Slope Stability Analysis
Using Finite Element Techniques

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LIMIT STATE ANALYSIS OF EARTHEN SLOPES
USING DUAL CONTINUUM/FEM APPROACHES

A. Review of Classical Methods

B. Proposed Slope Stability Analysis Methods
   * Gravity Increase Method
   * Strength Reduction Method

C. Comparison of the Methods for Total Stress Analysis

D. Application to Problems with Seepage

E. Assessment of Continuum/FEM Approaches to SSA
A. Review of Common Classical Methods

* Infinite Slope Analysis

* Mass Methods (Culmann’s method; Fellenius–Taylor method)

* Methods of Slices (Bishop’s simplified method, Ordinary method of slices, ...)

* Factor of Safety:

\[ FS = \frac{M_R}{M_D} \]

where \( M_R \) = The moment of ultimate resisting forces

\( M_D \) = The moment of driving forces
* Perceived shortcomings in classical methods:

1) Analysis of stresses within the soil mass is approximate.
   
   a) Using statics approximations for continuum system.
   b) Interslice forces?

2) Typically restricted to Mohr–Coulomb soil models

   * Other, more realistic soil models are presently available. (Critical state models; cap models; softening effects; etc)

3) Transient effects associated with pore pressure diffusion are difficult to incorporate.

* Research question:
  Can continuum/FEM methods be applied to improve state of the art in SSA?
B. Two Continuum/FEM Slope Stability Analysis Techniques

**Gravity Increase Method**

* Increase \( g \) until the slope becomes unstable and equilibrium solutions no longer exist. (W.F. Chen)

* \( g(t) = g_{\text{base}} \cdot f(t) \) where \( g_{\text{true}} \) is actual gravitational acceleration.

* \((F.S)_{gi} = \frac{g_{\text{limit}}}{g_{\text{true}}}\)

**Strength Reduction Method**

* Decrease the strength parameters of the slope until slope becomes unstable and equilibrium solutions no longer exist. (D.V. Griffiths, and O.C. Zeinkiewicz)

* \( Y(t) = Y_{\text{base}} \cdot f(t) \) where \( Y_{\text{base}} \) are actual strength parameters

* \((F.S)_{sr} = \frac{Y_{\text{base}}}{Y(t_{\text{limit}})} = \frac{1}{f(t_{\text{limit}})}\)
Fit of Drucker–Prager Yield Surface

with Sand Data of Desai and Sture.

\[ f(\sigma) = ||s|| - \left\{ \alpha + \lambda \left(1 - \exp\left[\beta I_1\right]\right) \right\} \leq 0 \]

\[ \lambda = 1.53 \text{ kPa}, \quad \beta = 3.48 \times 10^{-6} \text{ Pa}^{-1}, \quad \alpha = 0 \]
Application of Loads to Soil Mass (For gravity method)

Note: For purely frictional soils (non-cohesive), shear strength comes entirely from effective confining stresses.
C. Comparative Results (Total Stress Analysis)

1) Non-frictional Soil ($\alpha = 141\text{kPa}$)

(FS)$_{gi}$ = 3.03, (FS)$_{sr}$ = 3.04; Fellenius–Taylor Method; FS = 3.17
2) Frictional Soil: Slope angle 20° ($\lambda = 1.53$ kPa, $\beta = 3.48 \times 10^{-6}$ Pa$^{-1}$, $\alpha = 0$)

### Strength Reduction Method

<table>
<thead>
<tr>
<th>H (m)</th>
<th>$(FS)_{sr}$</th>
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<tbody>
<tr>
<td>6</td>
<td>3.14</td>
</tr>
<tr>
<td>12</td>
<td>2.49</td>
</tr>
<tr>
<td>30</td>
<td>1.23</td>
</tr>
<tr>
<td>60</td>
<td>0.64</td>
</tr>
</tbody>
</table>

### Gravity Increase Method

<table>
<thead>
<tr>
<th>H (m)</th>
<th>$(FS)_{gi}$</th>
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<tr>
<td>6</td>
<td>6.28</td>
</tr>
<tr>
<td>12</td>
<td>3.40</td>
</tr>
<tr>
<td>30</td>
<td>1.25</td>
</tr>
<tr>
<td>60</td>
<td>0.79</td>
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3) Heterogeneous Soil: Slope angle 30°

Clay: $\alpha = 141\text{kPa (dark region)}$

Sand: $\lambda = 1.53\text{kPa, } \beta = 3.48d^{-6}\text{ Pa}^{-1}, \alpha=0$

**Strength Reduction Method**

(FS)$_{sr}$=1.26

(FS)$_{sr}$=1.52

**Gravity Increase Method**

(FS)$_{gi}$=1.11

(FS)$_{gi}$=1.64
Steep Slope with tension crack 1
Clay: $\alpha = 141$ kPa

a) Undeformed slope.

b) Deformed slope at limit state.

Steep Slope with tension crack 2

a) Undeformed slope.

b) Deformed slope at limit state.
Slope under Pseudo−Static Earthquake Loading

$g = 9.81 \text{m/s}^2$ downward and leftward horizontal acceleration of 0.447g

<table>
<thead>
<tr>
<th>8m</th>
<th>30m</th>
<th>50m</th>
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</thead>
<tbody>
<tr>
<td>30m</td>
<td>9m</td>
<td>27m</td>
</tr>
<tr>
<td>54m</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) Undeformed state

b) Deformed limit state

Slope with Building (Clay: $\alpha = 141 \text{kPa}$) : $\text{(FS)}_{\text{grav}} = 2.47$
Comparative Summary of the Two Continuum/FEM Approaches

1) Two methods employ virtually identical computational FEM techniques.

2) Computational times are competitive compared to classical methods of slice type.

3) In total stress analysis, neither method is clearly superior over the other

   * For purely cohesive soils, both methods yield identical results.

   * For frictional soils, strength reduction method typically gives more conservative results and it guarantees the existence of a limit state.

4) Gravity Increase Method :

   This method is well suited for analyzing the stability of embankment constructed on saturated soil deposits, since the rate of construction of embankment can be simulated with the rate at which gravity loading on the embankment is increased.

5) Strength Reduction Method:

   This method appears well suited for analyzing the stability of existing slopes in which unconfined active seepage is occurring.
D.1 STABILITY ANALYSIS OF EMBANKMENTS ON SATURATED DEPOSITS

A) Use a coupled porous medium model
   * This model can capture the time dependent pore-pressure diffusion behaviors of a saturated porous medium.

B) Use the smooth elasto–plastic cap model
   * This model can account for coupled shear and compressive soil behaviors.

C) Use the Gravity Increase Method
   * This method can simulate the rate of embankment construction.
A. Continuum Formulation

Find $u^s$ and $v^w$, such that

$$\rho^s a^s = \nabla \cdot \sigma' - n^s \nabla p_w - \xi \cdot (v^s - v^w) + \rho^s b$$

$$\rho^w \frac{D}{Dt} (v^w) = -n^w \nabla p_w + \xi \cdot (v^s - v^w) + \rho^w b$$

Boundary Conditions

$$u^s = \bar{u}^s \quad \text{on} \quad \Gamma_{g_s}$$

$$u^w = \bar{u}^w \quad \text{on} \quad \Gamma_{g_w}$$

$$(\sigma' - n^s p_w \delta) n = \bar{h}^s \quad \text{on} \quad \Gamma_{h_s}$$

$$-n^s p_w n = \bar{h}^w \quad \text{on} \quad \Gamma_{h_w}$$

Initial Conditions

$$u^s (0) = u_o^s$$

$$\dot{u}^s (0) = \dot{u}_o^s$$

$$\dot{u}^w (0) = \dot{u}_o^w$$
Matrix Equations

\[
\begin{bmatrix}
M^s & 0 \\
0 & M^w
\end{bmatrix}
\begin{bmatrix}
a^s \\
a^w
\end{bmatrix} + \begin{bmatrix}
Z & -Z \\
-Z & Z
\end{bmatrix}
\begin{bmatrix}
v^s \\
v^w
\end{bmatrix} + \begin{bmatrix}
n^s(d^s, v) \\
n^w(v)
\end{bmatrix} = \begin{bmatrix}
f^{s(\text{ext})} \\
f^{w(\text{ext})}
\end{bmatrix}
\]

\[M^\alpha = \int N_A \rho^\alpha N_B \, d\Omega\]

\[Z = \int N_A \cdot \xi \cdot N_B \, d\Omega\]

\[
\begin{bmatrix}
n^s(d^s, v) \\
n^w(v)
\end{bmatrix} = \begin{bmatrix}
\int B_A \sigma' \, d\Omega + \int N_A n^s p^w \, d\Omega \\
-\int \nabla N_A n^w p^w \, d\Omega
\end{bmatrix}
\]

\[
\begin{bmatrix}
f^{s(\text{ext})} \\
f^{w(\text{ext})}
\end{bmatrix} = \begin{bmatrix}
\int N_A \rho^s b \, d\Omega + \int N_A h^s d\Gamma \\
\int N_A \rho^w b \, d\Omega + \int N_A h^w d\Gamma
\end{bmatrix}
\]

Tangent operator

\[
\begin{bmatrix}
M^s & 0 \\
0 & M^w
\end{bmatrix} + \begin{bmatrix}
\Delta t \gamma Z & -\Delta t \gamma Z \\
-\Delta t \gamma Z & \Delta t \gamma Z
\end{bmatrix} + \begin{bmatrix}
\Delta t^2 \beta (K + C^{ss}) & \Delta t^2 \beta C^{sw} \\
\Delta t^2 \beta C^{ws} & \Delta t^2 \beta C^{ww}
\end{bmatrix}
\]

\[K = \int B_A D B_B \, d\Omega\]

\[C^{\alpha\beta} = \int \lambda^w \frac{n^\alpha n^\beta}{n^w} \nabla N_A \nabla N_B \, d\Omega\]
B. Material Model Description

**Sandler–DiMaggio Cap Model**

Features:
* Five elasto–plastic subcases
* Singular tangent operators in the corner regions (no bulk stiffness)

**Smooth Cap Model**

Features:
* Three elasto–plastic subcases
* No problems with singular tangent operators
Yield functions
\[ f_1(\sigma, \xi) = |\eta| - F_e(I_1) \leq 0 \quad \text{where } F_e(I_1) = \alpha - \theta I_1 \]
\[ f_2(\sigma, \xi, \kappa) = |\eta|^2 - F_c(I_1, \kappa) \leq 0 \quad \text{where } F_c(I_1, \kappa) = R^2(\kappa) - (I_1 - \kappa)^2 \]
\[ f_3(\sigma, \xi) = |\eta|^2 - F_t(I_1) \leq 0 \quad \text{where } F_t(I_1) = T^2 - I_1^2 \]

Flow rule (associated)
\[ \dot{\varepsilon}^p = \sum \gamma^\alpha \frac{\partial f^\alpha}{\partial \sigma} \]

Non–associated hardening laws
\[ \dot{\rho} = H \dot{\varepsilon}^p \]
\[ \kappa = h'(\kappa) \text{tr}(\varepsilon^p) \quad \text{where } h'(\kappa) = \frac{\exp[-D\chi(\kappa)]}{WD\chi'(\kappa)} \quad \chi(\kappa) = \kappa - R(\kappa) \]

Karesh–Kuhn–Tucker Condition
\[ f^\alpha \leq 0 \quad \dot{\gamma}^\alpha \leq 0 \quad \dot{\gamma}^\alpha f^\alpha = 0 \]

Plastic consistency condition
\[ \dot{\gamma}^\alpha \dot{f}^\alpha = 0 \]
**1-D Compression test on sand**

**Drained triaxial compression test**

**Experimental data and Model Response for 1-D compression on dry sand**

**Experimental data and Model Response data for drained triaxial compression test**
C. Examples

*Cubzac-les-Point embankment in France*

foundation: $\alpha=12.3\,\text{kPa}$ $\theta=0.2003$, $w=0.15$, $D=3.2\times10^{-7}\,\text{Pa}^{-1}$
embankment: $\alpha=10.0\,\text{Pa}$ $\theta=0.2567$, $w=0.01$, $D=5.0\times10^{-7}\,\text{Pa}^{-1}$

* Experimental embankment constructed in 10 days up to failure in 1971.

* In 1982, Pilot et al analyzed the embankment’s stability by Bishop’s method of slices

$\gamma=21.2\,\text{kN/m}^3$
$c'=0$
$\phi'=35^\circ$

Mechanism and FS computed by Pilot

* Observation:
The computation method of SSA is more realistic (and conservative) than the classical method, since it accounts for the shear and compressibility behaviours of the clay soil.
Modeling of Sand Drains to Enhance Stability

without drains : $FS=0.675$
1 day
construction : $FS=0.968$

without drains : $FS=0.739$
10 days
construction : $FS=1.175$

without drains : $FS=0.909$
100 days
construction : $FS=1.64$

without drains : $FS=1.35$
1000 days
construction : $FS=2.61$
D.2 SLOPE STABILITY ANALYSIS WITH UNCONFINED SEEPAGE

A. Formulation

B. Example Solutions
A . Coupled Porous Medium
Free–Boundary Problem

1) Problem Geometry

2) Statement of the Problem

Steady state seepage and incompressible fluid are assumed

Find \( u_s \) and \( p \), such that

\[
\nabla \cdot (\sigma' - p \delta) + \rho b = 0 \quad \text{in } \Omega \quad \text{(Total Stress Equilibrium)}
\]

\[
\nabla \cdot v^s + \nabla \cdot v^w = 0 \quad \text{in } \Omega^w \quad \text{(Conservation of Fluid Mass)}
\]

Solid Boundary Conditions

\[
\begin{align*}
    u^s &= \bar{u}^s \quad \text{on } S_1 \\
    (\sigma' - p \delta) \cdot n &= \bar{h}^s \quad \text{on } S_1 \cup S_2
\end{align*}
\]

Fluid Boundary Conditions

\[
\begin{align*}
    p > 0 \quad \text{in } \Omega^w &; \quad p=0 \quad \text{elsewhere} \\
    n \cdot v^w &= 0 \quad \text{on } \Gamma_1 \\
    p=0 \quad \text{and} \quad n \cdot v^w &= 0 \quad \text{on } \Gamma_2 \\
    p=\bar{p} \quad \text{on } \Gamma_3 \\
    p=0 \quad \text{and} \quad -n \cdot v^w &\leq 0 \quad \text{on } \Gamma_4
\end{align*}
\]

where \( \bar{p} = \begin{cases} 
    \gamma_w (h_2-y) \text{ on the right side of dam} \\
    \gamma_w (h_1-y) \text{ on the left side of dam}
\end{cases} \)

\[
    v^w = -\kappa \cdot \text{grad} \left( \frac{p}{\gamma_w} + y \right) \quad \text{(Darcy’s Law)}
\]
Purely Cohesiveless Soil

**Strength Reduction Method** (dry slopes)

- $H=6\text{m}$ (12%)
  - $(FS)=2.56$

- $H=12\text{m}$ (14.7%)
  - $(FS)=2.24$

- $H=30\text{m}$ (18.5%)
  - $(FS)=1.13$

- $H=60\text{m}$ (19%)
  - $(FS)=0.58$

**Strength Reduction Method** (seepage effects included)

- $H=6\text{m}$
  - $(FS)=2.26$

- $H=12\text{m}$
  - $(FS)=1.91$

- $H=30\text{m}$
  - $(FS)=0.92$

- $H=60\text{m}$
  - $(FS)=0.47$
E. SUMMARY ON FEM SLOPE STABILITY ANALYSIS

1) Approximations required in SLICE type methods are not required.

2) The method can use virtually any realistic soil material model.

   * Usage of more sophisticated material models typically requires more laboratory testing.

3) For many applications, classical methods are suitable, given the uncertainty in soil properties.

4) FEM/SSA appears to hold an advantage over classical methods for problems involving seepage – as in embankment stability analysis.
5) Requirements for SSA with FEM are non-trivial.

* High-end PC or workstation
* FEM software (starting at $2k per year for commercial licenses)
* Understanding of soil mechanics, material models and FEM.

6) The 2D SSA examples presented here took between 15 minutes and a few hours to run on an engineering workstation (SGI Powerchallenge).

Presently, 3D SSA with FEM is too expensive to be feasible on PCs and workstations. In the future, this may become feasible as computing power advances.