53:243/58:251 Computational Inelasticity
The University of Iowa
College of Engineering
Fall 2005 Semester
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Assignment #1 Due: 16 Sept. 2005

1. The displacement field in a membrane is given by the following analytical expressions:

$$u_1(X_1, X_2) = \tfrac{L}{8}\sin(\lambda X_1)\sin(\lambda X_2);$$
$$u_2(X_1, X_2) = \tfrac{L}{8}\cos(\lambda X_1)\cos(\lambda X_2);$$
$$u_3(X_1, X_2) = 0;$$

for $X_1 \in \left[0, {}^{L}\!/_{2}\right]$; $X_2 \in \left[0, {}^{L}\!/_{2}\right]$; and $\lambda = {}^{2\pi}\!/_{L}$.

   a. Plot the shape of the deformed membrane;
   b. Compute the Green-Lagrange strain tensor at the following points:
      i. $(X_1, X_2) = (0,0)$;
      ii. $(X_1, X_2) = ({}^{L}\!/_{8}, {}^{L}\!/_{8})$;
      iii. $(X_1, X_2) = ({}^{L}\!/_{4}, {}^{L}\!/_{4})$;

2. A hyperelastic strain energy function for the isotropic membrane material of problem 1 is given as follows:

Volumetric strain energy: $U(J) = \dfrac{1}{2}K\left[\dfrac{1}{2}(J^2 - 1) - \ln(J)\right]$

Deviatoric strain energy: $\overline{W} = \dfrac{1}{2}\mu[tr(\overline{\boldsymbol{\theta}}) - 3]$

In the preceding expressions, J is the determinant of $\mathbf{F} = \mathbf{1} + {}^{\partial \mathbf{u}}\!/_{\partial \mathbf{X}}$; K is a constant bulk modulus; μ is a constant shear modulus; $\boldsymbol{\theta} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green deformation tensor; and $\overline{\boldsymbol{\theta}} = J^{-(2/3)}\boldsymbol{\theta}$ is its scaled counterpart with the characteristic that $\det(\overline{\boldsymbol{\theta}}) = 1$. For this model, the Kirchhoff stress $\boldsymbol{\tau}$ is related to deformation as follows: (see the background on the following pages to see how this was obtained)

$$\boldsymbol{\tau} = JU'\mathbf{1} + \mu\,\mathrm{dev}\left[\overline{\boldsymbol{\theta}}\right]$$

Compute expressions for the stress at the three specific points noted in Problem 1.

3. Derive the linear form of the above constitutive equation in the limit of infinitesimal deformation.

Background Information:

For problem #2 it was given that the strain energy density function (per unit volume in the undeformed configuration) has the following form:

$$W = U(J) + \overline{W}(\overline{\boldsymbol{\theta}}) \qquad \text{where:}$$

$$U(J) = \frac{K}{2}\left[\left(J^2 - 1\right) - \ln(J)\right] \text{ and } \qquad \overline{W}(\overline{\boldsymbol{\theta}}) = \frac{\mu}{2}\left[tr(\overline{\boldsymbol{\theta}}) - 3\right].$$

The rate of change of the strain energy density can be computed as follows:

$$\dot{W} = \frac{dU}{dJ}\dot{J} + \frac{\mu}{2}\mathbf{1} : \dot{\overline{\boldsymbol{\theta}}}$$

From continuum mechanics it can be shown that $\dot{J} = J(\mathbf{1} : \mathbf{d})$. In this expression, $\mathbf{d} = L_{(ij)} = \frac{1}{2}(L_{ij} + L_{ji}) = \frac{1}{2}(v_{i,j} + v_{j,i})$ and this is known as the *rate of deformation tensor*. The rate of deformation tensor **d** is the symmetric part of **L** which is usually called the *spatial velocity gradient*.

To evaluate $\dot{\overline{\boldsymbol{\theta}}}$, the chain rule should be employed. Doing so gives:

$$\dot{\overline{\boldsymbol{\theta}}} = \frac{d(J^{-2/3}\boldsymbol{\theta})}{dt} = -\frac{2}{3}\overline{\boldsymbol{\theta}}tr(\mathbf{d}) + J^{-2/3}\dot{\boldsymbol{\theta}}$$

To evaluate $\dot{\boldsymbol{\theta}}$ is fairly straightforward:

$$\begin{aligned}\dot{\theta}_{ij} &= \frac{d(F_{iK}F_{jK})}{dt} = \dot{F}_{iK}F_{jK} + F_{iK}\dot{F}_{jK} \\ &= \mathrm{L}_{\mathrm{im}}F_{mK}F_{jK} + F_{iK}L_{jm}F_{mK} \\ &= \mathrm{L}_{\mathrm{im}}\theta_{mj} + \theta_{im}L_{jm}\end{aligned}$$

Now, observe that $\mathbf{1} : \dot{\overline{\boldsymbol{\theta}}}$ reduces to the following:

$$\begin{aligned}\mathbf{1} : \dot{\overline{\boldsymbol{\theta}}} &= -\frac{2}{3}tr(\overline{\boldsymbol{\theta}})tr(\mathbf{d}) + \mathbf{1} : \left(\mathbf{L} \cdot \overline{\boldsymbol{\theta}} + \overline{\boldsymbol{\theta}} \cdot \mathbf{L}^T\right) \\ &= -\frac{2}{3}tr(\overline{\boldsymbol{\theta}})tr(\mathbf{d}) + 2\overline{\boldsymbol{\theta}} \cdot \mathbf{d} \\ &= 2\left[\overline{\boldsymbol{\theta}} - \frac{1}{3}tr(\overline{\boldsymbol{\theta}})\mathbf{1}\right] : \mathbf{d} \\ &= 2\ \mathrm{dev}[\ \overline{\boldsymbol{\theta}}\ ] : \mathbf{d}\end{aligned}$$

Finally, putting everything together, you'll get:

$$\begin{aligned}\dot{W} &= \frac{dU}{dJ}\dot{J} + \frac{\mu}{2}\mathbf{1} : \dot{\bar{\boldsymbol{\theta}}} \\ &= \frac{dU}{dJ} J\,\mathbf{1} : \mathbf{d} + \mu\,\mathrm{dev}[\bar{\boldsymbol{\theta}}] : \mathbf{d} \\ &= \boldsymbol{\tau} : \mathbf{d}\end{aligned}$$

And so, it follows that $\boldsymbol{\tau} = JU'\mathbf{1} + \mu\,\mathrm{dev}\left[\bar{\boldsymbol{\theta}}\right]$.