Period #9: Multi–dimensional Fluid Flow in Soils

A. Objective:

• To this point in time, we have considered one–dimensional flow in soils, where all fluid is flowing in the same direction.
• In most cases, however, fluid in different regions will be flowing in different directions. We call this multi–dimensional flow.
• Our objective here is to learn how to solve multi–dimensional flow problems such as the one shown below.

• To develop these capabilities, we will use two considerations:
  1) conservation of fluid mass in the soil (or continuity);
  2) multi–dimensional forms of Darcy’s Law.
B. Conservation of Fluid Mass (or Continuity)

Consider the case of fluid flowing through a region of soil in space as shown below:

Mass of water $M_w$ in $R$:

$$M_w = \int_R n \rho_w \, dR,$$

where:

- $\rho_w$ is the mass density of water (constant)
- $n$ is the porosity of the soil (constant)

Net rate of fluid mass outflow from $R$:

$$\text{rate of mass outflow} = \int_S l \cdot (\rho_w \, v) \, dS,$$

where:

- $v$ is the discharge velocity;
- $l$ is the unit normal vector.
A statement for the conservation of fluid mass in R is the following:

net rate of mass outflow = rate of fluid mass decrease

\[
\int_S \mathbf{l} \cdot (\rho_w \mathbf{v}) \, dS = -\frac{d(M_w)}{dt}
\]

\[
= -\int_R \partial_t (n\rho_w) \, /\partial t \, dR
\]

\[
= 0, \text{ since } n \text{ and } \rho_w \text{ are both assumed to be constant.}
\]

Assumption: both the soil and the fluid do not compress during the flow.

\[
\therefore \text{ rate of fluid mass outflow from } R = 0
\]

Restated, \( q_{in} = q_{out} \)

By Green’s Theorem from Vector Calculus:

\[
0 = \int_S \mathbf{l} \cdot (\rho_w \mathbf{v}) \, dS = \int_R \nabla \cdot (\rho_w \mathbf{v}) \, dR = \int_R [\mathbf{v} \cdot (\nabla \rho_w) + \rho_w \nabla \cdot \mathbf{v}] \, dR
\]

Side note: The quantity \( \nabla \) denotes a gradient operator. For example,

\[
\nabla \rho_w = \left[ \frac{\partial \rho_w}{\partial x_1} \right] \mathbf{e}_1 + \left[ \frac{\partial \rho_w}{\partial x_2} \right] \mathbf{e}_2 + \left[ \frac{\partial \rho_w}{\partial x_3} \right] \mathbf{e}_3.
\]

(Since \( \rho_w \) is assumed constant and uniform, \( \nabla \rho_w = 0 \).)
\[
0 = \int_{R} \rho_{w} \nabla \cdot \mathbf{v} \, dR \quad \text{for all regions } R
\]

\[
\therefore \text{ integrand itself vanish everywhere, or } \rho_{w} \nabla \cdot \mathbf{v} = 0.
\]

But since \( \rho_{w} \neq 0 \), then

\[
\nabla \cdot \mathbf{v} = 0, \quad \text{or}
\]

\[
0 = \frac{\partial v_{1}}{\partial x_{1}} + \frac{\partial v_{2}}{\partial x_{2}} + \frac{\partial v_{3}}{\partial x_{3}} : \textbf{Continuity Equation}
\]

For flow in saturated soils, where the fluid has uniform and constant density, this is the equation for conservation of fluid mass. It is commonly called the \textit{continuity equation}.

**Example:** One–dimensional flow in a soil deposit, in the \( x_{1} \) direction.

Since the flow is one–dimensional in the \( x_{1} \) direction, \( v_{2} = v_{3} = 0. \)

\[
\therefore \text{ the continuity equation reduces to:}
\]

\[
\frac{\partial v_{1}}{\partial x_{1}} = 0.
\]

**Interpretation:** \( v_{1} = \text{constant}. \)

This idea was used in considering flow through layered soil deposits.
C. Multi-Dimensional Forms of Darcy’s Law

In one-dimension, we have expressed Darcy’s Law as: \( v = k_i \)

In multiple dimensions, the same law is typically expressed: \( \mathbf{v} = -\mathbf{k} \cdot \nabla h \), where:

- \( \mathbf{v} \) is the discharge velocity vector;
- \( \mathbf{k} \) is the soil’s permeability tensor; and
- \( \nabla h \) is the gradient vector of \( h \).

In component form, this expression of Darcy’s Law is written:

\[
\begin{bmatrix}
  v_x \\
v_y \\
v_z
\end{bmatrix} =
- \begin{bmatrix}
  k_x & 0 & 0 \\
  0 & k_x & 0 \\
  0 & 0 & k_z
\end{bmatrix} \begin{bmatrix}
  \partial h/\partial x \\
  \partial h/\partial y \\
  \partial h/\partial z
\end{bmatrix} =
\begin{bmatrix}
  -k_x & \partial h/\partial x \\
  -k_y & \partial h/\partial y \\
  -k_z & \partial h/\partial z
\end{bmatrix}
\]

For anisotropic soils (such as layered soils), it is not generally true that \( k_x = k_y = k_z \). This is true only for isotropic soils.

If this multi-dimensional form of Darcy’s Law is inserted into the continuity equation, and one assumes a homogeneous and isotropic soil, then the resulting form of the continuity equation is:

\[
\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 : \text{common form of the Laplace Equation.}
\]