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Stochastic sensitivity analysis by dimensional decomposition and score functions

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ABSTRACT

This article presents a new class of computational methods, known as dimensional decomposition methods, for calculating stochastic sensitivities of mechanical systems with respect to probability distribution parameters. These methods involve a hierarchical decomposition of a multivariate response function in terms of variables with increasing dimensions and score functions associated with probability distribution of a random input. The proposed decomposition facilitates univariate and bivariate approximations of stochastic sensitivity measures, lower-dimensional numerical integrations or Lagrange interpolations, and Monte Carlo simulation. Both the probabilistic response and its sensitivities can be estimated from a single stochastic analysis, without requiring performance function gradients. Numerical results indicate that the decomposition methods developed provide accurate and computationally efficient estimates of sensitivities of statistical moments or reliability, including stochastic design of mechanical systems. Future effort includes extending these decomposition methods to account for the performance function parameters in sensitivity analysis.

1. Introduction

Sensitivity analysis provides an important insight about complex model behavior [1,2] so that one can make informed decisions on minimizing the variability of a system [3], or optimizing a system's performance with an acceptable risk [4]. For estimating the derivative or sensitivity[1] of a general probabilistic response, there are three principal classes of methods or analyses. The *finite-difference method* [5] involves repeated stochastic analyses for nominal and perturbed values of system parameters, and then invoking forward, central, or other differentiation schemes to approximate their partial derivatives. This method is cumbersome and often expensive, if not prohibitive, because evaluating probabilistic response for each system parameter, which constitutes a complete stochastic analysis, is already a computationally demanding task. The two remaining methods, the *infinitesimal perturbation analysis* [6,7] and the *score function method* [8], have been mostly viewed as competing methods, where both performance and sensitivities can be obtained from a single stochastic simulation. However, there are additional requirements of regularity conditions, in particular smoothness of the performance function or the probability measure [9]. For the infinitesimal perturbation analysis, the probability measure is fixed, and the derivative of a performance function is taken, assuming that the differential and integral operators are interchangeable. The score function method, which involves probability measure that continuously varies with respect to a design parameter, also requires a somewhat similar interchange of differentiation and integration, but in many practical examples, interchange in the score function method holds in a much wider range than that in infinitesimal perturbation analysis. Nonetheless, both methods, when valid, are typically employed in conjunction with the *direct* Monte Carlo simulation, a premise well-suited to stochastic optimization of discrete event systems. Unfortunately, in mechanical design optimization, where stochastic response and sensitivity analyses are required at each design iteration, even a single Monte Carlo simulation is impractical, as each deterministic trial of the simulation may require expensive finite-element or other numerical calculations. This is the principal reason why neither the infinitesimal perturbation analysis nor the score function method have found their way in to the design optimization of mechanical systems.

The direct differentiation method, commonly used in deterministic sensitivity analysis [10], provides an attractive alternative to the finite-difference method for calculating stochastic sensitivities. In conjunction with the first-order reliability method, Liu and Der Kiureghian [11] and their similar work has significantly contributed to the development of such methods for obtaining reliability sensitivities. The direct differentiation method, also capable of generating both reliability and its sensitivities from a single stochastic analysis, is particularly effective in solving finite-element-based reliability problems, when (1) the most probable point can be efficiently located and (2) a linear approximation of the performance function at that point is adequate. Therefore, the

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[1] The nouns "derivative" and "sensitivity" are used synonymously in this paper.

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direct differentiation method inherits high efficiency of the first-order reliability method, but also its limitations. In contrast, the three sensitivity methods described in the preceding are independent of underlying stochastic analysis.

This article presents a new class of computational methods, known as dimensional decomposition methods, for calculating stochastic sensitivities of mechanical systems with respect to probability distribution parameters. The idea of dimensional decomposition of a multivariate function, originally developed by the author's group for statistical moment [12,13] and reliability [14] analyses, has been extended to stochastic sensitivity analysis, which is the focus of the current paper. Section 2 describes a unified probabilistic response and sensitivity, and derives score functions associated with a number of probability distributions. Section 3 presents the dimensional decomposition method for calculating the probabilistic sensitivities, using either the numerical integration or the simulation method and score functions. The computational effort required by the decomposition method is also discussed. Four numerical examples illustrate the accuracy, computational efficiency, and usefulness of the sensitivity method in Section 4. Section 5 states the limitations of the proposed method. Finally, conclusions are drawn in Section 6.

2. Probabilistic response and sensitivity

Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω is a sample space and $\mathcal{F}$ is a σ-field on Ω. Defined over $(\Omega, \mathcal{F})$, consider a family $\{P_{\boldsymbol{\theta}} : \mathcal{F} \to [0, 1]\}$ of probability measures, where $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_M\}^T \in \mathbb{R}^M$ is an M-dimensional vector of deterministic parameters and $\mathbb{R}^M$ is an M-dimensional, real, vector space. In other words, a sample point $\omega \in \Omega$ obeys the probability law $P_{\boldsymbol{\theta}}(F)$ for any event $F \in \mathcal{F}$ and $\boldsymbol{\theta} \in \mathbb{R}^M$, so that the probability triple $(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$ depends on $\boldsymbol{\theta}$.

Let $\{\boldsymbol{X} = \{X_1, \ldots, X_N\}^T : (\Omega, \mathcal{F}) \to (\mathbb{R}^N, \mathcal{B}^N)\}$ with $\mathcal{B}^N$ as the Borel σ-field on $\mathbb{R}^N$ denote a family of $\mathbb{R}^N$-valued input random vector, which describes statistical uncertainties in loads, material properties, and geometry of a mechanical system. The probability law of $\boldsymbol{X}$ is completely defined by a family of joint probability density functions $\{f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}), \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{\theta} \in \mathbb{R}^M\}$ that are associated with probability measures $\{P_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \mathbb{R}^M\}$. Let $y(\boldsymbol{X})$, a real-valued, measurable transformation on $(\Omega, \mathcal{F})$, define a relevant performance function of a mechanical system. It is assumed that $y : (\mathbb{R}^N, \mathcal{B}^N) \to (\mathbb{R}, \mathcal{B})$ is not an explicit function of $\boldsymbol{\theta}$, although y implicitly depends on $\boldsymbol{\theta}$ via the probability law of $\boldsymbol{X}$. The objective of stochastic sensitivity analysis is to obtain the partial derivatives of a probabilistic characteristic of $y(\boldsymbol{X})$ with respect to a parameter $\theta_i, i = 1, \ldots, M$, given a reasonably arbitrary probability law of $\boldsymbol{X}$.

2.1. Statistical moments and reliability

Denote by $\mathcal{L}_q(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$ a collection of real-valued random variables including $y(\boldsymbol{X})$, which is defined on $(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$ such that $\mathbb{E}[|y^q(\boldsymbol{X})|] < \infty$, where $q \geq 1$ is an integer and $\mathbb{E}_{\boldsymbol{\theta}}$ represents the expectation operator with respect to the probability measure $\{P_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \mathbb{R}^M\}$. If $y(\boldsymbol{X})$ is in $\mathcal{L}_q(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$, then its qth moment, defined by the multifold integral

$$m_q(\boldsymbol{\theta}) := \mathbb{E}_{\boldsymbol{\theta}}\left[y^q(\boldsymbol{X})\right] := \int_{\mathbb{R}^N} y^q(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}; \quad q = 1, 2, \ldots, \quad (1)$$

exists and is finite. A similar integral appears in time-invariant reliability analysis, which entails calculating the failure probability

$$P_F(\boldsymbol{\theta}) := P_{\boldsymbol{\theta}}[\boldsymbol{X} \in \Omega_F] = \int_{\mathbb{R}^N} I_{\Omega_F}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x} := \mathbb{E}_{\boldsymbol{\theta}}\left[I_{\Omega_F}(\boldsymbol{X})\right], \quad (2)$$

where $\Omega_F := \{\boldsymbol{x} : y(\boldsymbol{x}) < 0\}$ is the failure set for component reliability analysis; and $\Omega_F := \{\boldsymbol{x} : \cup_{k=1}^K y^{(k)}(\boldsymbol{x}) < 0\}$ and $\Omega_F := \{\boldsymbol{x} : \cap_{k=1}^K y^{(k)}(\boldsymbol{x}) < 0\}$ are the failure sets for series-system and parallel-system reliability analyses, respectively, with $y^{(k)}(\boldsymbol{x})$ representing the kth out of K performance functions, and

$$I_{\Omega_F}(\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x} \in \Omega_F \\ 0, & \boldsymbol{x} \in \Omega \setminus \Omega_F \end{cases}; \quad \boldsymbol{x} \in \mathbb{R}^N \quad (3)$$

is an indicator function. Therefore, expressions of both integrals or expectations in Eqs. (1) and (2) can be consolidated into a generic probabilistic response

$$h(\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\theta}}[g(\boldsymbol{X})] := \int_{\mathbb{R}^N} g(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}, \quad (4)$$

where $h(\boldsymbol{\theta})$ and $g(\boldsymbol{x})$ are either $m_q(\boldsymbol{\theta})$ and $y^q(\boldsymbol{x})$, respectively, for statistical moment analysis or $P_F(\boldsymbol{\theta})$ and $I_{\Omega_F}(\boldsymbol{x})$, respectively, for reliability analysis.

2.2. Sensitivity analysis by score functions

Consider a distribution parameter $\theta_i, i = 1, \ldots, M$, and suppose that the derivative of a generic probabilistic response $h(\boldsymbol{\theta})$, which is either the statistical moment of a mechanical response or the reliability of a mechanical system, with respect to θ_i is sought. For such sensitivity analysis, the following assumptions are required [8].

1. The probability density function $f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})$ is continuous. Discrete distributions having jumps at a set of points, or a mixture of continuous and discrete distributions, can be treated similarly, but will not be discussed here.
2. The parameter $\theta_i \in \Theta_i \subset \mathbb{R}, i = 1, \ldots, M$, where Θ_i is an open interval on $\mathbb{R}$.
3. The partial derivative $\partial f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})/\partial\theta_i$ exists and is finite for all $\boldsymbol{x}$ and $\theta_i \in \Theta_i \subset \mathbb{R}$. In addition, $h(\boldsymbol{\theta})$ is a differentiable function of $\boldsymbol{\theta} \in \mathbb{R}^M$.
4. There exists a Lebesgue integrable dominating function $r(\boldsymbol{x})$ such that
$$\left| g(\boldsymbol{x}) \frac{\partial f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i} \right| \leq r(\boldsymbol{x}) \quad (5)$$
for all $\boldsymbol{\theta} \in \mathbb{R}^M$.

The assumptions 1–4 are known as the regularity conditions.

Taking the partial derivative of both sides of Eq. (4) with respect to θ_i gives

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} \int_{\mathbb{R}^N} g(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}. \quad (6)$$

By invoking assumption 4 and the Lebesgue dominated convergence theorem [15], the differential and integral operators can be interchanged, yielding

$$\begin{aligned} \frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} &= \int_{\mathbb{R}^N} g(\boldsymbol{x}) \frac{\partial f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i} \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} g(\boldsymbol{x}) \frac{\partial \ln f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i} f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x} \\ &= \mathbb{E}_{\boldsymbol{\theta}}\left[g(\boldsymbol{X}) \frac{\partial \ln f_{\boldsymbol{X}}(\boldsymbol{X}; \boldsymbol{\theta})}{\partial \theta_i} \right]; \quad i = 1, \ldots, M, \end{aligned} \quad (7)$$

provided $f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \neq 0$. Define

$$s^{(1)}_{\theta_i}(\boldsymbol{x}; \boldsymbol{\theta}) := \frac{\partial \ln f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial \theta_i}, \quad (8)$$

which is known as the first-order score function for the parameter θ_i [8]. Therefore, the first-order sensitivity of $h(\boldsymbol{\theta})$ can be expressed by

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \theta_i} = \mathbb{E}_{\boldsymbol{\theta}}\left[g(\boldsymbol{X}) s^{(1)}_{\theta_i}(\boldsymbol{X}; \boldsymbol{\theta}) \right]; \quad i = 1, \ldots, M. \quad (9)$$

Table 1
Log-derivatives for Gaussian and lognormal distributions

Distribution	$f_{X_i}(x_i;\mu_i,\sigma_i)$	$\frac{\partial \ln f_{X_i}(x_i;\mu_i,\sigma_i)}{\partial \mu_i}$	$\frac{\partial \ln f_{X_i}(x_i;\mu_i,\sigma_i)}{\partial \sigma_i}$
Gaussian	$\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right]$; $-\infty \le x_i \le +\infty$	$\frac{1}{\sigma_i}\left(\frac{x_i-\mu_i}{\sigma_i}\right)$	$\frac{1}{\sigma_i}\left[\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2-1\right]$
Lognormal[a]	$\frac{1}{\sqrt{2\pi}x_i\tilde{\sigma}_i}\exp\left[-\frac{1}{2}\left(\frac{\ln x_i-\tilde{\mu}_i}{\tilde{\sigma}_i}\right)^2\right]$; $0 < x_i \le +\infty$	$-\frac{1}{\tilde{\sigma}_i}\frac{\partial\tilde{\sigma}_i}{\partial\mu_i}+\frac{1}{\tilde{\sigma}_i^2}\left(\frac{\ln x_i-\tilde{\mu}_i}{\tilde{\sigma}_i}\right)\times\left[\tilde{\sigma}_i\frac{\partial\tilde{\mu}_i}{\partial\mu_i}+(\ln x_i-\tilde{\mu}_i)\frac{\partial\tilde{\sigma}_i}{\partial\mu_i}\right]$	$-\frac{1}{\tilde{\sigma}_i}\frac{\partial\tilde{\sigma}_i}{\partial\sigma_i}+\frac{1}{\tilde{\sigma}_i^2}\left(\frac{\ln x_i-\tilde{\mu}_i}{\tilde{\sigma}_i}\right)\times\left[\tilde{\sigma}_i\frac{\partial\tilde{\mu}_i}{\partial\sigma_i}+(\ln x_i-\tilde{\mu}_i)\frac{\partial\tilde{\sigma}_i}{\partial\sigma_i}\right]$

[a] $\tilde{\sigma}_i^2=\ln\left(1+\sigma_i^2/\mu_i^2\right)$ and $\tilde{\mu}_i=\ln\mu-\tilde{\sigma}_i^2/2$. The partial derivatives of $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ with respect to μ_i or σ_i can be easily obtained, so they are not reported here.

The extension to higher-order partial derivatives of $h(\boldsymbol{\theta})$ is straightforward. See Rubinstein and Shapiro [8], who pioneered the score function method.

In general, the sensitivity is not available analytically, since the response $h(\boldsymbol{\theta})$ is not either. Nonetheless, the probabilistic response $h(\boldsymbol{\theta})$ (Eq. (4)) and its sensitivities $\partial h(\boldsymbol{\theta})/\partial\theta_i$ (Eq. (9)) have been formulated as expectations of stochastic quantities with respect to the same density function, facilitating their concurrent evaluations in a single stochastic simulation or analysis. The main contribution of this work is to simultaneously evaluate both the stochastic response and sensitivities by an alternative route, known as the dimensional decomposition method, to the traditional Monte Carlo simulation.

The score function method requires differentiating only the probability density function $f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta})$. The resulting score functions can be easily determined and, in many cases, analytically. In contrast, the infinitesimal perturbation analysis in its original form requires derivatives or perturbation of the performance function, which is always expensive in stochastic-mechanics applications. Furthermore, if the performance function is not differentiable, the regularity condition that permits interchangeability of differential and integral operators is violated and the infinitesimal perturbation analysis will not work. In the score function method, $g(\boldsymbol{x})$ can be discontinuous – for example, the indicator function $I_{\Omega_F}(\boldsymbol{x})$ that comes from reliability analysis – but the method still allows evaluation of the sensitivity if the density function is differentiable. In reality, the density function is often smoother than the performance function, and therefore the regularity conditions are milder in the score function method than in the infinitesimal perturbation analysis.

2.3. Score functions

The score functions $s^{(1)}_{\theta_i}(\boldsymbol{x};\boldsymbol{\theta})$; $i=1,\ldots,M$ depend only on the probability distribution of random input $\boldsymbol{X}=\{X_1,\ldots,X_N\}^T$. When the distribution of X_i is either independent or both independent and identical, the expressions of the score functions simplify slightly. Since a major application of sensitivity analysis is design optimization, where the second-moment properties of random input play the role of design parameters, attention is confined to the score functions associated with the mean and standard deviations of input.

2.3.1. Independent distributions

Consider a random input $\boldsymbol{X}$, where the components $X_1,\ldots,X_N$ are independent random variables. Let X_i follow the probability density function $f_{X_i}(x_i;\mu_i,\sigma_i)$ for $i=1,\ldots,N$, with mean μ_i and standard deviation σ_i. The joint density of $\boldsymbol{X}$ is $f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta})=\prod_{i=1}^N f_{X_i}(x_i;\mu_i,\sigma_i)$, $\boldsymbol{\theta}=\{\mu_1,\sigma_1\ldots,\mu_N,\sigma_N\}^T$, and $M=2N$. Therefore, from Eq. (8), the first-order score functions for μ_i and σ_i become

$$s^{(1)}_{\mu_i}(\boldsymbol{x};\boldsymbol{\theta})=\frac{\partial\ln f_{X_i}(x_i;\mu_i,\sigma_i)}{\partial\mu_i};\quad i=1,\ldots,N \tag{10}$$

and

$$s^{(1)}_{\sigma_i}(\boldsymbol{x};\boldsymbol{\theta})=\frac{\partial\ln f_{X_i}(x_i;\mu_i,\sigma_i)}{\partial\sigma_i};\quad i=1,\ldots,N, \tag{11}$$

respectively.

2.3.2. Independent and identical distributions

Furthermore, consider $\boldsymbol{X}$, where the components $X_1,\ldots,X_N$ are not only independent, but also follow the common probability density function $f(x_i;\mu,\sigma)$ with the common mean μ and common standard deviation σ, so that $\boldsymbol{\theta}=\{\mu,\sigma\}^T$ and $M=2$. Such distributions frequently arise in manufacturing processes, where a sequence of events occur independently, but share the same distribution parameters. In that case, the joint density of $\boldsymbol{X}$ is $f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta})=\prod_{i=1}^N f(x_i;\mu,\sigma)$, yielding

$$s^{(1)}_{\mu}(\boldsymbol{x};\boldsymbol{\theta})=\sum_{i=1}^N\frac{\partial\ln f(x_i;\mu,\sigma)}{\partial\mu} \tag{12}$$

and

$$s^{(1)}_{\sigma}(\boldsymbol{x};\boldsymbol{\theta})=\sum_{i=1}^N\frac{\partial\ln f(x_i;\mu,\sigma)}{\partial\sigma} \tag{13}$$

as the first-order score functions. In either case, the log-derivatives of a marginal probability density function are required in determining the score functions. Table 1 presents explicit expressions of log-derivatives for Gaussian and lognormal distributions.

2.3.3. Dependent distributions

When $\boldsymbol{X}$ is dependent, the derivation of score functions is generally tedious, but not difficult. For example, when $\boldsymbol{X}$ is Gaussian with its mean $\boldsymbol{\mu}:=\mathbb{E}_{\boldsymbol{\theta}}[\boldsymbol{X}]=\{\mu_1,\ldots,\mu_N\}^T$, covariance matrix $\boldsymbol{\Sigma}:=\mathbb{E}_{\boldsymbol{\theta}}[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T]=[\rho_{ij}\sigma_i\sigma_j]$, and density $f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta})=\left[(2\pi)^{N/2}|\boldsymbol{\Sigma}|^{1/2}\right]^{-1}\exp\left[-(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})/2\right]$, where μ_i and σ_i are the mean and standard deviation, respectively, of X_i and ρ_{ij} is the correlation coefficient between X_i and X_j, the first-order score functions are

$$s^{(1)}_{\mu_i}(\boldsymbol{x};\boldsymbol{\theta})=\{0,\ldots,1,\ldots,0\}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \tag{14}$$

and

$$s^{(1)}_{\sigma_i}(\boldsymbol{x};\boldsymbol{\theta})=-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\frac{\partial\boldsymbol{\Sigma}^{-1}}{\partial\sigma_i}(\boldsymbol{x}-\boldsymbol{\mu})-\frac{1}{2}\frac{\partial\ln|\boldsymbol{\Sigma}|}{\partial\sigma_i}. \tag{15}$$

If the variables share the same mean μ, the same standard deviation σ, and the same correlation coefficient ρ between any two distinct variables, then $\boldsymbol{\mu}=\mu\mathbf{1}$, $\boldsymbol{\Sigma}=\sigma^2[1-(1-\delta_{ij})(1-\rho)]$, leading to simplified score functions

$$s^{(1)}_{\mu}(\boldsymbol{x};\boldsymbol{\theta})=\mathbf{1}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \tag{16}$$

and

$$s^{(1)}_{\sigma}(\boldsymbol{x};\boldsymbol{\theta})=\frac{1}{\sigma}\left[(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})-N\right], \tag{17}$$

where $\mathbf{1}=\{1,\ldots,1\}^T$ and δ_{ij} is the Kronecker delta. The score functions for dependent non-Gaussian distributions were not considered in this work.

3. Dimensional decomposition method

Consider a continuous, differentiable, real-valued, multivariate performance function $y(\boldsymbol{x}) := y(x_1, \ldots, x_N)$, which describes $g(\boldsymbol{x})$, a function that represents either $y^q(\boldsymbol{x})$ for statistical moment analysis or $I_{\Omega_F}(\boldsymbol{x})$ for reliability analysis. Let $\boldsymbol{c} = \{c_1, \ldots, c_N\}^T$ be a reference point of $\boldsymbol{X}$ and $y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}, c_{i_{S-k}+1}, \ldots, c_N)$ represent an $(S-k)$th dimensional component function of $y(\boldsymbol{x})$, where $1 \le S \le N$ and $k = 0, \ldots, S$. Then, an S-dimensional decomposition of $y(\boldsymbol{x})$ is [13,14]

$$\hat{y}_S(\boldsymbol{x}) := \sum_{k=0}^{S}(-1)^k \binom{N-S+k-1}{k} \times \sum_{i_1,\ldots,i_{S-k}=1;i_1<\cdots<i_{S-k}}^{N} y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}, c_{i_{S-k}+1}, \ldots, c_N). \tag{18}$$

Using a multivariate function theorem [13], it can be shown that $\hat{y}_S(\boldsymbol{x})$ in Eq. (18) consists of all terms of the Taylor series of $y(\boldsymbol{x})$ that have less than or equal to S variables. The expanded form of Eq. (18), when compared with the Taylor expansion of $y(\boldsymbol{x})$, indicates that the residual error in the S-variate approximation includes terms of dimensions $S+1$ and higher. All higher-order S- and lower-variate terms of $y(\boldsymbol{x})$ are included in Eq. (18), which should therefore generally provide a higher-order approximation of a multivariate function than equations derived from first- or second-order Taylor expansions. When $S = 1$ and 2, Eq. (18) degenerates to the univariate and bivariate approximations, respectively.

3.1. Decomposition with numerical integration

When $g(\boldsymbol{x})$ represents $y^q(\boldsymbol{x})$, it can also be decomposed by a convergent sequence of lower-variate approximations, as in Eq. (18). Therefore, for an independent random vector $\boldsymbol{X}$, an S-variate approximation of the qth moment $m_q(\boldsymbol{\theta})$ and its sensitivity $\partial m_q(\boldsymbol{\theta})/\partial\theta_i$ can be evaluated using standard numerical quadratures, leading to

$$m_q(\boldsymbol{\theta}) \cong \sum_{k=0}^{S}(-1)^k \binom{N-S+k-1}{k} \times \sum_{i_1,\ldots,i_{S-k}=1;i_1<\cdots<i_{S-k}}^{N} \sum_{j_{S-k}=1}^{n} \cdots \sum_{j_1=1}^{n} w_{i_1}^{(j_1)} \cdots w_{i_{S-k}}^{(j_{S-k})} \times y^q(c_1, \ldots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}^{(j_{S-k})}, c_{i_{S-k}+1}, \ldots, c_N) \tag{19}$$

and

$$\frac{\partial m_q(\boldsymbol{\theta})}{\partial\theta_i} \cong \sum_{k=0}^{S}(-1)^k \binom{N-S+k-1}{k} \times \sum_{i_1,\ldots,i_{S-k}=1;i_1<\cdots<i_{S-k}}^{N} \sum_{j_{S-k}=1}^{n} \cdots \sum_{j_1=1}^{n} w_{i_1}^{(j_1)} \cdots w_{i_{S-k}}^{(j_{S-k})} \times y^q(c_1, \ldots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}^{(j_{S-k})}, c_{i_{S-k}+1}, \ldots, c_N) s_{\theta_i}^{(1)}(x_{i_{t(i)}}^{(j_{t(i)})}; \boldsymbol{\theta}) \tag{20}$$

for $i = 1, \ldots, M$, where $x_{i_m}^{(j_m)}$ is the j_mth integration point of the i_mth variable, $w_{i_m}^{(j_m)}$ is the associated weight that includes the probability density, $m = 1, \ldots, S-k$, $1 \le t(i) \le N$ is an integer index such that θ_i is the distribution parameter of the $X_{i_{t(i)}}$th variable, and n is the number of integration points. The score function $s_{\theta_i}^{(1)}(x_{i_{t(i)}}; \boldsymbol{\theta})$ is associated with the parameter θ_i, which can be either the mean μ_i or the standard deviation σ_i of the random variable $X_{i_{t(i)}}$. Since $X_{i_{t(i)}}$ and $X_{i_{t(j)}}$ for any $t(i) \ne t(j)$ are independent, $s_{\theta_i}^{(1)}(x_{i_{t(i)}}; \boldsymbol{\theta})$ is a univariate function, as already demonstrated by Eq. (10) or (11). If $\boldsymbol{X}$ comprises both independent and identical random variables, the score function $s_{\theta_i}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta})$, according to Eq. (12) or (13), is a linear combination of univariate functions, and hence, effectively remains univariate. In the latter case, $s_{\theta_i}^{(1)}(x_{i_{t(i)}}; \boldsymbol{\theta})$ in Eq. (20) should be replaced by $s_{\theta_i}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta})$, which is just another univariate function. Therefore, the proposed equations, Eqs. (19) and (20), for independent and/or identical distributions entail evaluating at most S-dimensional integrals, which is substantially simpler and more efficient than performing one N-dimensional integration, particularly when $S \ll N$. Hence, the computational effort in conducting moment and its sensitivity analyses is significantly reduced using the dimensional decomposition. When $S = 1$, Eqs. (19) and (20) degenerate to the univariate approximation involving only one-dimensional integration. When $S = 2$, Eqs. (19) and (20) become the bivariate approximation entailing at most two-dimensional integration.

When $\boldsymbol{X}$ includes dependent random variables, the score function $s_{\theta_i}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta})$ is a multivariate function in general. Therefore, the formulation of the S-variate approximation of the sensitivity for dependent variables must include the S-variate decomposition of the product $y^q(\boldsymbol{x})s_{\theta_i}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta})$. Nonetheless, it is possible to generalize Eq. (20) for sensitivity analysis of a general stochastic system with dependent random variables, but with an additional layer of approximation from decomposition of the score functions.

3.2. Decomposition with simulation

The S-variate decomposition and associated numerical integration developed should not be applied to $g(\boldsymbol{x})$ when it represents $I_{\Omega_F}(\boldsymbol{x})$, the indicator function from reliability analysis. This is because $I_{\Omega_F}(\boldsymbol{x})$ is a discontinuous function, which takes on discrete values of 0 and 1 in the sample space Ω. In that case, consider the n-point Lagrange interpolation of $y(c_1, \ldots, c_{i_1-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}, c_{i_{S-k}+1}, \ldots, c_N)$ for $1 \le S \le N$ and $k = 0, \ldots, S$, yielding

$$\hat{y}_S(\boldsymbol{X}) \cong \sum_{k=0}^{S}(-1)^k \binom{N-S+k-1}{k} \times \sum_{i_1,\ldots,i_{S-k}=1;i_1<\cdots<i_{S-k}}^{N} \sum_{j_{S-k}=1}^{n} \cdots \sum_{j_1=1}^{n} \phi_{j_1}(X_{i_1}) \cdots \phi_{j_{S-k}}(X_{i_{S-k}}) \times y(c_1, \ldots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}^{(j_{S-k})}, c_{i_{S-k}+1}, \ldots, c_N), \tag{21}$$

where $x_{i_{S-k}}^{(j_{S-k})}$ is the j_{S-k}th sample of $X_{i_{S-k}}$, $\phi_{j_{S-k}}(X_{i_{S-k}})$ is the random Lagrange shape function, and $y(c_1, \ldots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1+1}, \ldots, c_{i_{S-k}-1}, x_{i_{S-k}}^{(j_{S-k})}, c_{i_{S-k}+1}, \ldots, c_N)$ is the associated deterministic coefficient. Eq. (21) provides a convergent sequence of lower-variate approximations of $y(\boldsymbol{X})$ if the Lagrange interpolations of component functions are convergent. Since $\hat{y}_S(\boldsymbol{x})$ represents an explicit approximation of $y(\boldsymbol{X})$, any probabilistic characteristic of $y(\boldsymbol{X})$, including its statistical moments and probability density function, can be easily evaluated by performing Monte Carlo simulation on Eq. (21).

Recall that $\hat{y}_S(\boldsymbol{x})$ or $\{\hat{y}_S^{(k)}(\boldsymbol{x}), k = 1, K\}$ are S-variate approximations of performance functions for component or system reliability analysis. Based on these approximations, let

$\hat{\Omega}_{F,S} := \{\boldsymbol{x} : \hat{y}_S(\boldsymbol{x}) < 0\}$, $\hat{\Omega}_{F,S} := \{\boldsymbol{x} : \cup_{k=1}^{K} \hat{y}_S^{(k)}(\boldsymbol{x}) < 0\}$, and $\hat{\Omega}_{F,S} := \{\boldsymbol{x} : \cap_{k=1}^{K} \hat{y}_S^{(k)}(\boldsymbol{x}) < 0\}$ define approximate failure sets in reliability analyses of the component, series system, and parallel system, respectively. Therefore, the Monte Carlo estimates of the failure probability $P_F(\boldsymbol{\theta})$ and its sensitivity $\partial P_F(\boldsymbol{\theta})/\partial\theta_i$, employing S-variate approximations of the failure sets, are

$$P_F(\boldsymbol{\theta}) \cong \mathbb{E}_{\boldsymbol{\theta}}\left[I_{\hat{\Omega}_{F,S}}(\boldsymbol{X})\right] = \lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L} I_{\hat{\Omega}_{F,S}}(\boldsymbol{x}^{(l)}) \tag{22}$$

and

$$\frac{\partial P_F(\boldsymbol{\theta})}{\partial\theta_i} \cong \mathbb{E}_{\boldsymbol{\theta}}\left[I_{\hat{\Omega}_{F,S}}(\boldsymbol{X})s_{\theta_i}^{(1)}(\boldsymbol{X};\boldsymbol{\theta})\right] = \lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L} I_{\hat{\Omega}_{F,S}}(\boldsymbol{x}^{(l)})s_{\theta_i}^{(1)}(\boldsymbol{x}^{(l)};\boldsymbol{\theta}), \tag{23}$$

respectively, where L is the sample size, $\boldsymbol{x}^{(l)}$ is the lth realization of $\boldsymbol{X}$, and

$$I_{\hat{\Omega}_{F,S}}(\boldsymbol{x}^{(l)}) = \begin{cases} 1, & \boldsymbol{x}^{(l)} \in \hat{\Omega}_{F,S} \\ 0, & \boldsymbol{x}^{(l)} \in \Omega \setminus \hat{\Omega}_{F,S} \end{cases} \tag{24}$$

is an approximate indicator function corresponding to S-variate decompositions of performance functions. Both independent and dependent random variables can be accounted for, provided that their realization can be generated. Setting $S = 1$ or 2 in Eqs. (22) and (23), the univariate or bivariate approximations of the failure probability and its sensitivity can be invoked.

A Monte Carlo simulation on an S-variate approximation $\hat{y}_S(\boldsymbol{X})$ also leads to its qth moment

$$m_q(\boldsymbol{\theta}) \cong \mathbb{E}_{\boldsymbol{\theta}}\left[\hat{y}_S^q(\boldsymbol{X})\right] = \lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L}\hat{y}_S^q(\boldsymbol{x}^{(l)}) \tag{25}$$

and its sensitivity

$$\frac{\partial m_q(\boldsymbol{\theta})}{\partial\theta_i} \cong \mathbb{E}_{\boldsymbol{\theta}}\left[\hat{y}_S^q(\boldsymbol{X})s_{\theta_i}^{(1)}(\boldsymbol{X};\boldsymbol{\theta})\right] = \lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L}\hat{y}_S^q(\boldsymbol{x}^{(l)})s_{\theta_i}^{(1)}(\boldsymbol{x}^{(l)};\boldsymbol{\theta}), \tag{26}$$

and therefore provides an alternative means to numerical integration.

The proposed methods involving univariate ($S = 1$) and bivariate ($S = 2$) approximations, where an n-point numerical integration yields sensitivity of moments or where an n-point Lagrange interpolation and associated Monte Carlo simulation produce sensitivity of moments or reliability, are defined as the *univariate* and *bivariate* decomposition methods in this paper.

3.3. Computational effort

The univariate and bivariate decomposition methods for calculating sensitivity of moment or reliability of a mechanical system require evaluation of deterministic coefficients: $y(\boldsymbol{c})$, $y(c_1,\ldots,c_{i-1},x_i^{(j)},c_{i+1},\ldots,c_N)$, and $y(c_1,\ldots,c_{i_1-1},x_{i_1}^{(j_1)},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2}^{(j_2)},c_{i_2+1},\ldots,c_N)$ for $i,i_1,i_2 = 1,\ldots,n$ and $j,j_1,j_2 = 1,\ldots,n$. Hence, the computational effort required by the decomposition method can be viewed as numerically calculating a mechanical response $y(\boldsymbol{x})$ at several deterministic inputs defined by either integration points (for moments) or user-selected sample points (for reliability). Therefore, the total cost for the univariate decomposition entails a *maximum* of $nN+1$ function evaluations. For the bivariate decomposition, a *maximum* of $N(N-1)n^2/2 + nN + 1$ function evaluations are required. If the integration or sample points include a common point in each coordinate x_i (see the forthcoming example section), the numbers of function evaluations reduce slightly.

4. Numerical examples

Four numerical examples involving mathematical functions and solid-mechanics/structural problems are presented to illustrate the proposed decomposition methods, for obtaining first-order sensitivity of the moment or reliability. Whenever possible, a forward finite-difference method, with one percent perturbation and the direct Monte Carlo simulation, was employed to evaluate the accuracy and computational efficiency of the decomposition methods. The sample sizes for the direct Monte Carlo simulation and the embedded Monte Carlo simulation of the decomposition method vary from 10^5 to 10^8, depending on the examples, but they are identical for a specific problem. The score functions associated with the probability distributions employed in these examples are provided in Section 2.3.

The dimensional decomposition was formulated in the original space ($\boldsymbol{x}$ space) of the random input. The reference point $\boldsymbol{c}$ associated with the decomposition methods was fixed at the mean input. A five-point Gauss–Hermite integration and three- to five-point Lagrange interpolation schemes were selected. For an n-point Lagrange interpolation, sample points $(c_1,\ldots,c_{i-1},x_i^{(j)},c_{i+1},\ldots,c_N)$ and $(c_1,\ldots,c_{i_1-1},x_{i_1}^{(j_1)},c_{i_1+1},\ldots,c_{i_2-1},x_{i_2}^{(j_2)},c_{i_2+1},\ldots,c_N)$ for univariate and bivariate decompositions, respectively, were uniformly distributed, where $x_k^{(l)} = c_k - (n-1)/2, c_k-(n-3)/2,\ldots,c_k,\ldots,c_k+(n-3)/2, c_k+(n-1)/2$, $k = i,i_1,i_2$, and $l = j,j_1,j_2$. For response or sensitivity analysis, $(n-1)N+1$ and $N(N-1)(n-1)^2/2+(n-1)N+1$ function evaluations are involved in the univariate and bivariate methods, respectively.

4.1. Example 1: Moments

The first example involves sensitivity analysis of moments of a quadratic performance function

$$y(\boldsymbol{X}) = X_1^2 + X_2^2 + X_3^2 + X_1X_2 + 2X_2X_3 + 4X_3X_1, \tag{27}$$

where $X_i \sim N(\mu,\sigma^2); i = 1,2,3$ are three independent and identically distributed Gaussian random variables with the common mean μ and common standard deviation σ and $\boldsymbol{\theta} = \{\mu,\sigma\}^T$. All finite-order moments of $y(\boldsymbol{X})$ can be obtained exactly, of which, the first three moments are: $m_1(\mu,\sigma) := \mathbb{E}_{\boldsymbol{\theta}}[y(\boldsymbol{X})] = 3\sigma^2+10\mu^2$, $m_2(\mu,\sigma) := \mathbb{E}_{\boldsymbol{\theta}}\left[y^2(\boldsymbol{X})\right] = 27\sigma^4+100\mu^4+198\sigma^2\mu^2$, and $m_3(\mu,\sigma) := \mathbb{E}_{\boldsymbol{\theta}}\left[y^3(\boldsymbol{X})\right] = 405\sigma^6+4914\mu^2\sigma^4+5040\mu^4\sigma^2+1000\mu^6$. Therefore, sensitivities of these moments with respect to μ or σ can be obtained exactly to evaluate the accuracy of the decomposition methods. When solving this problem by the decomposition methods, both numerical integration (Eqs. (19) and (20)) and simulation (Eqs. (25) and (26)) approaches were employed. A five-point Gauss–Hermite integration and a three-point Lagrange interpolation schemes were selected. The sample size $L = 10^6$.

Table 2 lists the first three moments $m_q := \mathbb{E}_{\boldsymbol{\theta}}[y^q(\boldsymbol{X})]$ and their first-order sensitivities $\partial m_q/\partial\mu$ and $\partial m_q/\partial\sigma$ for $q = 1,2,3$, when (1) $\mu = 1;\sigma = 0.3$ and (2) $\mu = 1;\sigma = 0.6$, representing moderate and large input uncertainties, respectively. The tabulated results came from the decomposition methods with both approaches and the exact solution. The agreement between the results of the decomposition methods using numerical integration or simulation and the exact solution is excellent. The results improve when switching from the univariate to the bivariate method, as expected. Regardless of the method selected, the magnitudes of moments and sensitivities for $\sigma = 0.6$, compared with those for $\sigma = 0.3$, increase, as they should for a larger standard deviation; but more importantly, a comparison

Table 2
Moments of a quadratic function and sensitivities

	Decomposition with integration		Decomposition with simulation		
	Univariate	Bivariate	Univariate	Bivariate	Exact
(1) Moderate input uncertainty ($\mu = 1, \sigma = 0.3$)					
m_1	10.27	10.27	10.27	10.27	10.27
m_2	117.89	118.11	117.94	118.1	118.04
m_3	1465.88	1495.94	1473.67	1495.18	1493.7
$\partial m_1/\partial\mu$	20	20	19.97	19.96	20
$\partial m_2/\partial\mu$	410.8	435.64	417.17	434.64	435.64
$\partial m_3/\partial\mu$	6595.89	7881.18	7286.03	7877.12	7894.01
$\partial m_1/\partial\sigma$	1.8	1.8	1.78	1.77	1.8
$\partial m_2/\partial\sigma$	119.77	122.69	120.41	122.42	121.72
$\partial m_3/\partial\sigma$	3187.95	3590.83	3299.99	3589.11	3560.62
(2) Large input uncertainty ($\mu = 1, \sigma = 0.6$)					
m_1	11.08	11.08	11.08	11.08	11.08
m_2	172.45	175.95	173.17	175.8	174.78
m_3	3012.46	3508.5	3143.39	3507.26	3470.15
$\partial m_1/\partial\mu$	20	20	19.96	19.96	20
$\partial m_2/\partial\mu$	443.2	542.56	471.05	541.0	542.56
$\partial m_3/\partial\mu$	8471.04	14326.02	11385.77	14548.95	14531.31
$\partial m_1/\partial\sigma$	3.6	3.6	3.61	3.59	3.6
$\partial m_2/\partial\sigma$	245.38	268.7	250.93	268.42	260.93
$\partial m_3/\partial\sigma$	7375.36	10749.53	8291.72	10795.5	10482.65

with the exact solution demonstrates that the decomposition methods, in particular the bivariate method, can indeed account for large input uncertainties, a desirable property of any approximate solution.

4.2. Example 2: Reliability

Consider two reliability problems: (1) a component reliability problem with the performance function

$$y(\boldsymbol{X}) = \frac{1}{1000 + \sum_{i=1}^{100} X_i} - \frac{1}{1000 + 3\sqrt{100}}, \tag{28}$$

where $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a 100-dimensional, Gaussian random vector with mean vector $\boldsymbol{\mu} = \mu\{1, \ldots, 1\}^T$ and covariance matrix $\boldsymbol{\Sigma} = \sigma^2[1 - (1 - \delta_{ij})(1 - \rho)]$, and $\boldsymbol{\theta} = \{\mu, \sigma\}^T$; and (2) a system reliability problem with three performance functions:

$$\begin{aligned} y^{(1)}(\boldsymbol{X}) &= X_2 + 2X_3 + X_4 - 1.15, \\ y^{(2)}(\boldsymbol{X}) &= X_1 + X_2 + X_4 + X_5 - 2.4, \text{ and} \\ y^{(3)}(\boldsymbol{X}) &= X_1 + 2X_3 + 2X_4 + X_5 - 3.55, \end{aligned} \tag{29}$$

where X_i is an independent, lognormal random variable with mean μ_i and standard deviation σ_i, and $\boldsymbol{\theta} = \{\mu_1, \sigma_1 \ldots, \mu_5, \sigma_5\}^T$. The performance functions in Eq. (29) represent three rigid-plastic failure mechanisms of a well-studied portal frame [16]. The objective of this example is to evaluate the accuracy of the decomposition method in calculating the component failure probability $P_{F,1} := P_{\boldsymbol{\theta}}[y(\boldsymbol{X}) < 0]$ and the series-system failure probability $P_{F,2} := P_{\boldsymbol{\theta}}[\boldsymbol{X} \in \Omega_F]$; $\Omega_F := \{\boldsymbol{x} : \cup_{k=1}^{3} y^{(k)}(\boldsymbol{x}) < 0\}$, and their respective sensitivities. The statistical properties of input are: (1) $\mu = 0, \sigma = 1$, and two distinct cases of $\rho = 0$ and $\rho = 0.5$ for the component reliability problem; and (2) $\mu_i = 1$ and $\sigma_i = 0.25$; $i = 1, \ldots, 5$ for the system reliability problem.

Since y for the component reliability problem is a multivariate function, the univariate or bivariate decomposition for a finite value of n, regardless how large, provides only an approximation. Nonetheless, using only $n = 3$ and $L = 10^6$, the univariate and bivariate estimates of $P_{F,1}$, $\partial P_{F,1}/\partial\mu$, and $\partial P_{F,1}/\partial\sigma$ are listed in Table 3 for both (1) uncorrelated ($\rho = 0$) and (2) correlated ($\rho = 0.5$) input. The exact solutions are

Table 3
Component failure probability and sensitivities for $\mu = 0, \sigma = 1$

	Univariate	Bivariate	Exact
(1) Uncorrelated input ($\rho = 0$)			
$P_{F,1}$	1.72×10^{-3}	1.34×10^{-3}	1.35×10^{-3}
$\partial P_{F,1}/\partial\mu$	5.52×10^{-2}	4.41×10^{-2}	4.43×10^{-2}
$\partial P_{F,1}/\partial\sigma$	1.55×10^{-2}	1.29×10^{-2}	1.33×10^{-2}
(2) Correlated input ($\rho = 0.5$)			
$P_{F,1}$	0.3414	0.3369	0.3365
$\partial P_{F,1}/\partial\mu$	0.5174	0.5148	0.5135
$\partial P_{F,1}/\partial\sigma$	0.1440	0.1482	0.1541

$P_{F,1} = \Phi(-\beta)$, $\partial P_{F,1}/\partial\mu = \phi(-\beta)\sqrt{N}/\sigma\sqrt{1 + (N-1)\rho}$, and $\partial P_{F,1}/\partial\sigma = \phi(-\beta)\left(3 - \sqrt{N}\mu\right)/\sigma^2\sqrt{1 + (N-1)\rho}$, where $\beta = \left(3 - \sqrt{N}\mu\right)/\sigma\sqrt{1 + (N-1)\rho}$, $\phi(u) = \left(1/\sqrt{2\pi}\right)\exp\left(-u^2/2\right)$, and $\Phi(u) = \int_{-\infty}^{u}\phi(\xi)\mathrm{d}\xi$. Compared with the exact results, also listed in Table 3, both versions of the decomposition method are satisfactory for solving reliability problems involving both independent and dependent Gaussian variables. However, the bivariate approximation provides a highly accurate solution for this high-dimensional, nonlinear problem.

Table 4 lists the system failure probability $P_{F,2}$, which is calculated by the proposed univariate decomposition method involving at most 11 ($n = 3$, $N = 5$) evaluations of each performance function and the direct Monte Carlo simulation involving 10^8 samples. Both methods provide an identical result: $P_{F,2} = 5.54 \times 10^{-5}$. This is possible, since each performance function in Eq. (29) is a univariate function, and is exactly represented by a three-point Lagrange interpolation, leading to the univariate approximation $\hat{\Omega}_{F,1}$ of the failure set that is the same as the exact failure set Ω_F. Therefore, there is no need to pursue the bivariate approximation. It is worth noting that a value of n as large as nine was required to produce a satisfactory univariate approximation of the failure set in the Gaussian space ($\boldsymbol{u}$ space), where the transformed performance functions become highly nonlinear [14]. Therefore, a higher-order interpolation can be avoided by decomposition in the original space in this problem.

The advantage of the univariate decomposition method in the $\boldsymbol{x}$ space extends to sensitivity analysis, as essentially the

Fig. 1. A through-wall-cracked cylinder under four-point bending; (a) geometry and loads; (b) cracked cross-section; (c) finite-element discretization.

Table 4
System failure probability and sensitivities for $\mu_i = 1, \sigma_i = 0.25$

	Univariate	Direct MCS/FD[a]
$P_{F,2}$	5.54×10^{-5}	5.54×10^{-5}
$\partial P_{F,2}/\partial \mu_1$	-5.9×10^{-4}	-5.94×10^{-4}
$\partial P_{F,2}/\partial \mu_2$	-3.23×10^{-4}	-3.18×10^{-4}
$\partial P_{F,2}/\partial \mu_3$	-5.14×10^{-4}	-4.68×10^{-4}
$\partial P_{F,2}/\partial \mu_4$	-8.32×10^{-4}	-7.82×10^{-4}
$\partial P_{F,2}/\partial \mu_5$	-5.91×10^{-4}	-5.92×10^{-4}
$\partial P_{F,2}/\partial \sigma_1$	7.34×10^{-4}	7.44×10^{-4}
$\partial P_{F,2}/\partial \sigma_2$	4.39×10^{-4}	4.48×10^{-4}
$\partial P_{F,2}/\partial \sigma_3$	7.98×10^{-4}	8.2×10^{-4}
$\partial P_{F,2}/\partial \sigma_4$	1.22×10^{-3}	1.31×10^{-3}
$\partial P_{F,2}/\partial \sigma_5$	7.27×10^{-4}	7.4×10^{-4}

[a] MCS/FD = Monte Carlo simulation/finite-difference.

same effort delivers the sensitivities $\partial P_{F,2}/\partial \mu_i$ and $\partial P_{F,2}/\partial \sigma_i$ for $i = 1, \ldots, 5$, which are also presented in Table 4. Alternative sensitivity estimates from the finite-difference method involving 10^8 samples for each direct simulation run were also developed, and can be found in the last column of Table 4. The agreement between the results of the univariate method and the finite-difference method is very good. It is worth noting that the finite-difference method typically gives biased sensitivity estimates, where slight fluctuations in the results are expected due to a finite variance of the estimator.

4.3. Example 3: Nonlinear fracture reliability

Consider a circumferential, through-wall-cracked (TWC), nonlinearly elastic cylinder, which is subjected to a four-point bending, as shown in Fig. 1(a). The cylinder has a mid-thickness radius $R = 50.8$ mm, a wall thickness $t = 5.08$ mm, and a symmetrically centered through-wall crack with the normalized crack angle $\theta/\pi = 1/8$. The outer span $L_o = 1.5$ m and the inner span $L_i = 0.6$ m. The cross-sectional geometry at the cracked section is shown in Fig. 1(b). The cylinder is composed of an ASTM Type 304 stainless steel, which follows the Ramberg–Osgood constitutive equation [17]

$$\epsilon_{ij} = \frac{1+\nu}{E} S_{ij} + \frac{1-2\nu}{3E}\sigma_{kk}\delta_{ij} + \frac{3}{2E}\alpha\left(\frac{\sigma_e}{\sigma_0}\right)^{m-1} S_{ij}, \tag{30}$$

where σ_{ij} and ϵ_{ij} are stress and strain components, respectively, E is the Young's modulus, ν is the Poisson's ratio, σ_0 is a reference stress, α is a dimensionless material coefficient, m is a strain hardening exponent, δ_{ij} is the Kronecker delta, $S_{ij} := \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ is the deviatoric stress, and $\sigma_e := \sqrt{3S_{ij}S_{ij}/2}$ is the von Mises equivalent stress. Table 5 lists the means, standard deviations, and probability distributions of tensile parameters (E, α, m), four-point bending load (F), and fracture toughness (J_{Ic}). All random variables are statistically independent, and $\boldsymbol{\theta}$ comprises all ten second-moment statistics of the random input. Also, $\sigma_0 = 154.78$ MPa