A. Review and Motivation

(1) Review:

- In most soils, changes in total volume are associated with reductions in void volume. The volume change of the soil grains is negligible.

- Changes in soil volume are produced by effective stresses.

- When loads are applied to soils, the increased compressive stresses can initially be taken up by the pore fluid.

- With time, though, the pore fluid pressure dissipates, and the increased compressive stresses are transferred to the soil skeleton.
  
  - As this gradually occurs, the soil will compress.

(2) Motivation:

- The important question that remains to be answered, is how long can this take?

- To better understand this, a common engineering consolidation model is presented here.
B. Rate of Consolidation in One-Dimension

- To begin, consider a very small element of soil being subjected to one-dimensional consolidation in the z-direction.

\[ v_z + [\frac{\partial(v_z)}{\partial z}] \Delta z = \text{exit discharge velocity} \]

\[ \Delta A = (\Delta x)(\Delta y) \]

\[ \Delta V = (\Delta x)(\Delta y)(\Delta z) = (\Delta A)(\Delta z) \]

\[ v_z = \text{entry discharge velocity} \]

- Rate of fluid mass outflow: \[ = \rho_w \Delta A \{ v_z + [\frac{\partial(v_z)}{\partial z}] \Delta z \} \]
- Rate of fluid mass inflow: \[ = \rho_w \Delta A v_z \]
- Net rate of fluid mass outflow: \[ = \text{outflow} - \text{inflow} = \rho_w \Delta A [\frac{\partial(v_z)}{\partial z}] \Delta z = \rho_w \Delta V \frac{\partial(v_z)}{\partial z} \]
• fluid mass conservation for this element:

net rate of fluid mass outflow = net rate of internal fluid mass decrease

\[ \rho_w \Delta V \frac{\partial (v_z)}{\partial z} = - \frac{\partial \{ M_w \}}{\partial t} = - \frac{\partial \{ \rho_w e \Delta V_s \}}{\partial t} = - \rho_w \Delta V_s \frac{\partial \{ e \}}{\partial t} \]

Aside: Note these assumptions:
a) fluid is incompressible (i.e. \( \frac{\partial (\rho_w)}{\partial t} = 0 \)); and
b) soil grains are also incompressible (i.e. \( \frac{\partial (\Delta V_s)}{\partial t} = 0 \)).

\[ \therefore \rho_w \Delta V \frac{\partial (v_z)}{\partial z} = -\rho_w \Delta V_s \frac{\partial \{ e \}}{\partial t}, \]

\[ \frac{\partial (v_z)}{\partial z} = - (\Delta V_s/\Delta V) \frac{\partial \{ e \}}{\partial t} \]

\[ \frac{\partial (v_z)}{\partial z} = - (1+e_0)^{-1} \frac{\partial \{ e \}}{\partial t} \quad \text{statement of fluid mass conservation.} \]

• For 1-D flow in the z-direction, Darcy’s Law gives \( v_z = -k_z \frac{\partial h}{\partial z} \), where:

\[ h = h_z + \frac{p_w}{\gamma_w} \]

\[ k_z = \text{permeability in the z-direction} \]
\[
\partial (-k_z \partial h/\partial z)/\partial z = - (1+e_0)^{-1} \partial e/\partial t
\]
\[
k_z \partial^2 h/\partial z^2 = (1+e_0)^{-1} \partial e/\partial t
\]

Note the assumption that \( k_z \) is constant in the z-direction.:

- This expression needs to be re-written in terms of pore pressure.

\[
h = h_z + (p_w/\gamma_w) = h_z + (p_{hyd} + u)/\gamma_w
\]

- Note: In the above expression, the fluid pressure \( p_w \) has been expressed as
  \( p_w = p_{hyd} + u \), where: \( p_{hyd} \) is the equilibrium hydrostatic pore-pressure in the soil before a load was applied, and \( u \) is the excess pore-pressure in the soil due to the application of a load.

- Since both \( h_z \) and \( p_{hyd} \) are linear functions of \( z \), \( \partial^2 h/\partial z^2 = (1/\gamma_w) \partial^2 u/\partial z^2 \).

\[
\therefore \left( k_z/\gamma_w \right) \partial^2 u/\partial z^2 = (1+e_0)^{-1} \partial e/\partial t
\]

- The remaining task is to relate the void ratio \( e \) to excess pore pressure \( u \).
This is done in two steps:

a) Note that when a constant and uniform load is applied to a soil deposit and consolidation is occurring, the vertical stress in the soil is constant.

\[ 0 = \frac{\partial (\sigma_v)}{\partial t} = \frac{\partial (\sigma'_v)}{\partial t} + \frac{\partial (p_w)}{\partial t} \]

Thus,
\[ \frac{\partial (\sigma'_v)}{\partial t} = -\frac{\partial (p_w)}{\partial t} \]
\[ = -\frac{\partial u}{\partial t} \]

That is, an increase in vertical effective stress \( \sigma'_v \) is achieved by a reduction of excess pore pressure \( u \).

b) Now to relate \( \sigma'_v \) to the void ratio \( e \), the \( e \) vs. \( \sigma'_v \) behavior of the soil is used.

- From the figure shown, by linearizing the soil behavior over a given range of \( \sigma'_v \) and \( e \):
  \[ \Delta e = -a_v \Delta \sigma'_v \]

- From the preceding relation between \( \sigma'_v \) and \( u \), it follows that:
  \[ \Delta e = a_v \Delta u \]

- Expressing this in rate form, \( \frac{\partial e}{\partial t} = a_v \frac{\partial u}{\partial t} \).
Putting it all together, the final statement of fluid mass conservation is:

\[ \left( \frac{k_z}{\gamma_w} \right) \frac{\partial^2 u}{\partial z^2} = \left[ \frac{a_v}{1 + e_0} \right] \frac{\partial u}{\partial t} \]

or

\[ c_v \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} : \quad \text{One-dimensional consolidation equation} \]

where \( c_v = \frac{k_z(1 + e_0)}{a_v \gamma_w} \): consolidation coefficient (units are \( L^2/T \))

C. Solutions of the Consolidation Equation

Initial conditions for excess pore-pressure in the low-permeability soil layer:

\[ u = u_0 = \Delta q \quad \text{for } z \in (0, 2H) \text{ at } t=0 \]

Boundary conditions for excess pore-pressure in the low-permeability soil layer:

\[ u = 0 \quad \text{at } z = 0 \quad \text{for } t \in [0, \infty) \]
\[ u = 0 \quad \text{at } z = 2H \quad \text{for } t \in [0, \infty) \]
The solution to the 1–D consolidation equation for these boundary conditions and initial conditions is:

\[
\begin{align*}
    u(z,t) &= \sum_{m=0}^{\infty} \left\{ \frac{2u_0}{M} \sin\left(\frac{Mz}{H}\right) \right\} \exp\left\{ -M^2T_v \right\} \\
    \text{where: } & \quad M = \frac{\pi}{2}(2m+1) \text{ and } \\
    T_v &= \frac{c_v}{H^2} : \text{non–dimensional time factor } \\
    H &= : \text{maximum drainage distance in the soil layer }
\end{align*}
\]

\[\text{"Excess" pore pressure } u(z,t)\]
• The next task is to relate the excess pore-pressure $u(z,t)$ to the actual compression of the soil layer.

• At a given time $t$, and a given location $z$ in the soil layer, define the local degree of consolidation as:

$$U_z(z,t) = \left[ u_0 - u(z,t) \right] / u_0 = 1 - \left[ u(z,t) / u_0 \right]$$

Observations:

- when $u(z,t) = u_0$, then $U_z(z,t) = 0$, which means that the soil at $(z,t)$ has not yet begun to consolidate; and
- when $u(z,t) = 0$, then $U_z(z,t) = 1$, which means that the soil at $(z,t)$ has fully consolidated.

• Now define the average degree of consolidation $U$ for the whole soil layer at a given time $t$ as:

$$U(t) = \left( \frac{1}{2H} \right) \int_{0}^{2H} U_z(z,t) \, dz = \left( \frac{1}{2H} \right) \int_{0}^{2H} \left[ 1 - u(z,t)/u_0 \right] \, dz$$

$$= 1 - \sum_{m=0}^{\infty} \left( \frac{2}{M^2} \right) \exp\left\{ -M^2 T_v \right\}$$

$U$ vs. $T_v$ is plotted in Figure 10.24 and tabulated in Table 10.5 of the textbook.
D. Applications

Example 16.1: A 3m thick double–drained layer of saturated clay soil was subjected to a surcharge loading and achieved 90% primary consolidation in 75 days. Find the coefficient of consolidation \( c_v \).

Solution:

• The dimensionless time factor for soils is: \( T_v = \frac{(c_v \cdot t)}{H^2} \).

\[ \therefore T_{90} = \frac{(c_v \cdot t_{90})}{H^2} = \frac{[c_v \cdot 75\text{days}]}{(1.5\text{m})^2} \]

• From the U vs. \( T_v \) plot on page 288 of the text, \( T_{90} = 0.85 \)

• Solving for \( c_v \) gives, \( c_v = 0.0255\text{m}^2/\text{day} = 2.95\times10^{-7}\text{m}^2/\text{sec} \).
Example 16.2: Consider the soil profile shown. The phreatic surface now coincides with the ground surface, but a long time ago it used to be at a depth of 5 feet below the found surface. Assume that a uniform pressure of 400 psf is to be applied over a large area.

a) Use the location of the phreatic surface a long time ago to compute the pre-consolidation vertical effective stress in the clay layer;

b) Estimate the ultimate settlement of the ground surface due only to primary consolidation of the clay layer; and

c) How long will it take to achieve 50% and 90% consolidation of the clay layer under the imposed loading?

Coarse, Dense Sand; $\gamma_{dry}=110$pcf; $\gamma_{sat}=130$pcf;

Clay Soil; $\gamma_{sat}=115$pcf;

$e_0 = 1.10$

$C_c = 0.60$

$C_s = 0.05$

$c_v = 0.05ft^2/day$

Bedrock
Solution:

a) First, compute the vertical effective stress at the center of the clay layer before the uniform pressure is applied.

\[(\sigma'_v)_0 = 5'*(130−62.4)pcf + 10'*(115−62.4)pcf\]
\[= 864 \text{ psf}\]

A long time ago, the phreatic surface was at the clay−sand interface. At that time, the vertical effective stress was:

\[(\sigma'_v) = 5'*(110)pcf + 10'*(115−62.4)pcf\]
\[= 1076 \text{ psf}\]

Since the vertical effective stress a long time ago was greater than what the current vertical effective stress is, the clay soil is over−consolidated, and the pre−consolidation stress \((\sigma'_v)_c= 1076\text{psf}\).

b) Estimate the ultimate settlement due to application of the load:

\[\Delta H = H_0\Delta e/(1+e_0)\]
\[= (20\text{ft}/2.1)*[-C_s\log[(\sigma'_v)_c/(\sigma'_v)_0] - C_c\log[(\sigma'_v)_f/(\sigma'_v)_c]]\]
\[= 9.524\text{ft} * [-0.05 \log(1076/864) − 0.6 \log(1264/1076)]\]
\[= 9.524\text{ft} * [−.0048 − .0420]\]

\[\Delta H = -0.445 \text{ ft} = -5.34\text{in} = \text{ultimate settlement.}\]
c) Solution: The dimensionless time factor for soils is: $T_v = \left( \frac{c_v t}{H^2} \right)$. 

From the plot of $U$ vs. $T_v$ on page 292 of the text: $T_{50} = 0.18$ and $T_{90} = 0.85$

Thus, $t_{50} = \frac{(T_{50}H^2)}{c_v} = 0.18 \times (20\text{ft})^2 / (0.05\text{ft}^2/\text{day})$

$= 1440 \text{ days (or about 4 years!)}$

In a similar way,

$t_{90} = \frac{(T_{90}H^2)}{c_v} = 0.85 \times (20\text{ft})^2 / (0.05\text{ft}^2/\text{day})$

$= 6800 \text{ days (or about 18.6 years!)}$

E. Summary:

- From this second example, we see that the times over which consolidation actually occurs can be over many years and even decades.

- The important factors that determine how long it will take for consolidation settlements to occur are:

  1) **the consolidation coefficient** $c_v$ which is proportional to a soil’s permeability $k$. The smaller $c_v$ (and the smaller $k$) the longer it will take for a soil to consolidate.

  2) **the maximum drainage distance** $H$. From the above examples, note that consolidation times are proportional to $H^2$. 