A parametric study on probabilistic fracture of functionally graded composites by a concurrent multiscale method

Arindam Chakraborty, Sharif Rahman *

College of Engineering, The University of Iowa, Iowa City, IA 52242, United States

A R T I C L E   I N F O

Article history:
Received 31 July 2008
Received in revised form 26 December 2008
Accepted 9 January 2009
Available online 21 January 2009

Keywords:
Level-cut
Filtered Poisson field
Random microstructure
Porosity
Stress intensity factors
Fracture toughness
Monte Carlosimulation
Fracture reliability
Crack propagation

A B S T R A C T

This article reports the results of a parametric study on the fracture behavior of a crack in functionally graded materials. The study involves stochastic descriptions of particle and void numbers; location, size, and orientation characteristics; and constituent elastic properties; a concurrent multiscale model for calculating crack-driving forces; and Monte Carlo simulation for fracture reliability analysis. A level-cut, inhomogeneous, filtered Poisson field describes the statistically inhomogeneous microstructures of graded composites. Numerical results for an edge-cracked, graded specimen show that the particle shape and orientation for the same phase volume fractions have negligible effects on fracture reliability, even for graded materials with a high modulus ratio. However, voids and the particle gradation parameter, if they exist or increase, can significantly raise the probability of fracture initiation. Limited crack-propagation simulations in graded composites containing brittle particles reveal that the fracture toughness of the matrix material can significantly influence the likelihood or the extent of crack growth.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

Assessing mechanical reliability of functionally graded materials (FGMs), which possess spatially varying material compositions and microstructures, mandates a fundamental understanding of their deformation and fracture behavior [1,2]. Most existing studies on FGM fracture [3–6] entail calculating crack-driving forces employing smoothly varying material properties that are derived from empirical rules of mixtures, classical bounds, or micromechanical homogenization [7,8]. However, an FGM is a multiphase, heterogeneous material with possibly distinct properties of individual phases. Depending on the crack-tip location and FGM microstructure, the resulting crack-driving forces can be markedly different when a significant mismatch exists in the properties of constituent material phases. Therefore, using homogenized properties in a macroscale analysis may lead to inaccurate or inadequate measures of crack-driving forces and fracture behavior of FGMs. The calculation of crack-driving forces becomes further complicated when accounting for a random microstructure, including spatial and random distributions of sizes, shapes, and orientations of constituent phases [9–12]. Therefore, in general, a stochastic fracture analysis incorporating random microstructural details, particularly in the crack-tip region, is required for high-fidelity reliability analysis [13].

A valuable insight can be gained by investigating how micromechanical parameters, such as the phase volume fraction, location, size, shape, and orientation of particles; porosity; and the fracture toughness properties of constituents, influence the fracture behavior of a particle-matrix FGM. An elaborate computational model, e.g., a microscale model that employs a discrete particle-matrix system in the entire domain of an FGM, can be invoked for such a parametric study. However, a microscale model, although capable of furnishing highly accurate solutions, constitutes a brute-force approach, and is therefore computationally expensive, if not prohibitive. An attractive alternative is multiscale analysis, where effective material properties are employed whenever possible, thereby solving a fracture problem of interest not only accurately, but also economically. For example, a concurrent multiscale model [13], recently developed by the authors, involves stochastic description of an FGM microstructure and constituent material properties, a two-scale algorithm including microscale and macroscale analyses for determining crack-driving forces, and the Monte Carlo simulation for fracture reliability analysis. Numerical results indicate that the concurrent multiscale model is sufficiently accurate, gives fracture probability solutions very close to those generated from the microscale model, and can reduce the computational effort of the latter model by more than a factor of...
two. Therefore, a detailed parametric study can be efficiently conducted using the concurrent multiscale model – the principal focus of this work. In general, the crack-driving forces experienced by an FGM can be very complex in practical scenarios involving a variety of combinations of microstructural parameters and constituent material properties. A clear understanding of the relationship between the microstructure and fracture behavior is vital to the successful application of FGM to the design of mechanical and structural components.

This paper presents the results of a parametric study on fracture behavior of two-dimensional, functionally graded composites. The study involves: (1) stochastic descriptions of particle and void numbers; location, size, and orientation characteristics; and constituent elastic properties; (2) a concurrent multiscale model for determining crack-driving forces under mixed-mode loading; and (3) Monte Carlo simulation for uncertainty propagation and fracture reliability analysis. Section 2 describes a generic fracture problem and a concurrent multiscale model for calculating various fracture response characteristics of interest, defines the random input parameters, and discusses crack-driving forces and fracture reliability. Section 3 describes the Monte Carlo simulation method for calculating statistical moments and probability densities of crack-driving forces, leading to the probability of fracture initiation. A numerical example comprising eight cases of FGM microstructure and three cases of fracture toughness of matrix, and the resultant fracture response, is presented in Section 4. Section 5 provides conclusions from this work and discusses future work.

2. Stochastic fracture mechanics

Consider a three-phase, functionally graded, heterogeneous solid with a rectilinear crack, domain \( D \subset \mathbb{R}^2 \), and a schematic illustration of its microstructure, as shown in Fig. 1. The microstructure in general includes three distinct material phases: one phase as particles, another phase as the matrix material, and the remaining phase as voids. The particle, matrix, and void subdomains are represented by \( D_p \), \( D_m \), and \( D_v \), respectively, where \( D_p \cup D_m \cup D_v = D \) and \( D_p \cap D_m = D_m \cap D_v = D_v \cap D_i = \emptyset \). A three-phase FGM, henceforth described as a two-phase, porous FGM or simply a porous FGM, can be reduced to a two-phase, non-porous FGM by discarding the void constituent. The particle and matrix represent isotropic and linear-elastic materials, and the elasticity tensors of individual phases, denoted by \( C^{(i)} \), are expressed as [14]

\[
C^{(i)} := \frac{v_i E_i}{(1 + v_i)(1 - 2v_i)} \mathbf{1} \oplus \frac{E_i}{1 + v_i} I; \quad i = p, m,
\]

where the symbol \( \oplus \) denotes the tensor product; \( E_i \) and \( v_i \) are the elastic modulus and Poisson’s ratio, respectively, of phase \( i \); and \( \mathbf{1} \) and \( I \) are second- and fourth-rank identity tensors, respectively. The superscripts or subscripts \( i = p \) and \( i = m \) refer to particle and matrix, respectively. At a spatial point \( x \in D \) in the macroscopic length scale, let \( \phi_p(x), \phi_m(x) \), and \( \phi_v(x) \) denote the volume fractions of particle, matrix, and void, respectively. Each volume fraction is bounded between 0 and 1 and satisfies the constraint: \( \phi_p(x) + \phi_m(x) + \phi_v(x) = 1 \). The crack faces are traction-free, and there is perfect bonding between the material phases.

Consider a linear-elastic solid with small displacements and strains. The equilibrium equation and boundary conditions for the quasi-static problem are

\[
\mathbf{V} \cdot \sigma + \mathbf{b} = 0 \quad \text{in} \; D_p \cup D_m \text{ or } D \setminus D_v \quad \text{and}
\]

\[
\sigma \cdot n = -\mathbf{t} \quad \text{on} \; \Gamma_1 \quad \text{[natural boundary conditions]}
\]

\[
u \cdot \mathbf{u} = \mathbf{u} \quad \text{on} \; \Gamma_0 \quad \text{[essential boundary conditions]},
\]

respectively, where \( \mathbf{u} : D \rightarrow \mathbb{R}^2 \) is the displacement vector; \( \sigma = C(x) \cdot \epsilon \) is the Cauchy stress tensor with \( C(x) \) and \( \epsilon := (1/2) (\nabla + \nabla^T) \mathbf{u} \) denoting the spatially variant elasticity tensor and strain tensor, respectively; \( \mathbf{n} \) is a unit outward normal to the boundary \( \Gamma \) of the solid; \( \Gamma_1 \) and \( \Gamma_0 \) are two disjoint portions of the boundary \( \Gamma \), where the traction vector \( \mathbf{t} \) and displacement \( \mathbf{u} \) are prescribed; \( \nabla^T := \{\partial/\partial x_1, \partial/\partial x_2\} \) is the vector of gradient operators; and symbols "+" and "-' denote dot product and tensor contraction, respectively.

The variational or weak form of Eqs. (2) and (3) is

\[
\int_D (C(x) : \epsilon) \cdot \delta \mathbf{u} \, \mathrm{d}x - \int_D \mathbf{b} \cdot \delta \mathbf{u} \, \mathrm{d}x - \int_{\Gamma_1} \mathbf{t} \cdot \delta \mathbf{u} \, \mathrm{d} \Gamma
\]

\[
= \sum_{x_k \in \Gamma_0} f(x_k) \cdot \delta \mathbf{u}(x_k) - \sum_{x_k \in \Gamma_0} \delta f(x_k) \cdot [\mathbf{u}(x_k) - \mathbf{u}(x_k)] = 0,
\]

where \( f(x_k) \) is the vector of reaction forces at a constrained node \( K \) on \( \Gamma_0 \), and \( \delta \) denotes the variation operator. The discretization of the weak form, Eq. (4), depends on how the elasticity tensor \( C(x) \) is defined, i.e., how the elastic properties of constituent material phases and their gradation characteristics are described. In the following section, a concurrent multiscale model is described to approximate \( C(x) \). Nonetheless, a numerical method, e.g., the finite-element method (FEM), is generally required to solve the discretized weak form, providing various response fields of interest.

2.1. Concurrent multiscale model

The FGM microstructure in Fig. 1 contains discontinuities in material properties at the interfaces between the matrix and particles. There exist two approaches with respect to defining the material property for fracture analysis of an FGM cracked structure. One approach involves stress analysis using effective material properties, often smooth and continuous, in the entire domain of the solid. This approach is referred to as macroscale analysis. The other approach, referred to as microscale analysis, entails stress analysis that is solely based on the exact but discrete material property information derived from the knowledge of explicit particle locations and their geometry. The concurrent multiscale model employed in this work includes both continuous and discrete material representations and requires a combined micromechanical and macromechanical stress analysis.
As depicted in Fig. 2, consider an arbitrary bounded subdomain \( \mathcal{D} \subseteq \mathcal{D} \), which surrounds the crack tip and contains finite numbers of particles and voids. Employing effective elastic properties at the crack-tip region may yield inadequate estimates of the resultant stress intensity factors (SIFs), particularly if there exists a significant mismatch between the matrix and particle properties. However, far from the crack tip, where the effect of crack-tip singularity vanishes rapidly, individual constituent properties may not be needed, and an appropriately derived effective material property should suffice. In other words, the subdomain \( \mathcal{D} \) is filled with individual particles and voids, whereas the remaining subdomain \( \mathcal{D} \setminus \mathcal{D} \) is assigned a continuously varying effective elasticity tensor \( \mathbf{C}(\mathbf{x}) \), derived from a suitable micromechanical homogenization. Using classical micromechanics, e.g., the self-consistent model, the Mori–Tanaka model, and the mean-field theory [7,8], \( \mathbf{C}(\mathbf{x}) \) can be easily calculated from the known phase volume fractions at \( \mathbf{x} \in \mathcal{D} \setminus \mathcal{D} \). Therefore, according to the concurrent model, Eq. (4) is discretized and solved using [13]

\[
\mathbf{C}(\mathbf{x}) \equiv \begin{cases} \mathbf{C}^{(p)}, & \text{if } \mathbf{x} \in \mathcal{D} \setminus \mathcal{D} \text{ and } \mathbf{x} \in \mathcal{D}_p \\
\mathbf{C}^{(m)}, & \text{if } \mathbf{x} \in \mathcal{D} \setminus \mathcal{D} \text{ and } \mathbf{x} \in \mathcal{D}_m \\
\mathbf{C}(\mathbf{x}), & \text{if } \mathbf{x} \in \mathcal{D} \setminus \mathcal{D} 
\end{cases}
\]

(5)

where discontinuity in material properties exists at the interfaces between \( \mathcal{D} \) and \( \mathcal{D} \setminus \mathcal{D} \) and between \( \mathcal{D}_p \) and \( \mathcal{D}_m \). Since the material representation in \( \mathcal{D} \) is discrete, the calculation of the resulting SIFs in the concurrent model is not straightforward. The interaction integral method, commonly employed for fracture analysis of homogeneous or smoothly inhomogeneous media [3], requires a constant or continuously varying material property inside the domain of a crack-tip contour. However, if a small, bounded, non-porous subdomain \( \mathcal{D}_c \subseteq \mathcal{D} \subseteq \mathcal{D} \) surrounding the crack tip is introduced in slightly modifying the elasticity tensor, for instance [13],

\[
\mathbf{C}(\mathbf{x}) \equiv \begin{cases} \mathbf{C}^{(p)}, & \text{if } (\mathbf{x} \in \mathcal{D}_c \text{ and } \mathbf{x}_p \in \mathcal{D}_p) \text{ or } (\mathbf{x} \in \mathcal{D} \setminus \mathcal{D}_c \text{ and } \mathbf{x} \in \mathcal{D}_p) \\
\mathbf{C}^{(m)}, & \text{if } (\mathbf{x} \in \mathcal{D}_c \text{ and } \mathbf{x}_m \in \mathcal{D}_m) \text{ or } (\mathbf{x} \in \mathcal{D} \setminus \mathcal{D}_c \text{ and } \mathbf{x} \in \mathcal{D}_m) \\
\mathbf{C}(\mathbf{x}), & \text{if } \mathbf{x} \in \mathcal{D} \setminus \mathcal{D}_c 
\end{cases}
\]

(6)

the result is a material representation that is locally homogeneous, i.e., \( \mathbf{C}(\mathbf{x}) \) is either \( \mathbf{C}^{(p)} \) or \( \mathbf{C}^{(m)} \), but constant for \( \mathbf{x} \in \mathcal{D}_c \). Therefore, mixed-mode SIFs can be readily calculated from the interaction integrals applicable to homogeneous materials, provided the crack-tip contour lies inside \( \mathcal{D}_c \). Applying Eq. (6) in solving the discretized weak form yields the interaction integrals \( \mathcal{M}^{(1,2)}_p \) and \( \mathcal{M}^{(1,2)}_m \), where \( \mathbf{C}(\mathbf{x}) = \mathbf{C}^{(p)}; \mathbf{x} \in \mathcal{D}_c \) and \( \mathbf{C}(\mathbf{x}) = \mathbf{C}^{(m)}; \mathbf{x} \in \mathcal{D}_c \), respectively. The SIFs are subsequently calculated from

\[
\mathbf{K}_i \approx \begin{cases} \frac{1}{2} \mathcal{M}^{(1,2)}_p \mathbf{E}^*_p, & \text{if } \mathbf{x}_{tip} \in \mathcal{D}_p \\
\frac{1}{2} \mathcal{M}^{(1,2)}_m \mathbf{E}^*_m, & \text{if } \mathbf{x}_{tip} \in \mathcal{D}_m. 
\end{cases}
\]

(7)

where \( \mathbf{E}^*_p = \mathbf{E}_p \) for the plane stress, \( \mathbf{E}^*_j = \mathbf{E}_j / (1 - v_j^2) \) for the plane strain conditions, and \( j = p, m \).

In calculating SIFs by the concurrent multiscale model, the size of \( \mathcal{D}_c \) should be comparable to the microstructural length scale, but large enough for the SIFs to be calculated accurately. The effects of discrete particles or voids in \( \mathcal{D} \), if they exist, should be propagated to the SIFs. The magnitude of the effect, however, depends on the size of \( \mathcal{D} \) relative to \( \mathcal{D}_c \). Further details of the concurrent model, including verification with a full microscale model, are available in the authors’ previous work [13].

2.2. Stochastic input

Uncertainties in FGM fracture analysis leading to stochastic properties of SIFs can come from a variety of sources. Two important sources are: (1) microstructural uncertainty that includes randomness in phase volume fractions, numbers of particles and voids, spatial arrangements of particles and voids, and the size, shape, and orientation properties of particles and voids, and (2) constitutive elastic material properties.

2.2.1. Random microstructure

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, where \( \Omega \) is the sample space, \( \mathcal{F} \) is the \( \sigma \)-algebra of subsets of \( \Omega \), and \( \mathbb{P} \) is the probability measure. Defined on the probability space and indexed by a spatial coordinate \( \mathbf{x} \in \mathcal{D} \subseteq \mathbb{R}^2 \), consider an inhomogeneous Poisson random field \( \mathcal{N} (\mathcal{D}') \) with an intensity measure \( \mu_{\mathcal{D}'} := \int_{\mathcal{D}'} \lambda(\mathbf{x}) d\mathbf{x} \), where \( \lambda(\mathbf{x}) \geq 0 \) is a spatially variant intensity function associated with the particle or void phase and \( \mathcal{D}' \in \mathcal{R}(\mathbb{R}^2) \) is a bounded Borel set such that points of \( \mathcal{N} \) falling in \( \mathcal{D}' \setminus \mathcal{D} \) do not contribute to particles or voids in \( \mathcal{D} \). The Poisson point field has the following properties: (1) the number \( \mathcal{N} (\mathcal{D}') \) of points in a bounded subset \( \mathcal{D}' \) has the Poisson distribution with intensity measure \( \mu_{\mathcal{D}'} \); and (2) random variables \( \mathcal{N} (\mathcal{D}_1'), \ldots, \mathcal{N} (\mathcal{D}_k') \) for any integer \( K \geq 2 \) and disjoint sets \( \mathcal{D}_1', \ldots, \mathcal{D}_k' \) are statistically independent. The Poisson field \( \mathcal{N} (\mathcal{D}') \) gives the number of points in \( \mathcal{D}' \) and is characterized by the probability

\[
P [ \mathcal{N} (\mathcal{D}') = n ] = \left( \int_{\mathcal{D}'} \lambda(\mathbf{x}) d\mathbf{x} \right)^n / n! \exp \left( - \int_{\mathcal{D}'} \lambda(\mathbf{x}) d\mathbf{x} \right); \quad n = 0, 1, 2, \ldots
\]

(8)

that \( n \) Poisson points exist in \( \mathcal{D}' \). The mean \( \mathbb{E}[\mathcal{N} (\mathcal{D}')] \) and variance \( \text{Var}[\mathcal{N} (\mathcal{D}')] \) of \( \mathcal{N} (\mathcal{D}') \) are both equal to \( \mu_{\mathcal{D}'} := \int_{\mathcal{D}'} \lambda(\mathbf{x}) d\mathbf{x} \).

Filtered Poisson field: An inhomogeneous, filtered Poisson random field can be described as [9,10]

\[
\mathbf{Z}(\mathbf{x}) := \sum_{i=1}^{N(\mathcal{D}')} \mathbf{Z}_i \mathbf{h} (\mathbf{F}(\mathbf{x} - \mathbf{G}_i)), \quad \mathcal{N}(\mathcal{D}') > 0;
\]

\[
\mathbf{x} \in \mathcal{D} \subset \mathcal{D} \subseteq \mathbb{R}^2,
\]

(9)

where \( \{\mathbf{G}_i\} \) is a collection of Poisson points, \( \mathbf{h} : \mathbb{R}^2 \to \mathbb{R} \) is a non-negative kernel function, \( \{\mathbf{Z}_i\} \) is a collection of independent
and identically distributed real-valued random variables, and \( \{ \Psi_i \} \) is a collection of independent and identically distributed rotation matrices in \( \mathbb{R}^2 \); for instance,

\[
\Psi_i = \begin{bmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{bmatrix}, \quad i = 1, \ldots, N; \quad \mathbf{x} \in \mathbb{R}^2, \tag{10}
\]

with \( \theta_i \) representing deterministic or random orientation of the \( i \)th particle or void. In Eq. (9), both \( \mathcal{D} \) and \( \mathcal{D}' \) are bounded subsets of \( \mathbb{R}^2 \) such that points of \( \mathcal{N} \) falling in the set difference \( \mathbb{R}^2 \setminus \mathcal{D}' \) do not contribute to the value of \( Z \) in \( \mathcal{D} \). Also, the subset \( \mathcal{D}' \) is bounded if, for example, \( h \) has a compact support. The filtered Poisson field \( Z(\mathbf{x}) \) can be viewed as the response of a filter with a transfer function at point \( \mathbf{x} \) that is subjected to a collection of random pulses arriving at Poisson points \( \{ \Gamma_i \} \).

**Level-cut, Filtered Poisson Field**: For an increasing function \( g : \mathbb{R} \rightarrow \mathbb{R} \), consider a real-valued translation random field

\[
Y(\mathbf{x}) := g(Z(\mathbf{x})), \tag{11}
\]

which describes a memoryless, measurable, nonlinear function of a real-valued random field \( Z(\mathbf{x}) \). The translation field \( Y(\mathbf{x}) \) is completely determined by mapping \( g \) and statistical properties of \( Z(\mathbf{x}) \). Consider a special form of

\[
g(z) = I(z; a) := \begin{cases} 1, & \quad z > a \\ 0, & \quad z \leq a; \quad z \in \mathbb{R}, \tag{12}
\end{cases}
\]

where \( a \) is a deterministic constant, known as level, and \( I(z; a) \) denotes an indicator function. The translation field \( Y \) in Eq. (11) with \( g \) in Eq. (12) is referred to as a level-cut random field. By clipping the inhomogeneous, filtered Poisson field \( Z(\mathbf{x}) \) defined in Eq. (9), the resulting translation field

\[
Y(\mathbf{x}) := I(Z(\mathbf{x}); a) \tag{13}
\]

then becomes a level-cut, inhomogeneous, filtered Poisson field, which provides a convenient model for a two-phase FGM microstructure in \( \mathcal{D} \). For example, the subsets \( \{ \mathbf{x} \in \mathcal{D} : Y(\mathbf{x}) = 1 \} \) and \( \{ \mathbf{x} \in \mathcal{D} : Y(\mathbf{x}) = 0 \} \) define particles (phase 1) and matrix (phase 2), respectively, which can be derived from the contour of \( Z(\mathbf{x}) \) at level \( a \). Fig. 3 depicts a schematic illustration of the two subsets, obtained from a cut (Fig. 3(b)) of a generic random field \( Z(\mathbf{x}) \) (Fig. 3(a)). The dark (charcoal) phase indicates particles embedded in the light (light blue) phase representing the matrix. The level-cut random field can also be employed for modeling porosity (void) in the matrix.

**Approximate level-cut, filtered Poisson field**: Let the kernel \( h(\mathbf{x}) \) be a compactly supported, non-negative function, for example,

\[
h(\mathbf{x}) = \begin{cases}
\exp(-\mathbf{\chi}^T \mathbf{\gamma} \mathbf{x}) = \exp\left(\frac{-\mathbf{\chi}_1^2}{\sigma_1^2} - \frac{\mathbf{\chi}_2^2}{\sigma_2^2}\right), & \text{for particles} \\
\mathbf{\chi}^T \mathbf{\gamma} \mathbf{x} = \frac{\mathbf{\chi}_1^2}{\sigma_1^2} + \frac{\mathbf{\chi}_2^2}{\sigma_2^2}, & \text{for voids}, \tag{14}
\end{cases}
\]

where \( \chi \in \mathbb{R}^2 \), \( \sigma_k > 0 \) are some constants, and \( \gamma = \text{diag}[1/\sigma_1, 1/\sigma_2] \in \mathbb{R}^{2 \times 2} \) is a diagonal matrix. Suppose that \( \lambda(\mathbf{x}), Z, \) and \( h \) are such that the values of \( Z \) in a small vicinity \( \mathcal{D}(\Gamma_i) \) of \( \Gamma_i \) can be approximated by

\[
\tilde{Z}(\mathbf{x}) = Z h(\Psi_i(\mathbf{x} - \Gamma_i)), \quad \mathbf{x} \in \mathcal{D}(\Gamma_i), \tag{15}
\]

leading to an approximate version

\[
\tilde{Y}(\mathbf{x}) = I(\tilde{Z}(\mathbf{x}); a) = I(Z h(\Psi_i(\mathbf{x} - \Gamma_i)); a), \quad \mathbf{x} \in \mathcal{D}(\Gamma_i) \tag{16}
\]

of the level-cut field \( Y(\mathbf{x}) \) in Eq. (13). Therefore, the particles or voids in the microstructure of a two-phase FGM can be conveniently approximated by the subsets

\[
\begin{align*}
(\mathbf{x} - \Gamma_i)^T \Psi_i^T \mathbf{\gamma} (\mathbf{x} - \Gamma_i) & \leq -\ln(a/Z_i), & \text{for particles} \\
(\mathbf{x} - \Gamma_i)^T \Psi_i^T \mathbf{\gamma} (\mathbf{x} - \Gamma_i) & = 1, & \text{for voids}, \tag{17}
\end{align*}
\]

of \( \mathcal{D} \) for \( Z_i > a \) and \( \mathbf{x} \in \mathbb{R}^2 \). The approximate level-cut field is capable of generating particles and voids with regular geometric shapes, e.g., circular, elliptical. Note that \( \tilde{Y}(\mathbf{x}) \) is strictly applicable when \( \lambda(\mathbf{x}) \) and \( \sigma_k \) are both small. While such smallness restriction limits the usefulness of \( \tilde{Y}(\mathbf{x}) \) somewhat, Eqs. (16) and (17) can be conveniently employed for finite-element discretization of fully penetrable particles even when \( \lambda(\mathbf{x}) \) and \( \sigma_k \) are not small. See Rahman [10] for further details.

**Algorithm**: The level-cut field \( \tilde{Y}(\mathbf{x}) \), described by Eq. (16), can be obtained from \( Z(\mathbf{x}) \) using the level-cut parameter \( a \). Once all the parameters of the level-cut field \( \tilde{Y}(\mathbf{x}) \) have been determined, samples of synthetic microstructures of two-phase FGMs can be generated based on the following algorithm:

1. **Step 1**: Define bounded subsets \( \mathcal{D} \) and \( \mathcal{D}' \) of \( \mathbb{R}^2 \). The bounded subset \( \mathcal{D}' \) must be such that points of \( \mathcal{N} \) falling in \( \mathbb{R}^2 \setminus \mathcal{D}' \) do not contribute to the value of \( Z(\mathbf{x}) \) in \( \mathcal{D} \). Specify the kernel function \( h \) and define its parameters.
2. **Step 2**: Generate a sample \( k^* \) of the homogeneous Poisson random variable \( N^*(\mathcal{D}) \), which has a constant intensity \( \lambda^* \), which has a constant intensity function in \( \mathcal{D}' \).
3. **Step 3**: Generate \( k \) independent uniformly distributed points in \( \mathcal{D}' \). Denote these points by \( \mathbf{x}_i \), \( i = 1, \ldots, k^* \).
4. **Step 4**: Perform thinning of the point set obtained in Step 3. In so doing, each point \( \mathbf{x}_i \), independently of the other, is kept with probability \( \lambda(\mathbf{x}_i)/\lambda^* \), which is equivalent to discarding the point with probability \( 1 - \lambda(\mathbf{x}_i)/\lambda^* \). The resulting point pattern...
with the size $k^{*} \leq k$ follows the inhomogeneous Poisson field $\mathcal{N}(D')$ with intensity function $\lambda(x)$. Denote these points by $I_{i}, i = 1, \ldots, k^{*}$

- Step 5: Generate $k^{*}$ independent samples of random rotation matrices $\{\psi_{i}\}$ and random variables $\{z_{i}\}$ from their relevant distributions.

- Step 6: Calculate the corresponding samples of the random fields $\tilde{Z}(x)$ from Eq. (15) and $\tilde{Y}(x)$ from Eq. (16) for a specified level $a$. The sample of $\tilde{Y}(x)$ yields a two-phase, statistically inhomogeneous microstructure in $D$.

Independent samples of $\tilde{Z}(x)$ and $\tilde{Y}(x)$ are delivered by repeated application of the above-stated algorithm.

Different functions and parameters involved in generating the level-cut field $\tilde{Y}(x)$ are:

1. a Poisson field $\mathcal{N}(D')$ that gives the number of points in $D'$,
2. a spatially invariant intensity function $\lambda(x)$ that is related to particle or void content fraction, $\lambda(x)$ for a non-negative kernel function $h$, which determines particle or void shape (e.g., spherical particles using $c_{1} \neq c_{2}$),
3. random rotation matrices $\{\psi_{i}\}$ that determine particle or void orientations, and
4. a variable $a$ denoting the level at which the cut is made, affecting also the size of particle or void. In the present study, $\lambda(x)$ as well as the non-negative kernel function $h$ and the level-cut variable $a$ are deterministic. The use of deterministic intensity function makes the phase volume fraction deterministic.

The level-cut, inhomogeneous, filtered Poisson random field and the associated algorithm for microstructures described in the preceding pertain to any two-phase FGM. Therefore microstructures of a non-porous, particle-matrix FGM can be readily generated. For a porous, particle-matrix FGM, the microstructures can be delivered by a three-step serial process: (1) generate a two-phase, non-porous, particle-matrix FGM microstructure; (2) generate a two-phase, void-matrix FGM microstructure; and (3) superimpose the microstructure from Step 2 over that from Step 1, forming a resultant microstructure of a porous, particle-matrix FGM. The resultant microstructure includes particles, voids, and matrix as constituents.

### 2.2.2. Random constituent elastic properties

In addition to a random microstructure, the constituent elastic (isotropic) properties of material phases can be stochastic, as assumed in the present study. Let $E_{p}$ and $v_{p}$ denote the elastic modulus and Poisson’s ratio, respectively, of the particle, and $E_{m}$ and $v_{m}$ denote the elastic modulus and Poisson’s ratio, respectively, of the matrix. Therefore, the random vector $[E_{p}, E_{m}, v_{p}, v_{m}]^{T} \in \mathbb{R}^{4}$ describes the stochastic elastic properties of both constituents. The constituent properties are invariant in the macroscopic length scale.

#### 2.2.3. Input random vector

Let $\lambda_{p}(x)$ or $\lambda_{v}(x)$ denote intensity functions associated with the particle or void, respectively, in $D$. Using these intensity functions, let the Poisson random variables $N_{p}$ and $N_{v}$ represent the number of particles and voids, respectively, in a bounded subset $D$ of $\mathbb{R}^{2}$, where $D$ is a subset such that points of $N_{p}$ and $N_{v}$ falling in $D = \mathbb{R}^{2} \setminus \overline{D}$ do not contribute to particles and voids, respectively, in $D$. Denote the coordinates of the centroids of particles and voids by $(U_{p1}, U_{p2})_{i}, i = 1, \ldots, N_{p}$ and $(U_{v1}, U_{v2})_{i}, i = 1, \ldots, N_{v}$, respectively. Correspondingly, the random scaling variables and orientations are: $z_{p_{i}}$ and $\theta_{p_{i}}; i = 1, \ldots, N_{p}$ for particles; and $z_{v_{i}}$ and $\theta_{v_{i}}; i = 1, \ldots, N_{v}$, for voids. Table 1 lists all stochastic input variables for both non-porous and porous FGMs. Let $\mathbf{R} \in \mathbb{R}^{n}$ denote an $N$-dimensional input random vector that contains uncertainties from all sources described in Sections 2.2.1 and 2.2.2. Therefore, the total number $N$ of random variables is $4N_{p} + 5$ for non-porous FGM and $4N_{p} + 4N_{v} + 6$ for porous FGM. The input random vector $\mathbf{R}$ characterizes uncertainties from all sources in an FGM and is completely described by its joint probability density function $f_{\mathbf{R}}(\mathbf{r})$, where $\mathbf{r}$ is a realization of $\mathbf{R}$.

### 2.3. Crack-driving forces and reliability

A major objective of stochastic fracture-mechanics analysis is to find probabilistic characteristics of crack-driving forces, such as SIFs $K_{I}(\mathbf{R})$ and $K_{II}(\mathbf{R})$ for modes I and II, respectively, and the $J$-integral and other fracture integrals, due to uncertain input $\mathbf{R}$. For a given input, the standard FEM can be employed to solve the discretized weak form (Eq. (4)), leading to the calculation of SIFs and other crack-driving forces.

Suppose that failure is defined when the crack propagation is initiated at a crack tip, i.e., when $K_{eff}(\mathbf{R}) = h(K_{I}(\mathbf{R}), K_{II}(\mathbf{R})) > K_{f}$, where $K_{eff}$ is an effective SIF with $h$ depending on a selected mixed-mode theory, and $K_{f}$ is a relevant mode-I fracture toughness of the material measured in terms of SIF. This requirement cannot be satisfied with certainty, since $K_{I}$ and $K_{II}$ are both dependent on $\mathbf{R}$, which is random, and $K_{f}$ itself may be a random variable or field. Hence, the performance of a cracked FGM should be evaluated by the conditional reliability or its complement, the conditional probability of failure $P_{F}$, defined as the multifold integral

$$P_{F}(K_{f}) := P[y(\mathbf{R}) < 0] := \int_{\mathbb{R}^{N}} I_{y}(\mathbf{r}) f_{\mathbf{R}}(\mathbf{r}) \, d\mathbf{r},$$

where

$$y(\mathbf{R}) = K_{f} - h(K_{I}(\mathbf{R}), K_{II}(\mathbf{R}))$$

is another indicator function. In this work, the maximum circumferential stress theory was invoked to describe mixed-mode fracture initiation [15].

The evaluation of the multidimensional integral in Eq. (18), either analytically or numerically, is not possible because the total number of random variables $N$ is large, $f_{\mathbf{R}}(\mathbf{r})$ is generally non-Gaussian, and $y(\mathbf{r})$ is a highly nonlinear function of $\mathbf{r}$. Therefore, Monte Carlo simulation was employed for calculating the probabilistic characteristics of crack-driving forces and the probability of fracture initiation.

### 3. Monte Carlo simulation

Recall that $K_{I}(\mathbf{r}) = K_{I}(r_{1}, \ldots, r_{N}); i = 1, II$ and $y(\mathbf{r}) = y(r_{1}, \ldots, r_{N})$ represent crack-driving forces or a performance function that depends on crack-driving forces. The number of input random variables $N$, the dimension of the stochastic problem, and the input–output mapping $K_{I} : \mathbb{R}^{N} \to \mathbb{R}$ or $y : \mathbb{R}^{M} \to \mathbb{R}$ depends on the concurrent multiscale model. The objective is to evaluate the probabilistic characteristics of generic output responses $K_{I}(\mathbf{R})$ or $y(\mathbf{R})$, when the probability distribution of the random input $\mathbf{R} \in \mathbb{R}^{N}$ is prescribed.

Consider a generic $N$-dimensional random vector $\mathbf{R} = [R_{1}, \ldots, R_{N}]^{T}$, which characterizes uncertainty in all input parameters under consideration with its known joint PDF $f_{\mathbf{R}}(\mathbf{r})$. Suppose that $r^{(1)}, \ldots, r^{(l)}$ are $l$ realizations of the input vector $\mathbf{R}$, which can be generated independently. Let $K_{I}(r^{(1)}), \ldots, K_{I}(r^{(l)}) ; i = 1, II$ and $y(r^{(1)}), \ldots, y(r^{(l)})$ be the output samples of mixed-mode SIFs
Table 1
List of stochastic inputs.

<table>
<thead>
<tr>
<th>Microstructural parameters</th>
<th>Two-phase FGM (non-porous)</th>
<th>Three-phase FGM (porous)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}_p$</td>
<td>$(U_{b1}, U_{b2})$; $i = 1, \ldots, \mathcal{N}_p$</td>
<td>$(U_{b1}, U_{b2})$; $i = 1, \ldots, \mathcal{N}_p$</td>
</tr>
<tr>
<td>$Z_p$</td>
<td>$i = 1, \ldots, \mathcal{N}_p$</td>
<td>$i = 1, \ldots, \mathcal{N}_p$</td>
</tr>
<tr>
<td>$\phi_p$</td>
<td>$i = 1, \ldots, \mathcal{N}_p$</td>
<td>$i = 1, \ldots, \mathcal{N}_p$</td>
</tr>
</tbody>
</table>

Constituent properties

Input random vector ($\mathbf{R}$)

$E_p, E_m, v_p, v_m$

$(U_{p1,1}, U_{p1,2}), \ldots, (U_{p1,1}, U_{p1,2})$

Table 1

$Z_p, \phi_p, i = 1, \ldots, \mathcal{N}_p$

Total number of random variables ($N$)

$4\mathcal{N}_p + 5$

$4\mathcal{N}_p + 6$

---

$K_i(\mathbf{R})$ and performance function $y(\mathbf{R})$ that correspond to the input $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(i)}$, which can be obtained from repeated deterministic fracture-mechanics evaluations of $K_i$ and $y$. The $i$th moment of $K_i(\mathbf{R})$, estimated by Monte Carlo simulation [16] is

$$E[K_i(\mathbf{R})] = \lim_{L \to \infty} \frac{1}{L} \sum_{l=1}^{L} K_i(\mathbf{r}^{(l)}) \quad i = I, II.$$  \hspace{1cm} (21)

By setting $l = 1$ and $2$, the mean and standard deviation of $K_i(\mathbf{R})$ and $K_2(\mathbf{R})$ can be obtained.

For reliability analysis, the direct Monte Carlo estimate $P_{F,MCS}$ of the conditional failure probability is

$$P_{F,MCS}(\mathbf{X}_i) = \lim_{L \to \infty} \frac{1}{L} \sum_{l=1}^{L} I_p(\mathbf{r}^{(l)}) \quad \text{where } I_p(\mathbf{r}^{(l)}) \text{ denotes the } j\text{th realization of the indicator function defined in Eq. (20).}$$  \hspace{1cm} (22)

---

4. An edge-cracked FGM specimen

Consider a two-dimensional, square, FGM specimen with domain $\Omega = 10 \text{ cm} \times 10 \text{ cm}$, which contains randomly dispersed, fully penetrable, elliptical, silicon carbide (SiC) particles and circular voids in an aluminum (Al) matrix. Fig. 4 illustrates a microstructural sample of the specimen, which was created by the level-cut, filtered, random field $Y(\mathbf{x})$ over $\Omega = 10.5 \text{ cm} \times 10.5 \text{ cm}$ and random orientations of the particles. The center of $\Omega$ coincides with that of $\Omega$. The specimen contains a horizontally placed, $a_c = 5 \text{ cm}$ long edge crack with the crack-tip location $\mathbf{x}_{\text{tip}} = [x_{\text{tip}}, y_{\text{tip}}]^T = [5, 5]^T \text{ cm}$ and is subjected to a far-field tensile stress $\sigma^\infty$ and a far-field shear stress $\tau^\infty$. For stationary crack: $\sigma^\infty = \tau^\infty = 1 \text{ kN/cm}^2$, and for crack propagation: $\sigma^\infty = \tau^\infty = 1.8 \text{ kN/cm}^2$. The subdomains $\Omega$ and $\Omega'$ are $5 \text{ cm} \times 5 \text{ cm}$ and $5.5 \text{ cm} \times 5.5 \text{ cm}$ squares with the center coinciding with the crack tip. The subdomain $\Omega_1$ is a circle with radius equal to half of the radius of a circle with an area equal to that of the average particle area in $\Omega$. The center of $\Omega_1$ is located at the crack tip. The sizes of the subdomains $\Omega$ and $\Omega_1$ are deemed adequate based on the results of the authors' previous study [13]. A plane strain condition was assumed.

4.1. Inputs

Particles and voids were generated using the level-cut, inhomogeneous, filtered Poisson random field $Y(\mathbf{x})$ described in Section 2.2.1. The aspect ratio, defining the length-to-width ratio of each particle or void, and hence the shape, whether elliptical or circular, is deterministic and identical for all particles in a particular sample. The particles and voids both follow horizontally varying Poisson intensity functions

$$\lambda_p(\mathbf{x}) = c_p \ln \frac{1 + \epsilon}{1 + \epsilon - \left(\frac{x_1}{10.5}\right)^{\nu_p}}$$  \hspace{1cm} (23)

and

$$\lambda_v(\mathbf{x}) = \begin{cases} c_v \ln \frac{5.25(1 + \epsilon) - x_1}{5.25(1 + \epsilon)}, & x_1 \leq 5.25 \\ c_v \ln \frac{5.25(1 + \epsilon) - (10.5 - x_1),} & x_1 > 5.25, \end{cases}$$  \hspace{1cm} (24)

respectively, where $\mathbf{x} = [x_1, y_1]^T \in \mathbb{R}^2$, $0 \leq x_1 \leq 10.5 \text{ cm}$; $i = 1, 2; \epsilon = 10^{-5}$; $n_p$ is an exponent representing particle gradation; $c_p = 6 \text{ cm}^{-2}$ is the intensity parameter for particles; and $c_v$ is the intensity parameter for voids. The intensity parameters and the corresponding volume fractions ($\phi(\mathbf{x})$, say) are related via
\[ \phi_i(\mathbf{x}) \cong \mathbb{E}_i[\tilde{Y}_i(\mathbf{x})] = P[\tilde{Z}_i(\mathbf{x}) > a] \quad [9, 10]; \quad i = p, v, \] denoting particle and void phases, where the expectation of \( \tilde{Y}_i(\mathbf{x}) \) and the probability of \( \tilde{Z}_i(\mathbf{x}) \) both depend on the intensity functions expressed by Eqs. (23) and (24) and hence, on the intensity parameters. Three values of \( n_p = 0.5, 1, \) and 2 were examined. For voids, three values of \( c_v \) were selected: (1) \( c_v = 0 \), representing non-porous FGM; (2) \( c_v = 2.52 \text{ cm}^{-2} \), yielding a void volume fraction of nearly 0.1 at the crack tip; and (3) \( c_v = 5.46 \text{ cm}^{-2} \), leading to a void volume fraction of nearly 0.2 at the crack tip. The exponential kernel function for particles is given by Eq. (14), where \( \sigma_1 = 0.3 \text{ cm}; \quad \sigma_2 = 0.1 \text{ cm} \) for elliptical particles and \( \sigma_1 = \sqrt{0.3 \times 0.1} \text{ cm} \) for circular particles. A cylindrical kernel function, also represented by Eq. (14), was used for voids, all of which are equal-size circles with \( \sigma_1 = \sigma_2 = \sqrt{0.1 \times \text{ average particle area}/\pi} \text{ cm} \). The random variables \( Z_p = 10 |U_i|; \quad i = 1, \ldots, N_p \) and \( Z_v = 10 |U_i|; \quad i = 1, \ldots, N_v \), where \( |U_i| \) is a collection of independent, standard Gaussian random variables. The orientation of the ith elliptical particle is either random (e.g., \( \Theta_{pi} = U(0, 2\pi) \)), i.e., uniformly distributed between 0 and \( 2\pi \) or deterministic (e.g., \( \Theta_{pi} = 0 \) or \( \pi/2 \)).

Finally, the material phases SiC and Al are both linear-elastic and isotropic. However, the elastic moduli \( E_{\text{SiC}} \) and \( E_{\text{Al}} \) and the Poisson’s ratios \( \nu_{\text{SiC}} \) and \( \nu_{\text{Al}} \) of SiC and Al, respectively, are random variables; their means, standard deviations, and probability distributions are listed in Table 2. The lognormal distribution of the elastic properties, although chosen arbitrarily in this work, was restricted to small coefficients of variation. Therefore, a lognormal or similar other distributions of the elastic properties should reveal the influence of microstructure, if any, on a fracture response of interest. The mode-I fracture toughnesses of SiC and Al, which are deterministic, are as follows: \( K_{c, \text{SiC}} = 5 \text{ MPa} \sqrt{\text{m}} \) and \( K_{c, \text{Al}} = 20, \ 30 \), and \( 40 \text{ MPa} \sqrt{\text{m}} \). Therefore, the matrix is relatively more ductile than the particles.

### 4.2. Parameters studied

Four microstructural features and the fracture toughness of Al were considered for a parametric study on fracture behavior of FGMs. They include: (1) particle shape parameters described by \( \sigma_1 \) and \( \sigma_2 \) in Eq. (14); (2) particle orientation represented by \( \Theta_{pi} \) (i.e., \( \Theta_i \) for particles in Eq. (10)); (3) porosity quantified by \( c_v \) in Eq. (24); (4) particle gradation characterized by \( n_p \) in Eq. (23) and (5) porous FGM mode-I fracture toughness \( K_{c, \text{Al}} \) of Al (matrix). Table 3 summarizes these five parameters, leading to eleven distinct cases for the parametric study. The cases 1–8 account for the effect of FGM microstructural features on the mixed-mode SIFs, leading to fracture initiation. Therefore, they do not include \( K_{c, \text{Al}} \) as it relates to crack propagation, examined in cases 9–11.

Fig. 5(a)–(c) present spatial distributions of particle and void volume fractions for both non-porous and porous FGMs. The volume fractions, \( \phi_i(\mathbf{x}) \cong \mathbb{E}_i[\tilde{Y}_i(\mathbf{x})] \), were calculated from ten independent microstructural samples of \( \tilde{Y}_i(\mathbf{x}) \), generated using the algorithm in Section 2.2.1. Fig. 5(a) comprises plots of particle volume fractions for four non-porous FGMs with \( n_p = 1 \): one with circular (\( \sigma_1 = \sigma_2 = \sqrt{0.3 \times 0.1} \text{ cm} \)) particles (case 1) and three with elliptical (\( \sigma_1 = 0.3 \text{ cm}; \quad \sigma_2 = 0.1 \text{ cm} \)) particles entailing \( \Theta_{pi} = U(0, 2\pi) \) (case 2), 0 (case 3), and \( \pi/2 \) (case 4). All four distributions of particle volume fractions from Fig. 5(a) are practically coincident. Therefore, the individual effects of particle shape and orientation on FGM fracture response can be evaluated. Similarly, Fig. 5(b) shows distributions of particle and void volume fractions for two porous FGMs with \( n_p = 1 \), obtained by selecting \( c_v = 2.52 \text{ cm}^{-2} \) (case 5) and \( c_v = 5.46 \text{ cm}^{-2} \) (case 6). From Fig. 5(b), the maximum void volume fractions that occur at \( x_1 = 5 \text{ cm} \) are nearly 0.1 and 0.2, respectively. Finally, Fig. 5(c) depicts three markedly different distributions of particle volume fractions for non-porous FGMs, when the particle gradation parameters are \( n_p = 0.5 \) (case 7), 1 (case 1), and 2 (case 8). Therefore, \( n_p \) controls the spatial distribution of particle volume fractions significantly.

#### Table 2

**Statistical properties of constituents in SiC–Al FGM.**

<table>
<thead>
<tr>
<th>Elastic property</th>
<th>Mean</th>
<th>Coefficient of variation (%)</th>
<th>Probability distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{\text{SiC}} ) (GPa)</td>
<td>419.2</td>
<td>15</td>
<td>Lognormal</td>
</tr>
<tr>
<td>( E_{\text{Al}} ) (GPa)</td>
<td>69.7</td>
<td>10</td>
<td>Lognormal</td>
</tr>
<tr>
<td>( \nu_{\text{SiC}} )</td>
<td>0.19</td>
<td>Lognormal</td>
<td></td>
</tr>
<tr>
<td>( \nu_{\text{Al}} )</td>
<td>0.34</td>
<td>Lognormal</td>
<td></td>
</tr>
</tbody>
</table>

\( E_p = E_{\text{SiC}}; \quad E_m = E_{\text{Al}}; \quad v_p = \nu_{\text{SiC}}; \quad v_m = \nu_{\text{Al}}. \)
4.3. Finite element modeling

Fig. 6(a)–(h) present eight finite-element discretizations of samples of non-porous and porous FGM specimens for the first eight cases listed in Table 3, all employed in conjunction with the concurrent multiscale model. The numbers and types of elements for these eight finite-element meshes are listed in Table 4. All finite-element meshes are conforming and were developed using the commercial code ABAQUS [17]. Full 3 × 3 and 3-point Gauss quadrature rules were employed for the quadrilateral and triangular elements, respectively, for numerical integration.

For finite-element models of porosity, a void falling anywhere on the crack was redistributed in the $x_2$ direction by splitting the void into two equivalent parts and placing them just above and just below the crack line in such a way that the equivalent voids do not intersect with the crack. Also, since $D_2$ is non-porous and voids are small compared with particle sizes, any portion of a void falling inside $D_2$ was ignored and was replaced by the particle or the matrix phase depending on the crack-tip material of the underlying particle-matrix microstructure of the non-porous FGM.

The stochastic analysis was conducted by the Monte Carlo simulation with a sample size of 10,000. Therefore, the smallest value of the failure probability calculated was limited to $10^{-3}$. A failure probability of $10^{-3}$, although not a very small number, was deemed adequate to conduct the parametric study.

4.4. Results and discussions

Fig. 7(a)–(h) plot von Mises stress contours generated using the concurrent multiscale model for the six non-porous (cases 1, 2, 3, 4, 7, and 8) and two porous (cases 5 and 6) FGM samples in Fig. 6(a)–(h). The effective elastic properties required by the concurrent multiscale model were calculated using the Mori–Tanaka approximation [18], which incorporates orthotropic effective elastic properties due to the alignment of elliptical particles in a particular direction and the aspect ratio. Voids are represented using degenerative material properties in the Mori–Tanaka approximation. The overall stress responses for different FGM samples, indicated by the contour patterns, are similar. However, there are also differences in the local stress fields that may have significant implications in determining crack-driving forces and eventually in statistical and reliability predictions for different microstructures. The second-moment characteristics and fracture reliability results pertaining to parametric study are presented next.

### Table 3

Parameters studied and associated cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>Shape ($\sigma_1$, $\sigma_2$, cm)</th>
<th>Orientation ($\phi_{\alpha}$)</th>
<th>Porosity ($c$, cm$^{-3}$)</th>
<th>Particle gradation ($a_p$)</th>
<th>Fracture toughness of Al ($K_{Ic}$, MPa m$^{1/2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>0</td>
<td>1</td>
<td>NA*</td>
</tr>
<tr>
<td>2</td>
<td>$\sigma_1 = 0.3$, $\sigma_2 = 0.1$</td>
<td>$U(0, 2\pi)$</td>
<td>0</td>
<td>0</td>
<td>NA*</td>
</tr>
<tr>
<td>3</td>
<td>$\sigma_1 = 0.3$, $\sigma_2 = 0.1$</td>
<td>$\pi/2$</td>
<td>0</td>
<td>1</td>
<td>NA*</td>
</tr>
<tr>
<td>4</td>
<td>$\sigma_1 = 0.3$, $\sigma_2 = 0.1$</td>
<td>$\pi/2$</td>
<td>0</td>
<td>1</td>
<td>NA*</td>
</tr>
<tr>
<td>5</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>2.52</td>
<td>1</td>
<td>NA*</td>
</tr>
<tr>
<td>6</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>5.46</td>
<td>1</td>
<td>NA*</td>
</tr>
<tr>
<td>7</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>0</td>
<td>0.5</td>
<td>NA*</td>
</tr>
<tr>
<td>8</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>0</td>
<td>2</td>
<td>NA*</td>
</tr>
<tr>
<td>9</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>0</td>
<td>1</td>
<td>NA*</td>
</tr>
<tr>
<td>10</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>0</td>
<td>1</td>
<td>NA*</td>
</tr>
<tr>
<td>11</td>
<td>$\sigma_1 = \sigma_2 = 0.3 \times 0.1$</td>
<td>NA*</td>
<td>0</td>
<td>1</td>
<td>30</td>
</tr>
</tbody>
</table>

* Not applicable.

### Table 5

The means and coefficients of variations of $K_I$ and $K_{II}$ for six non-porous FGMs (cases 1, 2, 4, 7, and 8) and two porous FGMs (cases 5 and 6) described in Fig. 6(a)–(h). The results for the mean values of $K_I$ for cases 2–4 are comparable with that for case 1, with a maximum of 3.5% difference observed in case 3. The variability in the $K_I$ values for cases 1–4, expressed in terms of coefficient of variation, shows a similar trend. These results indicate that the statistical properties of mode I SIF do not alter appreciably due to particle shape or orientation. However, the amount of porosity (cases 5 and 6) enhances mode I SIF, as Table 5 shows an increase of 17.6% in the mean value of $K_I$ from case 1 to case 6. However, an introduction of voids does not increase the coefficient of variation of $K_I$ values appreciably. This is because by increasing the void content (which replaces particles and matrix in the FGM microstructure), no substantial change is made in the total variability due to the location characteristics of the constituents (in terms of particle and void locations). Furthermore, the particle gradation parameter (cases 7 and 8) also has a noticeable effect on the $K_I$ statistics. For case 7, the mean value of $K_I$ increases from that for case 1. This can be explained by the following facts. As the overall particle content for case 7 ($\eta_p = 0.5$) is much higher than that for case 1 ($\eta_p = 1$), it lowers the individual $K_I$ values for case 7. At the same time, due to the higher value of $\phi_{\alpha}$ at the crack-tip location for case 7, more samples contain particle at the crack tip. The $K_I$ value for an FGM sample containing particle at the crack tip is much higher than that for a sample containing matrix at the crack tip. Therefore, the mean of $K_I$ for case 7 is higher than that for case 1. A similar argument explains why the mean of $K_{II}$ for case 8 ($\eta_p = 2$), which has a significantly lower value of $\phi_{\alpha}$ at the crack-tip location compared with case 1, is lower than that for case 1. Since case 7 contains a greater number of particles and particles are fully penetrable, individual particles coalesce to form larger particle clusters and in the process lose their individual location characteristics, thereby reducing the variability of $K_I$.

Results for the means and coefficients of variation of $K_{II}$ for cases 1–4 shows a similar trend as that observed for $K_I$. Therefore, from the results of both $K_I$ and $K_{II}$, it can be concluded that statistical properties of SIFs do not alter appreciably due particle shape and orientation. By introducing porosity (cases 5 and 6), the mean values of $K_{II}$ change appreciably, as can be observed by a 29.7% increase in the value for case 5 compared with case 1. Variability in the $K_{II}$ values also increases substantially due to the presence of voids. Although the total variability due to location characteristics of the constituents remains comparable even with an increase in the void content, $K_{II}$ is more dependent on the variability in void locations, which increases with the increase in porosity. Hence, results for both $K_I$ and $K_{II}$ indicate that the amount of porosity substantially enhances crack-driving forces. Results for
the means and coefficients of variation of $K_\text{II}$ for cases 7 and 8 show a similar trend as that observed for $K_\text{I}$, which indicates that particle gradation parameter also has a noticeable impact on the SIF statistics.

4.4.2. Fracture reliability

While the second moment statistics are useful, a more meaningful stochastic response is the conditional probability of fracture initiation, $P_F(K_{\text{Ic}}) := P[y(R) < 0]$, described in Section 2.3. The failure probability $P_F(K_{\text{Ic}})$ derived from the maximum circumferential stress theory was calculated by the concurrent multiscale model for different FGM microstructures. Fig. 8(a) plots how $P_F(K_{\text{Ic}})$ varies as a function of the fracture toughness $K_{\text{Ic}}$ for non-porous FGMs with different microstructures containing circular and elliptical particles with different orientations, all generated using $n_p = 1$ (cases 1, 2, 3, and 4). The results indicate that the effects of particle shape and orientation on fracture reliability are small. This can be attributed to the following reason. Due to the fully penetrable particle model, particles of different shapes and orientations coalesce to achieve a target volume fraction and
therefore lose their geometric identities, except at very low volume fractions. In the present study, $\phi_p$ is close to 0.5 near the crack tip. Therefore, close to the crack tip, overall particle domains are similar for different particle shapes and orientations. Since SIFs are strongly influenced by the particles near the crack tip, effects of particle shape and orientation on fracture reliability are negligible. Also, due to the bi-axial nature of stresses surrounding the crack tip, particle orientation has little or no contribution in shielding the crack tip from applied loads and therefore has negligible effect on SIFs, and hence on fracture reliability.

To further investigate the effect of particle orientation, an additional analysis was performed using a very high stiffness ratio between particle and matrix. For the SiC–Al FGM, the ratio of the mean value of elastic modulus of SiC to that of Al, defined as the modular ratio, is $419.2/69.7 \approx 6.0$. A new FGM, made of SiC as particles and Polyimide as the matrix, was considered, which has a modular ratio of 40. Statistical properties of the elastic moduli $E_{\text{SiC}}$ and $E_{\text{Polyimide}}$ and the Poisson’s ratios $\nu_{\text{SiC}}$ and $\nu_{\text{Polyimide}}$, of SiC and Polyimide, respectively, are listed in Table 6. The motivation for choosing samples of FGM with a very high modular ratio

---

**Fig. 7.** von Mises stress contours for sample specimens of SiC–Al FGM employing concurrent multiscale model: (a) case 1; (b) case 2; (c) case 3; (d) case 4; (e) case 5; (f) case 6; (g) case 7; (h) case 8.
is that the orientation of particles will strongly influence the effective orthotropic material properties of an FGM with high elastic mismatch. Therefore, the effect of particle orientation on FGM fracture, if any, can be enhanced. However, fracture reliability results of SiC–Polyimide FGM depicted in Fig. 8(b) show no statistically significant differences among the failure probability curves with circular and elliptical particles entailing different orientations. Hence, it can be concluded that due to the fully penetrable particles and biaxial nature of stresses around the crack tip, particle shape and orientation have negligible effects on fracture reliability, even for FGMs with a very high angular ratio.

Table 4

<table>
<thead>
<tr>
<th>Case</th>
<th>No. and type of elements in D \ D</th>
<th>No. and type of elements in D \ D</th>
<th>No. and type of elements in D \ D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>818 Type-1</td>
<td>10,302 Type-2</td>
<td>168 Type-1</td>
</tr>
<tr>
<td>2</td>
<td>835 Type-1</td>
<td>10,249 Type-2</td>
<td>12 Type-3</td>
</tr>
<tr>
<td>3</td>
<td>832 Type-1</td>
<td>10,485 Type-2</td>
<td>168 Type-1</td>
</tr>
<tr>
<td>4</td>
<td>840 Type-1</td>
<td>10,222 Type-2</td>
<td>12 Type-3</td>
</tr>
<tr>
<td>5</td>
<td>1093 Type-1</td>
<td>21,302 Type-2</td>
<td>168 Type-1</td>
</tr>
<tr>
<td>6</td>
<td>1052 Type-1</td>
<td>20,497 Type-2</td>
<td>12 Type-3</td>
</tr>
<tr>
<td>7</td>
<td>816 Type-1</td>
<td>11,120 Type-2</td>
<td>168 Type-1</td>
</tr>
<tr>
<td>8</td>
<td>822 Type-1</td>
<td>9,802 Type-2</td>
<td>12 Type-3</td>
</tr>
</tbody>
</table>

* Type-1: eight-noded, non-singular, quadrilateral elements. Type-2: six-noded, non-singular, triangular elements. Type-3: eight-noded, quarter-point (singular), collapsed quadrilateral elements.

Table 5

<table>
<thead>
<tr>
<th>Case</th>
<th>KI (MPa√m) Mean</th>
<th>KI (MPa√m) Mean</th>
<th>Coefficient of variation (%)</th>
<th>Coefficient of variation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25.170</td>
<td>2.070</td>
<td>44.736</td>
<td>98.599</td>
</tr>
<tr>
<td>2</td>
<td>25.237</td>
<td>2.085</td>
<td>45.192</td>
<td>95.779</td>
</tr>
<tr>
<td>3</td>
<td>26.052</td>
<td>2.096</td>
<td>43.095</td>
<td>90.410</td>
</tr>
<tr>
<td>4</td>
<td>24.780</td>
<td>2.038</td>
<td>45.192</td>
<td>141.095</td>
</tr>
<tr>
<td>5</td>
<td>26.069</td>
<td>2.678</td>
<td>43.654</td>
<td>183.308</td>
</tr>
<tr>
<td>6</td>
<td>29.588</td>
<td>2.684</td>
<td>45.451</td>
<td>141.095</td>
</tr>
<tr>
<td>7</td>
<td>26.462</td>
<td>2.678</td>
<td>42.902</td>
<td>183.308</td>
</tr>
<tr>
<td>8</td>
<td>23.955</td>
<td>2.379</td>
<td>43.095</td>
<td>82.682</td>
</tr>
</tbody>
</table>

Fig. 10 plots how PF (KP) varies as a function of the fracture toughness KP for non-porous FGMs with microstructures containing circular particles generated using three different values of the particle gradation parameter np : 0.5 (case 7), 1 (case 1), and 2 (case 8). From the comparison between Figs(c) and 10, the greater the volume fraction of particles close to the crack tip (obtained by decreasing np), the less the probability of failure. Since an increase in particle content close to the crack tip enriches the particle-to-particle distance, the crack from applied external loads more effectively, the crack in an FGM with a higher particle volume fraction at the crack-tip region has a lower propensity to propagate.

4.4.3. Crack propagation

A limited crack-propagation simulation was conducted to evaluate the effect of material fracture toughness of a non-porous FGM. The microstructure contains circular particles (σ1 = σ2 = √0.3 × 0.1 cm) and the particle gradation parameter np = 1 (cases 9, 10, and 11). Crack trajectories were determined based on the maximum circumferential stress theory with an incremental crack length of Δx = 0.5 cm. Since particles and matrix are perfectly bonded, a crack can propagate from matrix to particle without changing direction. The initial crack tip is located at xtip = [xtip,1, xtip,2] = (5, 5) cm. The applied external loads were kept constant during the crack propagation, and crack propagation was stopped when KEff(R) < Keff(xtip), where Keff(xtip) is the mode-I material fracture toughness of the crack tip, i.e., Keff(xtip) = Keff(xtip) if xtip ∈ Dm and Keff(xtip) = Keff(xtip) if xtip ∈ Dp. Due to random microstructures, crack-propagation analyses were performed for three independent samples.

Fig. 11(a) shows crack propagation paths in three samples when KSC = 20 MPa√m. Due to low toughness, cracks in all three samples of Fig. 11(a) propagated all the way through the plate. When the toughness of Al was doubled, as in case 10 (KSC = 40 MPa√m) – the results of which are displayed in
Fig. 8. Probability of fracture initiations of non-porous FGMs with different particle shapes and orientations: (a) SiC–Al FGM; (b) SiC–Polyimide FGM.

Fig. 9. Probability of fracture initiations of SiC–Al non-porous FGMs with different gradations of intensity functions.

4.5. Computational cost

For a complete stochastic fracture analysis of a given random FGM microstructure, Monte Carlo simulation entailing 10,000 samples were conducted. Therefore, 10,000 deterministic finite-element analyses were involved, leading to a conditional failure probability curve in Figs. 8–10. The underlying deterministic analyses were performed using the concurrent multiscale model and an HP XW8400 workstation. The absolute CPU time required by a single, deterministic multiscale analysis, including all preprocessing efforts, was 50 s. This leads to a full stochastic analysis time of approximately 5.8 days. Therefore, a more efficient but sufficiently accurate stochastic method, if it can be developed, will reduce the computational burden. The need for a computationally efficient stochastic method becomes critical when analyzing three-dimensional FGMs, which are not included in the present study.

5. Conclusions and future work

A parametric study on fracture behavior of a crack in two-dimensional, functionally graded composites was conducted. The study involves stochastic descriptions of particle and void numbers; location, size, and orientation characteristics; and constituent elastic properties; a concurrent multiscale model for determining crack-driving forces under mixed-mode loading; and Monte Carlo simulation for uncertainty propagation and fracture reliability analysis. A level-cut, inhomogeneous, filtered Poisson field describes the statistically inhomogeneous random microstructure of a non-porous or porous, graded composite. The concurrent multiscale model involves simultaneously performed microscale and macroscale analyses.

A numerical problem involving an edge-cracked, functionally graded specimen under a mixed-mode deformation was analyzed to calculate the statistical moments of crack-driving forces, the conditional probability of fracture initiation, and sample properties of crack-propagation. Eight distinct varieties of graded microstructure encompassing circular and elliptical particles; deterministic and random particle orientations; low and high porosity contents; and low, medium, and high particle gradation parameters, were analyzed. In addition, three distinct values (low, medium, and high) of fracture toughness properties of the matrix material were examined. The results pertaining to a stationary crack demonstrates that (1) particle shape and orientation for the same phase volume fractions have negligible effects on fracture reliability, even for graded media with a high modular ratio; (2) voids strongly influence crack-driving forces by rendering the graded microstructure weaker, thereby significantly
Fig. 11. Crack paths for SiC-Al FGM samples with different $K_{Ic,Al}$: (a) $K_{Ic,Al} = 20\,\text{MPa}\sqrt{\text{m}}$; (b) $K_{Ic,Al} = 40\,\text{MPa}\sqrt{\text{m}}$; (c) $K_{Ic,Al} = 30\,\text{MPa}\sqrt{\text{m}}$.

raising the probability of fracture initiation; and (3) increasing the particle gradation parameter, which is inversely related to particle volume fraction, significantly increases the probability of fracture initiation. The crack propagation results for perfectly bonded composites with brittle particles reveal that under a fixed external loading, the fracture toughness of the matrix material can
have a significant effect on the likelihood or the extent of crack growth.

The stochastic analysis conducted using Monte Carlo simulation requires a large number of deterministic finite-element analyses. Further research is necessary to develop alternative stochastic methods that are sufficiently accurate but generate probabilistic solutions in a computationally efficient way.

Acknowledgments

The authors would like to acknowledge financial support from the US National Science Foundation under Grant No. CMS-0409463.

References