sum_{n=-p}^{p} F_n e^{i n \Omega t} \]

     \[ F_n = \frac{\Omega}{2\pi} \int_0^T f(t) e^{-i n \Omega t} \, dt; \quad n = 0, 1, 2, \ldots; \quad F_{-n} = \text{Conj}(F_n) \]
For each component, the solution amplitude is

\[ U_n = \left[ -n^2 \Omega^2 m + i n \Omega c + k \right]^{-1} F_n = H_n F_n \]

\[ H_n = \frac{1}{-n^2 \Omega^2 m + i n \Omega c + k} \]

here

\[ \text{Natural frequency, } \omega = \sqrt{\frac{k}{m}} \implies m = \frac{1}{\omega^2} \]

\[ \text{Damping ratio, } \xi = \frac{c}{2m \omega} \implies \frac{c}{k} = \frac{2m \omega \xi}{k} = \frac{2 \xi}{\omega} \]

\[ \text{Frequency ratio, } r_n = \frac{n \Omega}{\omega} \]

Therefore,

\[ H_n = \frac{1}{k(-r_n^2 + 2i r_n \xi + 1)} \]

Complete solution

\[ u(t) = \sum_{n=-P}^{P} U_n e^{in \Omega t} \]

Summary of the procedure

a. Use the direct Fourier transform to determine the complex frequency amplitudes for the applied force – transform force into the frequency domain

\[ F_n = \frac{\Omega}{2\pi} \int_0^T f(t) e^{-in \Omega t} \, dt; \quad n = 0, 1, 2, \ldots; \quad F_{-n} = \text{Conj}(F_n) \]
b. Determine the complex frequency amplitudes of the solution

\[ H_n = \frac{1}{-\eta^2 \Omega^2 \omega_n + i \eta \Omega \omega_n + k} = \frac{1}{k(-\eta_n^2 + 2i r_n \xi + 1)} \]

\[ U_n = H_n F_n; \quad \dot{U}_n = i \eta \Omega H_n F_n; \quad \ddot{U}_n = -\eta^2 \Omega^2 H_n F_n \]

c. Use the inverse Fourier transform to take the frequency amplitudes of the solution back into the time domain

\[ u(t) = \sum_{n=-P}^{P} U_n e^{i \eta n \Omega t}; \quad iu(t) = \sum_{n=-P}^{P} i \eta \Omega U_n e^{i \eta n \Omega t}; \quad ii(t) = -\sum_{n=-P}^{P} \eta^2 \Omega^2 U_n e^{i \eta n \Omega t} \]

Example: Determine the steady-state motion of the water tower when it is subjected to a triangular periodic load. 10% damping

Load period, \( T = 0.64 \quad \Rightarrow \quad \Omega = \frac{2\pi}{0.64} = 9.81748 \)

\[ f(t) = \begin{cases} 
750. (t - 0.16) + 120 & t \leq 0.16 \\
-750. (t - 0.48) - 120 & t \leq 0.48 \\
750. (t - 0.64) & t \leq 0.64
\end{cases} \]

\( \Omega = 9.81748 \)
\[ m = 0.1; \quad k = 120; \quad \omega = 34.641 \]

\[ \zeta = 0.1; \quad c = 0.69282; \]

\[ F_{-3} = (1/0.64)^{0.64} \int_0^{0.64} f(t) e^{-i(3\omega t)} \, dt = -5.4038 \, t; \quad H_{-3} = 0.0218456 + 0.0134042 \, t; \quad U_{-3} = 0.0724338 - 0.118049 \, t \]

\[ F_{-2} = (1/0.64)^{0.64} \int_0^{0.64} f(t) e^{-i(2\omega t)} \, dt = 0; \quad H_{-2} = 0.019447 + 0.00199504 \, t; \quad U_{-2} = 0 \]

\[ F_{-1} = (1/0.64)^{0.64} \int_0^{0.64} f(t) e^{-i(\omega t)} \, dt = 48.6342 \, t; \quad H_{-1} = 0.00902683 + 0.000556336 \, t; \quad U_{-1} = -0.0270569 + 0.439012 \, t \]

\[ F_0 = (1/0.64)^{0.64} \int_0^{0.64} f(t) e^{i(0)\omega t}) \, dt = 0; \quad H_0 = 0.0083333; \quad U_0 = 0 \]

\[ F_1 = (1/0.64)^{0.64} \int_0^{0.64} f(t) e^{i(\omega t)} \, dt = -48.6342 \, t; \quad H_1 = 0.00902683 - 0.000556336 \, t; \quad U_1 = -0.0270569 - 0.439012 \, t \]

\[ F_2 = (1/0.64)^{0.64} \int_0^{0.64} f(t) e^{i(2\omega t}) \, dt = 0; \quad H_2 = 0.019447 - 0.00199504 \, t; \quad U_2 = 0 \]

\[ F_3 = (1/0.64)^{0.64} \int_0^{0.64} f(t) e^{i(3\omega t)} \, dt = 5.4038 \, t; \quad H_3 = 0.0218456 - 0.0134042 \, t; \quad U_3 = 0.0724338 + 0.118049 \, t \]

\[ u(t) = (-0.0270569 + 0.439012 \, t) \, e^{-0.81748 \, t} - (0.0270569 + 0.439012 \, t) \, e^{0.81748 \, t} \]

\[ + (0.0724338 - 0.118049 \, t) \, e^{-29.4524 \, t} + (0.0724338 + 0.118049 \, t) \, e^{29.4524 \, t} \]
2. Using discrete Fourier Transform for numerical force data

- Use the direct Discrete Fourier Transform to compute the complex frequency amplitudes for the applied force

\[ F_n = \frac{1}{N} \sum_{s=0}^{N-1} f_s e^{-i(2\pi s n / N)}; \quad n = 0, 1, 2, \ldots, N - 1 \]

- Determine the complex frequency amplitudes of the solution

\[ H_n = \frac{1}{-n^2 \Omega^2 m + i n \Omega c + k} \]

\[ U_n = H_n F_n; \quad \dot{U}_n = i n \Omega H_n F_n; \quad \ddot{U}_n = -n^2 \Omega^2 H_n F_n \]

- Use the inverse Fourier transform to transform frequency amplitudes of the solution back into the time domain

\[ u_s = \sum_{n=0}^{N-1} U_n e^{i(2\pi s n / N)}; \quad s = 0, 1, 2, \ldots, N \]

\[ \dot{u}_s = \sum_{n=0}^{N-1} \dot{U}_n e^{i(2\pi s n / N)}; \quad s = 0, 1, 2, \ldots, N \]

\[ \ddot{u}_s = \sum_{n=0}^{N-1} \ddot{U}_n e^{i(2\pi s n / N)}; \quad s = 0, 1, 2, \ldots, N \]

3. General Non-Periodic loading

- General loading can be treated in the frequency domain approach by extending the load period to include a large interval of zero force to the end of the actual force. Mathematically the load is still periodic.

- The method is used only for large duration loading where the initial built-up of the loading is relatively slow.
4. Impulsive force

A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period of time and then stops.

The simplest form is the impulsive force – a force that has a large magnitude \( F \) and acts for a very short period of time.
5. Multiple degrees of freedom frequency domain solution

Coupled Equation of motion

\[ m \ddot{u} + c \dot{u} + k u = f(t) \]

Rayleigh damping

\[ c = \alpha m + \beta k \]

Undamped free vibration modes

\[ k \phi_i = \lambda_i m \phi_i; \quad i = 1, 2, \ldots, n \]

Mass orthogonality:

\[ \phi_j^T m \phi_i = 0; \quad i \neq j \]

Stiffness orthogonality:

\[ \phi_j^T k \phi_i = 0; \quad i \neq j \]

Uncoupled equations

Modal coordinates

\[ z = (z_1 \quad z_2 \quad \ldots \quad z_n)^T \]

\[ u(t) = \sum_i z_i(t) \phi_i \]

Damped modal equations

\[ M_i \ddot{z}_i(t) + (\alpha M_i + \beta K_i) \dot{z}_i(t) + K_i z_i(t) = R_i; \quad i = 1, 2, \ldots \]
\[ z_i(0) = \frac{1}{M_i} (\phi_i^T m u^0); \quad \dot{z}_i(0) = \frac{1}{M_i} (\phi_i^T m v^0); \]

\[ M_i = \phi_i^T m \phi_i; \quad K_i = \phi_i^T k \phi_i; \quad \omega_i = \sqrt{K_i / M_i}; \quad R_i = \phi_i^T F \]

\[ C_i = \alpha M_i + \beta K_i \]

\[ M_i \ddot{z}_i(t) + C_i \dot{z}_i(t) + K_i z_i(t) = R_i; \quad i = 1, 2, \ldots \]

Solution of modal equations

a. Use the direct Discrete Fourier transform to compute the complex frequency amplitudes for the applied force

\[ F_{i,n} = \frac{1}{N} \sum_{n=0}^{N-1} R_{i,n} e^{-in(2\pi s n / N)}; \quad n = 0, 1, 2, \ldots, N - 1 \]

b. Determine the complex frequency amplitudes of the solution

\[ H_{i,n} = \frac{1}{-n^2 \Omega^2 M_i + i n \Omega C_i + K_i} \]

\[ Z_{i,n} = H_{i,n} F_{i,n}; \quad \dot{Z}_{i,n} = i n \Omega H_{i,n} F_{i,n}; \quad \ddot{Z}_{i,n} = -n^2 \Omega^2 H_{i,n} F_{i,n} \]

c. Use the Inverse Fourier transform to transform frequency amplitudes of the solution back into the time domain

\[ z_{i,s} = \sum_{n=0}^{N-1} Z_{i,n} e^{i(2\pi s n / N)}; \quad s = 0, 1, 2, \ldots, N \]

\[ \dot{z}_{i,s} = \sum_{n=0}^{N-1} \dot{Z}_{i,n} e^{i(2\pi s n / N)}; \quad s = 0, 1, 2, \ldots, N \]

\[ \ddot{z}_{i,s} = \sum_{n=0}^{N-1} \ddot{Z}_{i,n} e^{i(2\pi s n / N)}; \quad s = 0, 1, 2, \ldots, N \]

Final solution

\[ u = \sum_i z_i \phi_i \]