Markov Model for Seismic Reliability Analysis of Degrading Structures

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Abstract: A Markov model is proposed to evaluate seismic performance and sensitivity to initial state of structural systems and determine the vulnerability of structures exposed to one or more earthquakes. The method of analysis is based on the seismic hazard modeled by a filtered Poisson process, nonlinear dynamic analysis for estimating structural response to earthquakes, uncertainty in initial damage state, and failure conditions incorporating damage accumulation during consecutive seismic events. Simple structures designed by the seismic design code are used to illustrate the proposed method. Effects of uncertainty in the initial state of these systems on seismic reliability are also investigated.

Introduction

A major objective of seismic design is the generation of structures that can survive earthquakes. Current methods for evaluating the overall seismic performance of structural systems are based on global damage indices and lifetime maximum seismic hazard. The global indices are obtained by heuristic combinations of local damage measures, and the seismic hazard is modeled without any consideration for cumulative damage during consecutive seismic events. Such a global measure of damage cannot characterize structural state uniquely, provides only a crude estimate of structural performance during seismic events, and cannot be used to assess structural vulnerability to future loadings. Since most structures are designed to resist several earthquakes during their exposure time, the lifetime largest ground motion may not be meaningful as a design load parameter, due to accumulation of damage between consecutive seismic events. This is particularly true and unavoidable for a series of earthquakes including preshocks, main events, and aftershocks, during which repairs of structural systems cannot be performed.

In addition to the preceding limitations, current estimates of seismic reliability analysis of building structures are based on: (1) Elementary approximations of seismic hazard, e.g., by the lifetime maximum peak ground acceleration a_{10} that is exceeded at least once in 50 years with probability 10%; (2) static method for structural stress analysis; and (3) assumption that the local failure (e.g., cross-section failure) yields system collapse. These simplified methods have also been used in studies (Ellingwood et al. 1980; O'Connor and Ellingwood 1987; Rahman and Grigoriu 1989) to determine reliability indices for code-designed buildings subject to seismic ground shaking. Some of these simplifications can significantly affect seismic reliability (Rahman and Grigoriu 1989; Rahman 1991).

Another important issue in the evaluation of seismic performance is the

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lack of exact knowledge in the initial state of structural systems. This uncertainty is primarily caused by manufacturing processes, errors in design, inadequate construction, unsatisfactory quality control for new structures, and lack of information concerning damage caused by previous seismic events for existing buildings. Reliability analysis based solely on current definitions of global damage indices cannot be applied to determine sensitivity to initial state of structural systems. Hence, any rational assessment of structural performance should simultaneously account for the mechanical degradation process of critical cross sections and components.

The objectives of this paper are to evaluate the seismic performance and sensitivity to initial state of structural systems, and determine the vulnerability of structures exposed to one or more earthquakes. A new methodology based on a Markov model is proposed for seismic reliability analysis. The method of analysis is based on (1) Simple but realistic characterization of seismic hazard; (2) nonlinear dynamic analysis for estimating structural response to earthquakes; (3) uncertainty in initial state of structural systems; and (4) failure conditions incorporating damage accumulation during consecutive seismic events. Simple structures designed by the *Uniform Building Code* (1988) are used to illustrate the proposed method. Effects of uncertainty in the initial state of these systems on seismic reliability are also investigated.

Seismic and Mechanical Models

Seismic Hazard

For simplicity, consider a site that is affected by a single seismic source characterized by a mean rate of earthquake occurrence λ. It is assumed that (1) The earthquake arrivals follow a homogeneous Poisson process with mean rate λ; (2) ground motions in different seismic events are independent stochastic processes $W^i(t)$, $i = 1, 2, \ldots, N(\tau)$ where $N(\tau)$ represents the random number of seismic events during lifetime period τ; and (3) seismic event i has random duration t^i. The supposition of stationary Poisson process has the implication that the interarrival times are independent and follow the same exponential distribution. Although this representation provides an elementary model of the seismic environment, it has been found to be consistent with historical occurrences for ground motions associated with earthquakes that are of engineering interest in structural applications (Algermissen 1983). Consequently, the Poisson assumption may still serve as a useful but simple model of seismic hazard (Cornell 1968). Fig. 1 shows the schematics of seismic environment at a site.

Nonlinear Degrading Systems

Consider a general multistory framed structure with n_c critical cross sections, each of which has n_p parameters to describe the restoring-force model. The stochastic seismic modeling of this multi-degree-of-freedom, hysteretic,

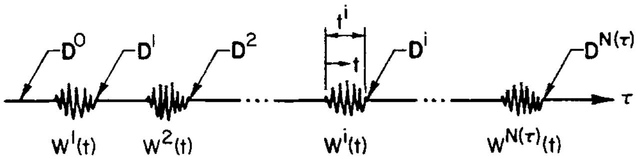

FIG. 1. Seismic Hazard at Site

and degrading system leads to the differential equations of the form (Rahman 1991)

$$\mathbf{m}\ddot{\mathbf{X}}^i(t) + \mathbf{g}^i(\{\mathbf{X}^i(s), \dot{\mathbf{X}}^i(s), 0 \leq s \leq t\}; t) = -\mathbf{m}\mathbf{d}W^i(t) \qquad (1)$$

with the initial conditions

$$\mathbf{X}^i(0) = \mathbf{0} \qquad (2a)$$

and

$$\dot{\mathbf{X}}^i(0) = \mathbf{0} \qquad (2b)$$

where t = local time coordinate originating at the beginning of seismic event i; $\mathbf{X}^i(t)$ = vector of generalized displacements; $\mathbf{g}^i$ = vector functional representing nonlinear hysteretic restoring forces; $\mathbf{m}$ = constant mass matrix; $\mathbf{d}$ = vector of influence coefficients; and $W^i(t)$ = stochastic process representation of ith seismic event. In earthquake engineering, the total restoring force $\mathbf{g}^i$ is usually modeled by the superposition of a nonhysteretic component

$$\mathbf{g}^i_{nh} = \mathbf{c}\dot{\mathbf{X}}^i(t) + \mathbf{k}^i_{nh}(\mathbf{X}^i(t))\mathbf{X}^i(t) \qquad (3)$$

and a hysteretic component

$$\mathbf{g}^i_h = \mathbf{k}^i_h(\mathbf{Z}^i(t))\mathbf{X}^i(t) \qquad (4)$$

where $\mathbf{c}$ = constant viscous damping matrix; $\mathbf{k}^i_{nh}$ = nonhysteretic part of stiffness matrix; $\mathbf{k}^i_h$ = hysteretic part of stiffness matrix; and $\mathbf{Z}^i(t)$ = vector of additional hysteretic variables the time evolution of which can be modeled by a set of general nonlinear ordinary differential equations

$$\dot{\mathbf{Z}}^i(t) = \mathbf{F}^i(\mathbf{X}^i(t), \dot{\mathbf{X}}^i(t), \mathbf{Z}^i(t), t; \mathbf{A}^i(t)) \qquad (5)$$

in which $\mathbf{F}^i$ = general nonlinear vector function the explicit expression of which depends on the hysteretic rule governed by a particular constitutive law; and $\mathbf{A}^i(t) \in \Re^n$ = damage state vector that has $n = n_c n_p$ components equal to the parameters of restoring forces at all critical cross section of a structural system at time t during seismic event i. A wide variety of the explicit form of (5) is available in Rahman (1991). Following the state vector approach (Hurty and Rubinstein 1964; Meirovitch 1967) with the designation of

$$\boldsymbol{\theta}^i_1(t) = \mathbf{X}^i(t) \qquad (6a)$$

$$\boldsymbol{\theta}^i_2(t) = \dot{\mathbf{X}}(t) \qquad (6b)$$

$$\boldsymbol{\theta}^i_3(t) = \mathbf{Z}^i(t) \qquad (6c)$$

the equivalent system of first-order nonlinear differential equations in state variables become

$$\dot{\boldsymbol{\theta}}^i_1(t) = \boldsymbol{\theta}^i_2(t) \qquad (7a)$$

$$\dot{\boldsymbol{\theta}}^i_2(t) = -\mathbf{m}^{-1}[\mathbf{c}\boldsymbol{\theta}^i_2(t) + \mathbf{k}^i_{nh}(\boldsymbol{\theta}^i_1(t))\boldsymbol{\theta}^i_1(t) + \mathbf{k}^i_h(\boldsymbol{\theta}^i_3(t))\boldsymbol{\theta}^i_1(t)] - \mathbf{d}W^i(t) \qquad (7b)$$

$$\dot{\boldsymbol{\theta}}^i_3(t) = \mathbf{F}^i(\boldsymbol{\theta}^i_1(t), \boldsymbol{\theta}^i_2(t), \boldsymbol{\theta}^i_3(t), t; \mathbf{A}^i(t)) \qquad (7c)$$

which can be recast in a more compact form

$$\dot{\boldsymbol{\theta}}^i(t) = \mathbf{h}^i(\boldsymbol{\theta}^i(t), t; \mathbf{A}^i(t)) \quad \text{(8)}$$

with the initial conditions

$$\boldsymbol{\theta}^i(0) = \mathbf{0} \quad \text{(9)}$$

where $\mathbf{h}^i(\ )$ = vector function; $\boldsymbol{\theta}^i(t)$ = response state vector; $\mathbf{A}^i(t) \in \Re^n$ = damage state vector representing state of parameters in the restoring force with $\Re^n$ denoting n-dimensional real vector space, and are given by

$$\boldsymbol{\theta}^i(t) = \begin{Bmatrix} \boldsymbol{\theta}_1^i(t) \\ \boldsymbol{\theta}_2^i(t) \\ \boldsymbol{\theta}_3^i(t) \end{Bmatrix} \quad \text{(10a)}$$

and

$$\mathbf{A}^i(t) = \begin{Bmatrix} A_1^i(t) \\ A_2^i(t) \\ \vdots \\ A_n^i(t) \end{Bmatrix} \quad \text{(10b)}$$

When the excitation is random, $\mathbf{A}^i(t)$ = vector stochastic process, and it characterizes structural state uniquely.

Markov Model

Damage State Vector

Consider a damage state vector $\mathbf{A}^i$, which has $n = n_c n_p$ components equal to parameters of restoring forces at all critical cross sections of a structural system at the end of ith seismic event. It can be obtained from

$$\mathbf{A}^i = \mathbf{A}^i(t^i) \quad \text{(11)}$$

where t^i = duration of ith seismic event and $\mathbf{A}^i(t)$ was defined earlier in (5). State vector $\mathbf{A}^i$ can be conveniently mapped into a normalized damage state vector $\mathbf{D}^i$ by the relation

$$D_j^i = 1 - \frac{A_j^i}{A_j^0} \quad \text{(12)}$$

where $j = 1, 2, \ldots, n$ represents an index for the component of vectors $\mathbf{A}^i \in \Re^n$ and $\mathbf{D}^i \in \Re^n$. This simple transformation permits the domain of each component of $\mathbf{D}^i$ to lie between 0 and 1. Note that the state vector $\mathbf{D}^i$ provides complete characterization of structural state at the end of earthquake i. Hence, one needs only $\mathbf{D}^i$ to perform dynamic analysis and determine structural performance through a new seismic event.

A duty cycle (DC) is a repetitive period of operation in the life of a structure that causes an increase in damage. For earthquake-resistant structure, each seismic event corresponds to a DC. If the earthquake is modeled as a filtered Poisson process, with each seismic event assumed to be an independent random process, damage-state vector $\mathbf{D}^i$ at the end of an ith DC depends only on initial state $\mathbf{D}^{i-1}$ at the start of the DC, and on that DC itself. It is independent of damage and loading history up to the start of that DC. In other words, the propagation of damage-state vector $\mathbf{D}^i$ can be treated as Markov process evolving on a discrete time scale (Rahman and Grigoriu 1990a, 1990b; Rahman 1991). The evolution of a discrete version of $\mathbf{D}^i$ can be described by one-step transition matrix $\mathbf{T}(i)$ with the element $T_{pq}(i)$ representing the probability that damage changes from state p to state q due to seismic event i. This is explained in the forthcoming section.

Transition Matrix

Consider a domain $\mathcal{D} \subseteq \Re^n$ (as shown in Fig. 2) having $\text{Prob}(\mathbf{D}^i \in \mathcal{D}) \simeq 1$ with $K = \Pi_{j=1}^n l_j$ cells (states) $\{C_p\}$ such that $\mathcal{D} = \cup_{p=1}^K C_p$, $C_p \cap C_q = \emptyset$ $(p \neq q)$; and l_j represents the number of discretized states of jth component of $\mathbf{D}^i \in \Re^n$. Consider the change in stochastic vector process $\mathbf{D}^i$, $i = 0, 1, 2, \ldots, N(\tau)$, taking values in a finite, or countable, number of cells C_1, $C_2, \ldots, C_K$. Let $\text{Prob}(\mathbf{D}^i \in C_p)$ be the probability that damage-state vector $\mathbf{D}^i$ is in cell C_p after i seismic events. Then the row vector $\mathbf{P}(i) = \{\text{Prob}(\mathbf{D}^i \in C_1), \text{Prob}(\mathbf{D}^i \in C_2), \ldots, \text{Prob}(\mathbf{D}^i \in C_K)\}$ represents a K-dimensional probability vector with pth $(p = 1, \ldots, K)$ component defining the probability that $\mathbf{D}^i$ belongs to the cell C_p after i seismic events.

Suppose the seismic events constitute a sequence of independent random processes. Then the probability $\text{Prob}(\mathbf{D}^i \in C_q | \mathbf{D}^{i-1} \in C_p$, past history of structural loading and damage) is equal to $T_{pq}(i) = \text{Prob}(\mathbf{D}^i \in C_q | \mathbf{D}^{i-1} \in C_p)$ because system performance is completely specified by the value of damage-state vector $\mathbf{D}^{i-1}$ at the application of earthquake i. Denote $\mathbf{T}(i) = \{T_{pq}(i)\}$; $p, q = 1, 2, \ldots$; and K = the one-step transition matrix from time $i - 1$ to time i associated with the ith DC. Hence, $\{\mathbf{D}^i, i = 0, 1, 2, \ldots, N(\tau)\}$ is a discrete-state (DS), discrete-time (DT) Markov vector process, where $N(\tau)$ is the total (random) number of seismic events at a site. Fig. 3 shows the diagram of transition probabilities for Markov process $\mathbf{D}^i$.

The estimation of transition probability $T_{pq}(i)$ involves computation of conditional probability density of the random vector $\mathbf{D}^i | \mathbf{D}^{i-1} \in C_p$ for all the cells C_p, $p = 1, 2, \ldots, K$. The method of Monte Carlo simulation can

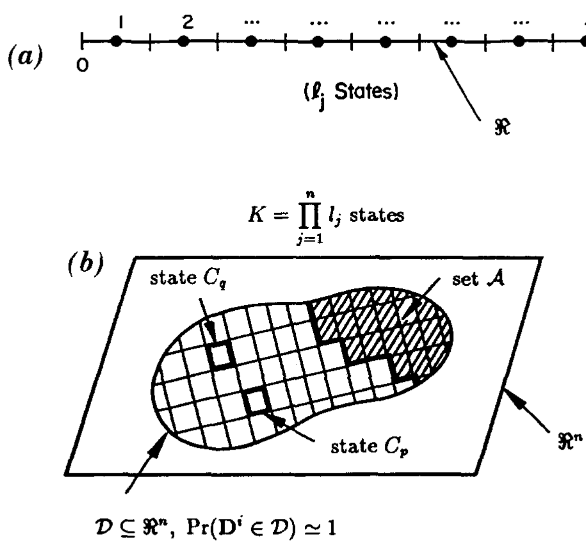

FIG. 2. Discretization of Sample Space

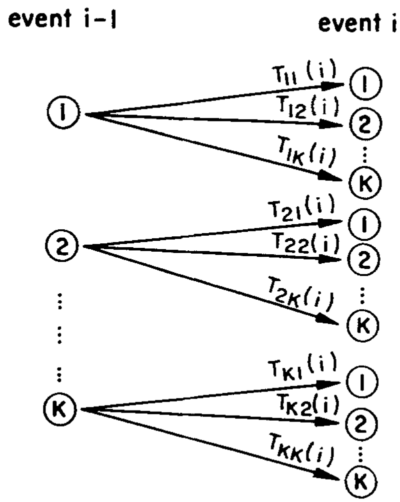

FIG. 3. Diagram for Transition Probabilities

be used for this purpose due to the unavailability of analytic solutions. Each deterministic trial in the simulaton method requires nonlinear dynamic analysis. Mathematically, this corresponds to the computational effort for solving the deterministic initial-value problem in (8) and (9). Various numerical integrators such as Runge-Kutta method (Kutta 1901; Runge 1895), Adam's or Gear's method (Gear 1971; Shampine and Gear 1979), Bulirsch-Stoer extrapolation method (Bulirsch and Stoer 1966), and many others can be applied to obtain the solution. The selection of a particular method, however, depends on its computational efficiency, numerical accuracy and stability, and the stiffness of the nonlinear system of differential equations. In this study, several numerical schemes are tested, and the fifth- and sixth-order Runge-Kutta integrators are determined to be satisfactory and used for structural analysis.

For a small increase in the dimension of damage-state vector, there is a correspondingly large increase in the order of transition matrix. For example, when the dimension n of $\mathbf{D}^i$ is increased to $n + 1$, the order of $\mathbf{T}(i)$ increases from $\Pi_{j=1}^{n} l_j = K$ to $\Pi_{j=1}^{n+1} l_j = l_{n+1}K$. This observation regarding rapid increase in computational involvement suggests the initial use of a Markov model for reduced degree-of-freedom models such as the shear beam idealization (Rahman 1991).

Evolution of Distribution of D^i

Consider a K-dimensional row vector that prescribes the joint probability mass function of the random vector $\mathbf{D}^i$ denoting damage after ith seismic event. The probability of $\mathbf{D}^i$ following i seismic events is (Parzen 1962; Rahman 1991)

$$\mathbf{P}(i) = \mathbf{P}(i - 1)\mathbf{T}(i) \qquad i = 1, 2, \ldots, N(\tau) \quad \text{(13)}$$

When this equation is used recursively, the distribution of probability of being in state C_p, $p = 1, 2, 3, \ldots, K$ after i seismic events becomes

$$\mathbf{P}(i) = \mathbf{P}(0) \prod_{j=1}^{i} \mathbf{T}(j) \quad \text{(14)}$$

where $\mathbf{P}(0)$ = initial vector representing probability distribution of $\mathbf{D}^0$. In general, (14) defines a nonstationary Markov Process due to differential severities of DCs. However, if one assumes independent and identically distributed random processes for earthquakes with same deterministic duration, the Markov process becomes stationary, and (14) takes the form

$$\mathbf{P}(i) = \mathbf{P}(0)\mathbf{T}^i \quad \text{(15)}$$

where the index j is dropped due to the invariance of transition matrix to severities of DCs.

Lifetime Distribution

The lifetime probability distribution $\mathbf{P}(\tau)$, defined as the distribution of damage-index vector $\mathbf{D}^{N(\tau)}$ in lifetime τ, can be obtained from the theorem of total probability

$$\mathbf{P}(\tau) = \sum_{i=0}^{\infty} \mathbf{P}(i)\text{Prob}[N(\tau) = i] \quad \text{(16a)}$$

$$\mathbf{P}(\tau) = \sum_{i=0}^{\infty} \mathbf{P}(i) \frac{(\lambda\tau)^i}{i!} \exp(-\lambda\tau) \quad \text{(16b)}$$

$$\mathbf{P}(\tau) \simeq \sum_{i=0}^{i^*} \mathbf{P}(i) \frac{(\lambda\tau)^i}{i!} \exp(-\lambda\tau) \quad \text{(16c)}$$

in which i^* = finite real integer to be determined from the observation that the i^*th component of the preceding summation in (16) is negligibly small. For stationary Markov process, a more compact form of lifetime distribution can be obtained as

$$\mathbf{P}(\tau) = \sum_{i=0}^{\infty} \mathbf{P}(0)\mathbf{T}^i \frac{(\lambda\tau)^i}{i!} \exp(-\lambda\tau) \quad \text{(17a)}$$

$$\mathbf{P}(\tau) = \mathbf{P}(0)\exp(-\lambda\tau) \sum_{i=0}^{\infty} \frac{(\lambda\tau\mathbf{T})^i}{i!} \quad \text{(17b)}$$

$$\mathbf{P}(\tau) = \mathbf{P}(0)\exp[-\lambda\tau(\mathbf{I} - \mathbf{T})] \quad \text{(17c)}$$

where $\mathbf{I}$ = K-dimensional identity matrix. Determination of preceding probability requires computation of $e^{\mathbf{U}}$ where $\mathbf{U} = -\lambda\tau(\mathbf{I} - \mathbf{T})$. Appendix I describes the evaluation procedures of linear algebra to calculate $e^{\mathbf{U}}$ for $\mathbf{U} \in \mathcal{L}(\Re^K \times \Re^K)$, where $\mathcal{L}(\Re^K \times \Re^K)$ denotes a set of linear mapping from $\Re^K$ to $\Re^K$.

Note that (15) and (17) are simplifed expressions of the event and the lifetime probabilities for stationary Markov process $\mathbf{D}^i$. This is true when the seismic events during lifetime of a structure are independent and identical random processes. This does not mean that the sample events (e.g., what actually happens in structures) are all identical; it only suggests that the probabilistic characteristics of these events are similar. To evaluate seismic performance due to one or more earthquakes, this assumption will be used in the numerical examples as an initial application of the proposed Markov model. When the probabilistic characteristics of these seismic events are different, more generic versions of the preceding equations, such as (14) and (16), should be used. In so doing, however, one will require enormous amount of data (which may be lacking) to quantitatively define each of these random processes during lifetime of a structure. They are not considered in this paper.

Mean First-Passage Time

Another quantity of engineering interest in seismic performance evaluation is the mean number of earthquakes before absorption to any undesirable damage state(s). Considering homogeneous Markov process with stationary transition probabilities, let $\mu_{\mathcal{A}}(p)$ denote the mean number of seismic events before the system enters a damage set $\mathcal{A} \subseteq \mathcal{D}$ with $\mathcal{A} \cup \mathcal{A}^c = \mathcal{D} \subseteq \Re^n$ (Fig. 2) if the initial damage state is $C_p \subseteq \mathcal{A}^c$. Then, the mean first-passage time is given by (Rahman and Grigoriu 1990a, 1990b; Rahman 1991)

$$\mu_{\mathcal{A}}(p) = E(\text{Absorption time}|\mathbf{D}_0 \in C_p) \quad (18a)$$

$$\mu_{\mathcal{A}}(p) = \sum_{C_q \in \mathcal{A}^c} E(\text{Absorption time}|\mathbf{D}^0 \in C_p, \mathbf{D}^1 \in C_q)$$

$$\cdot \operatorname{Prob}(\mathbf{D}^1 \in C_q|\mathbf{D}^0 \in C_p) \quad (18b)$$

$$\mu_{\mathcal{A}}(p) = 1 + \sum_{C_q \in \mathcal{A}^c} E(\text{Absorption time}|\text{initial state is } C_q)T_{pq} \quad (18c)$$

$$\mu_{\mathcal{A}}(p) = 1 + \sum_{C_q \in \mathcal{A}^c} \mu_{\mathcal{A}}(q)T_{pq} \quad (18d)$$

When the initial states are uncertain, the mean first-passage time can still be obtained from $\mu_{\mathcal{A}}(p)$ by averaging relative to the probability of $\mathbf{D}^0$. Let $\mu_{\mathcal{A}}$ represent the mean number of events that the system, starting at initial state $C_p \subseteq \mathcal{A}^c$ with probability $\operatorname{Prob}(\mathbf{D}^0 \in C_p)$, has to wait before absorption to damage set $\mathcal{A} \subseteq \mathcal{D}$. It is given by (Rahman 1991)

$$\mu_{\mathcal{A}} = \sum_{C_p \in \mathcal{A}^c} \mu_{\mathcal{A}}(p)\operatorname{Prob}(\mathbf{D}^0 \in C_p) \quad (19)$$

Numerical Example

Seismic Hazard

Consider two sites A and B in the western U.S. with mean earthquake-arrival rates $\lambda_A = 0.92$/year and $\lambda_B = 0.024$/year, respectively (Algermissen et al. 1982; Rahman and Grigoriu 1989; Rahman 1991). The sites are shown in Fig. 4. Site A is located in Riverside and San Diego counties, and site B is located mostly in Orange county in California. Both sites lie in the same seismic zone 4 of the *Uniform Building Code* (1988), and have the same value of $a_{10} = 0.4g$, which is defined as the 10% upper fractile of lifetime largest peak ground acceleration (Algermissen and Perkins 1976; Algermissen et al. 1982). It is assumed that the ground motions in different seismic events are (1) Independent and identically distributed zero-mean band-limited white stationary Gaussian processes $W(t)$, which has the one-sided power spectral density

$$G(\omega) = G_0, \qquad 0 < \omega < \bar{\omega} \quad (20a)$$

$$G(\omega) = 0, \qquad \text{otherwise} \quad (20b)$$

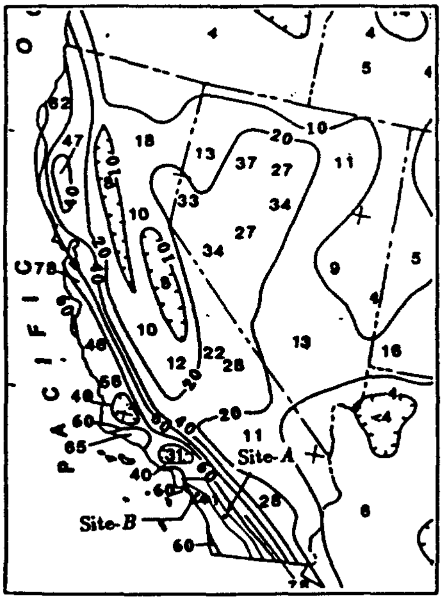

FIG. 4. Probabilistic Map of a_{10} for Western U.S.

with intensity G_0 and bandwidth $\bar{\omega}$; and (2) have the same deterministic strong motion duration T_s.

The distribution of the peak ground acceleration during a seismic event can be approximated by

$$F(w) \overset{\text{def}}{=} \text{Prob}\left[\max_{T_s}|W(t)| < w\right] \simeq \exp\left[-\frac{T_s}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{w^2}{2\lambda_0}\right)\right] \quad \ldots\ldots (21)$$

where $\lambda_k = \int_0^\infty \omega^k G(\omega)\, d\omega = k$th spectral moment of $W(t)$. Therefore, the cumulative-distribution function of the largest peak ground acceleration W_τ during a lifetime period τ is

$$F_\tau(w) = \exp[-\lambda\tau\{1 - F(w)\}] \quad \ldots\ldots (22)$$

From (21) and (22) with $\lambda_0 = G_0\bar{\omega}$ and $\lambda_2 = G_0\bar{\omega}^3/3$, and the condition $F_{\tau=50\text{yr}}(\omega = a_{10}) = 0.9$ (Rahman and Grigoriu 1989)

$$G_0 = \frac{a_{10}^2}{-2\bar{\omega}\ln\left[-\dfrac{\pi\sqrt{3}}{T_s\bar{\omega}}\ln\left(1 + \dfrac{\ln 0.9}{50\lambda}\right)\right]} \quad \ldots\ldots (23)$$

It is equal to 10,026 mm^2/s^3 when $\lambda = \lambda_A = 0.92$/year and 16,090 mm^2/s^3 when $\lambda = \lambda_B = 0.024$/year for a deterministic strong motion duration $T_s = 30\exp\{-3.254(a_{10}/g)^{0.35}\} = 2.83$ s and bandwidth $\bar{\omega} = 25\pi$ rad/s as proposed by Lai (1982). Sites A and B are characterized by frequent small seismic events and rare large earthquakes, respectively. However, designs at both sites are identical according to the *Uniform Building Code* (1988).

Structural System

Consider a special moment resisting framed structure (Rahman and Grigoriu 1989, 1990a), which is modeled here as a hysteretic, degrading Bouc-Wen oscillator (Bouc 1967; Wen 1976, 1980) with linear damping ratio $\zeta = 0.05$; initial natural frequency $\omega_0 = 20.944$ rad/s; mass m; and subjected to the ith seismic event $W^i(t) = W(t)$ giving the equation of motion

$$m\ddot{X}^i(t) + g^i(\{X^i(s), \dot{X}^i(s), 0 \le s \le t\}; t) = -mW(t) \quad (24)$$

where $X^i(t)$ = relative displacement of oscillator with respect to ground motion at time t during seismic event i. The total restoring force g^i is assumed to admit an additive decomposition of nonhysteretic component

$$g^i_{nh}(X^i(t), \dot{X}^i(t)) = 2\zeta\omega_0 m\dot{X}^i(t) + \alpha\omega_0^2 mX^i(t) \quad (25)$$

and hysteretic component

$$g^i_h(\{X^i(s), \dot{X}^i(s), 0 \le s \le t\}; t) = (1 - \alpha)\omega_0^2 mZ^i(t) \quad (26)$$

in which α quantifies the participation of linear restoring force; and the hysteretic variable $Z^i(t)$ satisfies the ordinary nonlinear differential equation (Bouc 1967; Wen 1980; 1976)

$$\dot{Z}^i(t) = A^i(t)\dot{X}^i(t) - \beta|\dot{X}^i(t)|\,|Z^i(t)|^{\mu-1}Z^i(t) - \gamma\dot{X}^i(t)|Z^i(t)|^{\mu} \quad (27)$$

in which the model parameters β, γ, and $A^i(t)$ govern the amplitude and shape of hysteretic loops; and the parameter μ controls the smoothness of transition from elastic to inelastic region. It is assumed here that μ, β, γ are constants, and $A^i(t)$, which also controls system degradation, has the following implicit time dependency through the dissipated hysteretic energy $\varepsilon(t)$ at local time t (Baber and Wen 1980)

$$A^i(t) = A^{i-1}(T_s) - \delta_A\varepsilon(t) \quad (28)$$

where δ_A signifies constant rate of system deterioration with $\varepsilon(t)$ satisfying the differential equation

$$\dot{\varepsilon}(t) = (1 - \alpha)\omega_0^2 mZ^i(t)\dot{X}^i(t) \quad (29)$$

The degradation law in (28) and (29) is defined here quite arbitrarily. It is obtained from one of the main choices available in the current literature. Further study with more realistic buildings need to be undertaken to make decisions regarding the proper selection of structural deterioration. The time-invariant parameters governing hysteresis are chosen as $\alpha = 0.04$, $\mu = 1$, $\beta = 0.1505$, and $\gamma = 0.1505$, consistent with the initial stiffness and strength values of the oscillator (Sues et al. 1988; Rahman 1991). Structural deterioration is permitted by assigning a small value of $\delta_A = 1.0 \times 10^{-6}$ in (28). The structural characteristics are assumed to be deterministic.

The state of structure is represented by $A^i = A^i(T_s) \in \Re$ denoting the value of parameter $A^i(t)$ of restoring force model at the end of ith seismic event. The corresponding normalized damage index $D^i = 1 - A^i/A^0$, which varies from 0 to 1, is discretized into $K = l_1 = 16$ distinct cells (states) of equal length 0.0625 (shown in Fig. 5). When this index is calibrated to the observed seismic damage in actual structures, each or group of these cells can be correlated with common engineering measures such as minor, medium, severe, reparable, nonreparable, collapse damage states, and so forth. Regardless, the discretized cells $C_1, C_2, \ldots, C_{16}$ in succession denote progressive states of structural damage. Since the damage is an irreversible

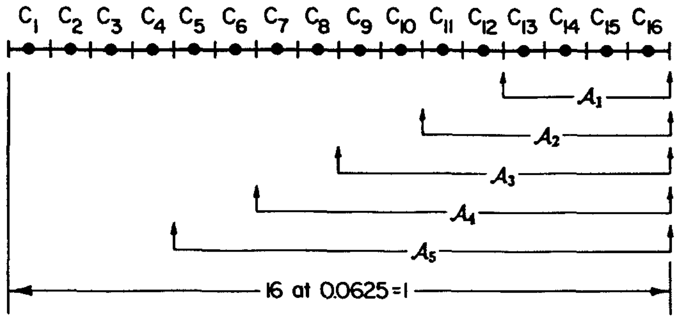

FIG. 5. Discretization of Sample Space of $D^i \in \Re$

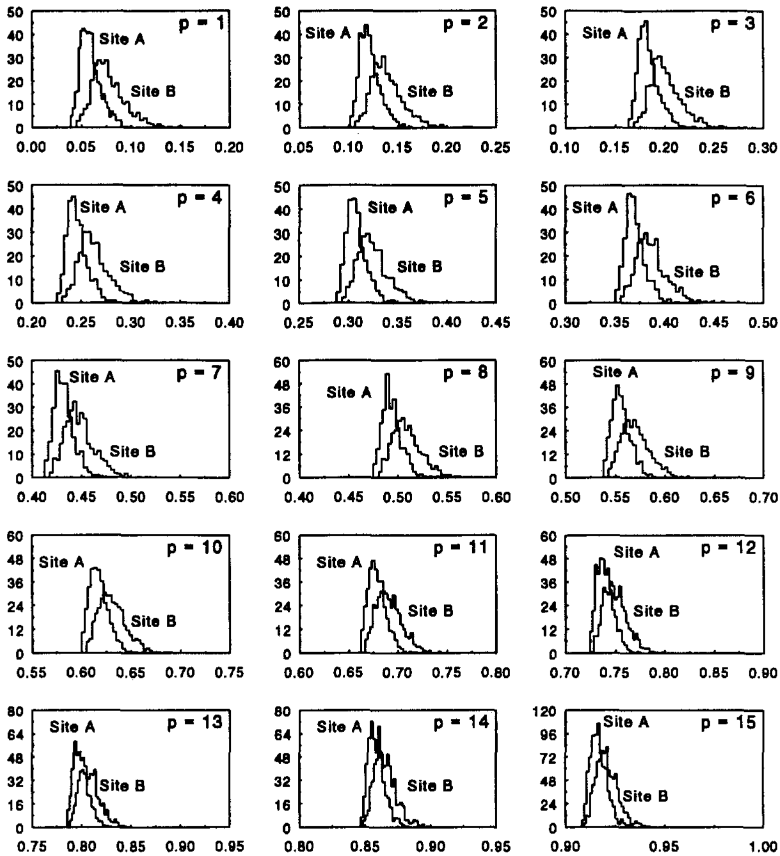

FIG. 6. Histogram of $D^i|D^{i-1} \in C_p, p$ = 1–15

process, after each seismic event without any subsequent repair, the structural state advances only to any of the higher numbered damage states, or it may remain in the same state. In other words, once $D^{i-1} \in C_p$, there is a zero probability that $D^i \in C_q$ when $q < p$ for all $p = 1, 2, \ldots, 16$.

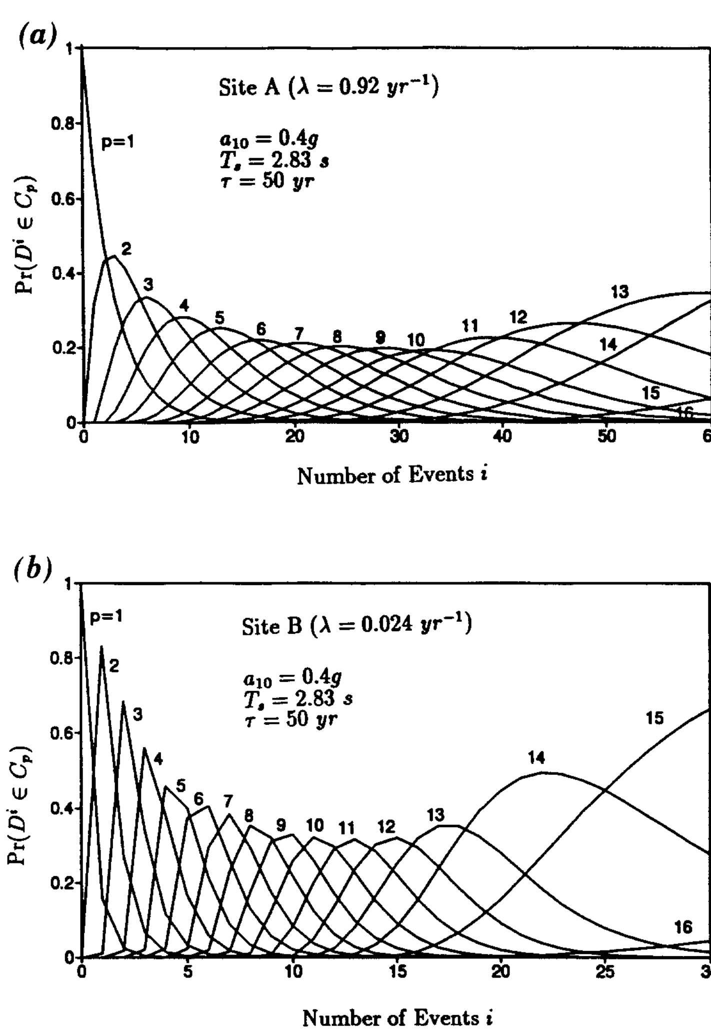

FIG. 7. Evolution of Damage Probability Belonging to State C_p with Deterministic Initial State: (*a*) Site A; (*b*) Site B

Specially for $p = 16$, i.e., for the cell C_{16}, which represents state of largest possible damage, if the damage process ever enters that state, the probability of remaining in that state becomes unity. It is known as the absorbing or trapping state, since once entered the process is never left.

Structural Response and Reliability

As mentioned previously, (24), (27), and (29) can be rewritten as a system of first-order ordinary differential equations analogous to (8) and (9). This nonlinear system of equations for the initial value problem is then solved by using step-by-step numerical integration. Fifth- and sixth-order explicit Runge-Kutta integrators are used to obtain such solutions.

The transition matrix **T** is constructed by performing several conditional

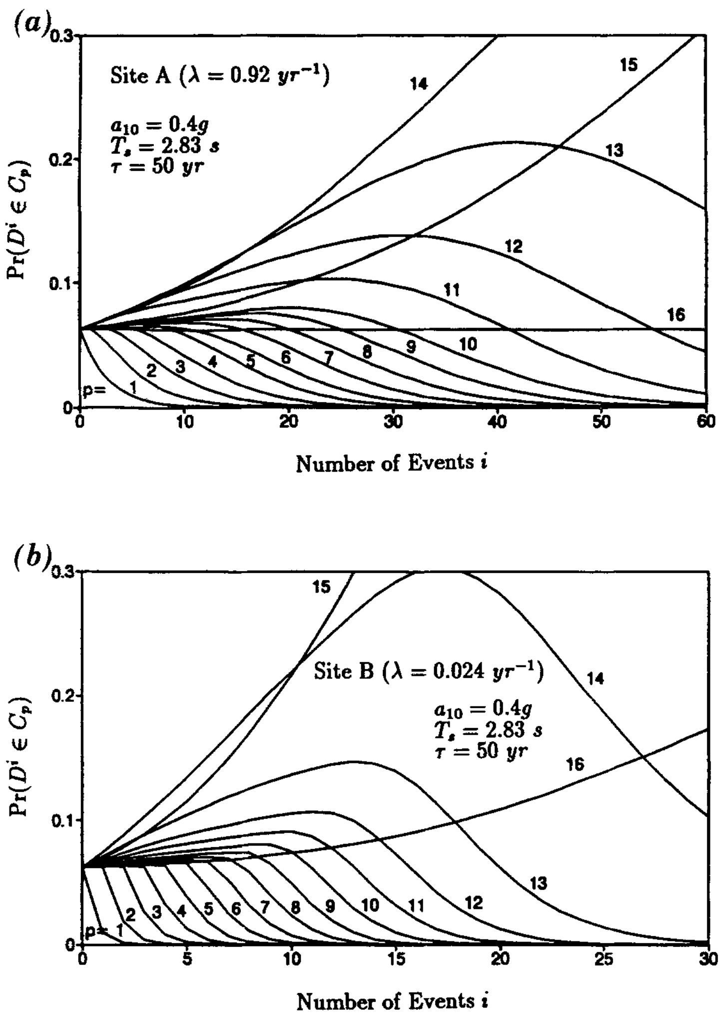

FIG. 8. Evolution of Damage Probability Belonging to State C_p with Uncertain Initial State: (*a*) Site A; (*b*) Site B

Monte Carlo simulations each with 1,000 samples. In brief, the effort in the simulation consists of the following three steps. First, the oscillator is preassigned a damage index (before seismic event i), which is associated with the damage state C_p. A representative value, such as the midpoint of the cell C_p, can be used to define this deterministic damage index. This also defines the initial value $A^i(0)$ of the degrading parameter $A^i(t)$ of the hysteretic model during the ith seismic event. Second, with the condition $D^{i-1} \in C_p$, 1,000 samples of random excitation representing the ith seismic event $W(t)$ are artificially generated. Third, 1,000 deterministic nonlinear dynamic analyses are carried out with the oscillator subjected to each of these realizations of $W(t)$. This generates 1,000 samples of conditional damage index $D^i|D^{i-1} \in C_p$ following seismic event i, from which its histogram can be

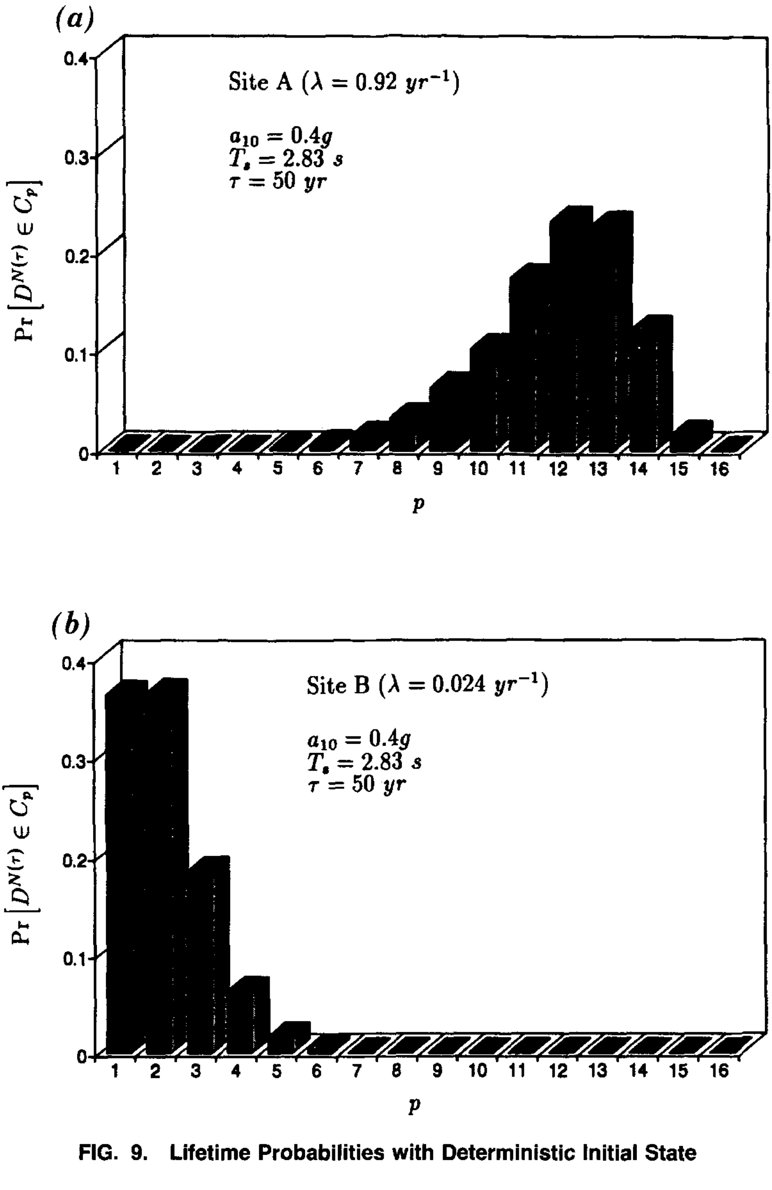

FIG. 9. Lifetime Probabilities with Deterministic Initial State

developed. Fig. 6 shows the histograms of $D^i | D^{i-1} \in C_p$ for the cells C_p, $p = 1, 2, \ldots, 15$ obtained for both the sites A and B. Due to larger spectral intensity G_0, the shapes of preceding histograms for site B exhibit more spread than those for site A. These histograms, which estimate the conditional probability densities, are used to construct the first 15 rows of corresponding transition matrix **T**. Since the cell C_{16} is absorbing state, the last row of the transition matrix is calculated by setting $T_{16,q} = 1$ for $q = 16$ and zero otherwise. Here, no repairs of structural systems are considered following each seismic event. This has the implication that **T** is an upper-

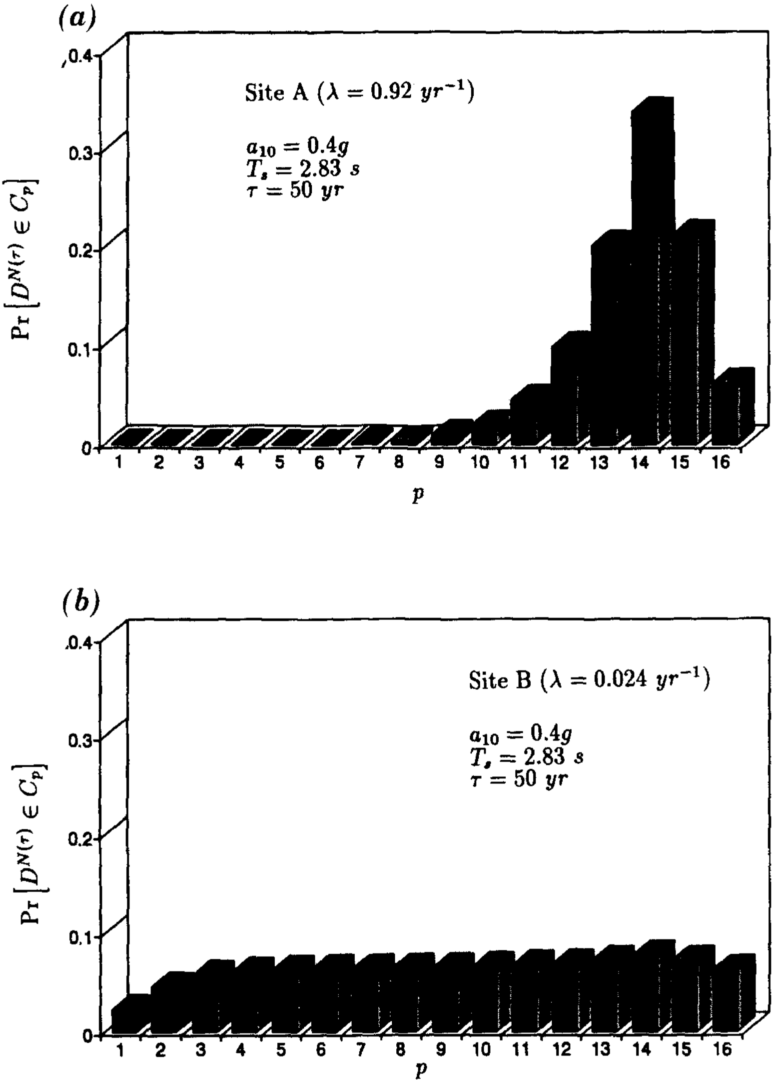

FIG. 10. Lifetime Probabilities with Uncertain Initial State

triangular 5matrix. In case there is a systematic maintenance program after each seismic event, the transition matrix will need to be modified based on inspection and repair methodologies.

The event distribution of damage, starting from any damaged state of system, can be obtained from the transition matrices described earlier. Fig. 7 shows the evolution of this distribution of D^i, with respect to seismic event i, according to (15) for both sites A and B starting with deterministic initial state $C_p = C_1$ of structural system [i.e., when $P_p(0)$ representing the pth component of $\mathbf{P}(0)$ is 1 for $p = 1$ and zero otherwise]. However, if the initial state is uncertain and particularly if it has uniform distribution with

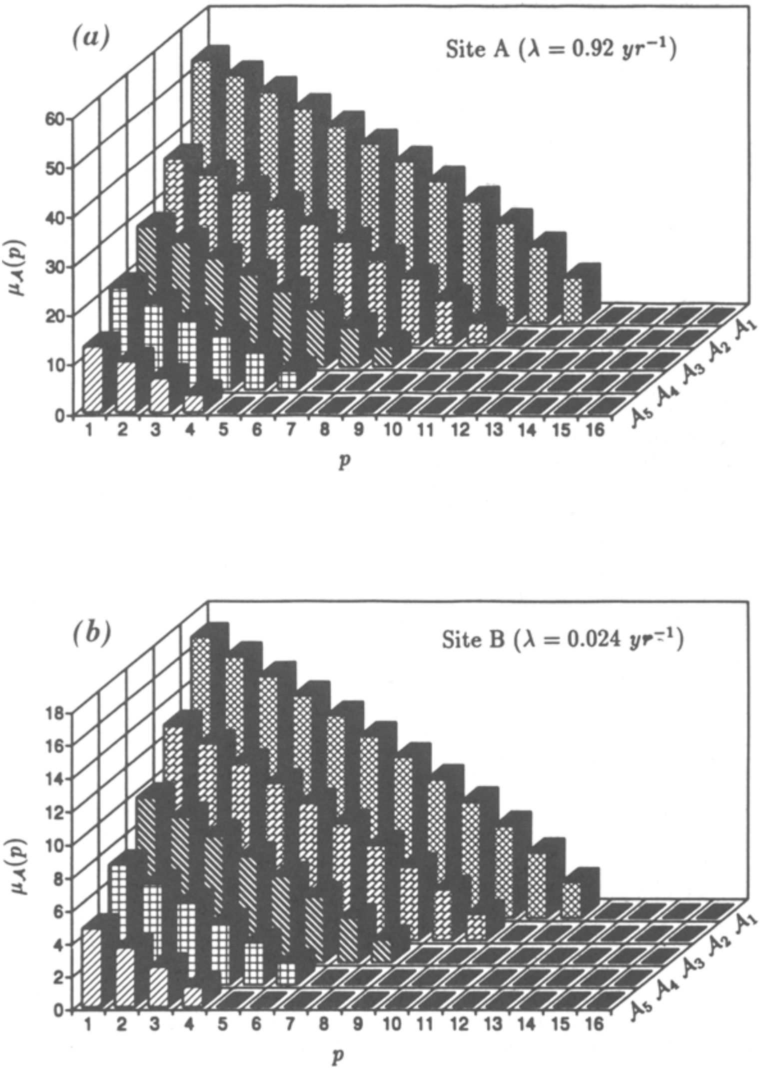

FIG. 11. Mean First-Passage Times with Deterministic Initial States

$P_p(0) = 1/16$ for all $p = 1, 2, \ldots, 16$, the same equation can be used to obtain the preceding evolution of damage probability $\mathbf{P}(i)$. Fig. 8 exhibits such probabilities for both sites A and B.

The lifetime probability distribution of damage after $N(\tau)$ seismic events are computed using (17) with the assumption of initially undamaged deterministic state of system, i.e., when $P_p(0) = 1$ for $p = 1$ and zero otherwise. Fig. 9 shows the lifetime probability mass function of $D^{N(\tau)}$ with $\tau = 50$ years for both sites A and B. Based on these case-specific studies, buildings at sites with infrequent large earthquakes appear to sustain less damage than those at sites with frequent small seismic events. Similar results were also found in Rahman and Grigoriu (1989) and Rahman (1991) for linear and nonlinear nondegrading models of structural systems. However, more studies need to be undertaken to make a generic conclusion.

Fig. 10 shows the lifetime probabilities for $\tau = 50$ years, starting from a uniform distribution of initial damage state for the sites A and B. Due to change in initial condition, the reliabilities can still be obtained directly from (17) and previous transition matrices. Results show that the uncertainty regarding initial condition can yield significant variation on seismic reliability estimates.

Consider several damage sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, and $\mathcal{A}_5$ ($\mathcal{A}_i = \cup_{p=15-2i}^{16} C_p$, $i = 1, 2, \ldots, 5$), which are defined in Fig. 5. These damage sets may represent collections of undesirable damage states, which may be prescribed for a specific design condition. The mean first-passage time providing the number of seismic events before absorption to these several sets of undesirable damage state(s) starting from any deterministic initial-damage state is exhibited in Fig. 11. For example, when site B is considered, if the deterministic initial state is C_4 (i.e., $p = 4$), the structure will require the following number of earthquakes on the average to enter the following damage sets $\mathcal{A}_1 = 13.4$, $\mathcal{A}_2 = 9.46$, $\mathcal{A}_3 = 6.42$, $\mathcal{A}_4 = 3.74$, and $\mathcal{A}_5 = 1.23$ (13.4 is the mean first-passage time for the damage set $\mathcal{A}_1$, and so on). They are computed from (18), and are obtained for both sites A and B. Due to a large difference in the mean arrival times of the two sites, the mean first-passage time for site A is found to be considerably higher than that for site B. When the initial state is uncertain, and the probability of D^0 is uniformly distributed among all states, the corresponding mean absorption times for the sites A and B can still be calculated from (18) and (19). They are found to be $\mathcal{A}_1 = 24.89$, $\mathcal{A}_2 = 13.78$, $\mathcal{A}_3 = 8.34$, $\mathcal{A}_4 = 4.59$, and $\mathcal{A}_5 = 2.09$ events for site A; and $\mathcal{A}_1 = 7.53$, $\mathcal{A}_2 = 4.69$, $\mathcal{A}_3 = 2.88$, $\mathcal{A}_4 = 1.62$, and $\mathcal{A}_5 = 0.75$ events for site B. All these results provide useful information to make decisions for optimal inspection and repair of structural systems.

The Markov model can also be applied to evaluate seismic performance of existing structures that have been exposed to past earthquakes. The analysis, however, requires calculation of transition matrix, which can be performed by two approaches. In the first approach, the transition probabilities can be computed explicitly by carrying out stochastic dynamic analysis of new structures, as done here. One can then use the same transition matrix with an appropriate initial state characterizing damage state of the existing structures. In the second approach, an estimation procedure can be developed by obtaining the preceding probabilities from a suitable data base involving observed performance of existing structures.

Conclusions

A new methodology based on a Markov model is proposed to evaluate seismic performance and sensitivity to the initial state of structural systems and determine the vulnerability of structures exposed to one or more earthquakes. The analysis accounts for simple but realistic characterization of seismic hazard, nonlinear dynamic analysis for estimating structural response, uncertainty in the initial state of structural systems, and failure conditions incorporating damage accumulation during consecutive seismic events.

The method is based on theoretical development using general hysteretic restoring force characteristics that can be applied to both reinforced concrete and steel structures. It estimates both event and lifetime reliabilities, thus providing a designer more control in seismic performance evaluation. It can be used to determine the damage-probability evolution during several earthquakes, allowing investigation on seismic vulnerability of new and existing structures. The model facilitates computation of mean first-passage time determining average number of seismic events before the structure will suffer potential damage. It also evaluates sensitivity of seismic reliability due to variability in the initial state of structural systems.

The Markov model developed in this paper has been applied to evaluate seismic reliability measures of simple code-designed structures. Results suggest that designs by the *Uniform Building Code* have different reliabilities at sites with frequent small earthquakes and infrequent large earthquakes, although the sites are characterized by the same value of a_{10}. Similar findings were also obtained in Rahman and Grigoriu (1989) when the reliabilities are calculated for nondegrading systems. However, more studies need to be undertaken to make a generic conclusion.

The uncertainty regarding initial condition can yield significant variation on seismic reliability. Since variability regarding initial conditions can play a significant role in seismic reliability estimate, it is essential that any reliability scheme has provisions of uncertain initial condition(s). Using the Markov structure, this is accomplished here with little effort.

A small increase in the dimension of damage-state vector representing state of structural systems is associated with a comparatively large increase in the order of transition matrix. Correspondingly, the computational involvement in obtaining transition probabilities may become significant.

Appendix I. Evaluation of $e^{\mathbf{U}}$

Consider a real $K \times K$ square matrix $\mathbf{U}$. A nonzero vector $\mathbf{x} \in \mathscr{C}^K$, satisfying the relation $\mathbf{Ux} = \Lambda \mathbf{x}$ for some scalar $\Lambda \in \mathscr{C}$ is called the right eigenvector of $\mathbf{U}$ with the associated eigenvalue Λ where $\mathscr{C}^K$ is K-dimensional complex vector space. When $\mathbf{xU} = \Lambda \mathbf{x}$, the vector $\mathbf{x}$ is known as the left eigenvector of $\mathbf{U}$. Suppose there are K linearly independent complete family $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(K)}$ of either right and left eigenvectors of $\mathbf{U}$. Then there exists linearly independent right eigenvectors $\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}, \ldots, \boldsymbol{\phi}^{(K)}$, and linearly independent left eigenvectors $\boldsymbol{\psi}^{(1)}, \boldsymbol{\psi}^{(2)}, \ldots, \boldsymbol{\psi}^{(K)}$, which satisfy the orthogonality condition

$$(\boldsymbol{\phi}^{(i)}, \boldsymbol{\psi}^{(j)}) \overset{\text{def}}{=} \sum_{k=1}^{K} \phi_{ik}\bar{\psi}_{jk} = \delta_{ij} \quad \text{(30)}$$

where $\boldsymbol{\phi}^{(i)} = \{\phi_{i1}, \phi_{i2}, \ldots, \phi_{iK}\}$; $\boldsymbol{\psi}^{(j)} = \{\psi_{j1}, \psi_{j2}, \ldots, \psi_{jK}\}$; $\bar{\psi}_{jk}$ = complex conjugate of ψ_{jk}; and δ_{ij} = Kronecker delta. Assume that $\Lambda_1, \Lambda_2, \ldots, \Lambda_K$ are the eigenvalues (which may not be distinct) corresponding to the eigenvectors $\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}, \ldots, \boldsymbol{\phi}^{(K)}$. Then the matrix $\mathbf{U}$ can be represented by

$$\mathbf{U} = \boldsymbol{\Phi\Lambda\Psi} \quad \text{(31)}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{K1} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{K2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{1K} & \phi_{2K} & \cdots & \phi_{KK} \end{bmatrix} \quad \text{(32a)}$$

$$\mathbf{\Psi} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1K} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{K1} & \psi_{K2} & \cdots & \psi_{KK} \end{bmatrix} \quad \text{(32}b\text{)}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \Lambda_K \end{bmatrix} \quad \text{(32}c\text{)}$$

From (30), it can be shown that

$$\mathbf{\Psi\Phi} = \mathbf{\Phi\Psi} = \mathbf{I} \quad (33)$$

where $\mathbf{I}$ is the K-dimensional identity matrix. This immediately gives

$$\mathbf{U}^m = \mathbf{\Phi\Lambda}^m\mathbf{\Psi} \quad (34)$$

with

$$\mathbf{\Lambda}^m = \begin{bmatrix} \Lambda_1^m & 0 & \cdots & 0 \\ 0 & \Lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \Lambda_K^m \end{bmatrix} \quad (35)$$

Consider now the expansion of $e^{\mathbf{U}}$ given by

$$e^{\mathbf{U}} = \sum_{m=0}^{\infty} \frac{\mathbf{U}^m}{m!} \quad (36)$$

which, when combined with (34) and (35), reduces to

$$e^{\mathbf{U}} = \sum_{m=0}^{\infty} \frac{\mathbf{\Phi\Lambda}^m\mathbf{\Psi}}{m!} \quad \text{(37}a\text{)}$$

$$e^{\mathbf{U}} = \mathbf{\Phi}\left(\sum_{m=0}^{\infty} \frac{\mathbf{\Lambda}^m}{m!}\right)\mathbf{\Psi} \quad \text{(37}b\text{)}$$

$$e^{\mathbf{U}} = \mathbf{\Phi}e^{\mathbf{\Lambda}}\mathbf{\Psi} \quad \text{(37}c\text{)}$$

where

$$e^{\mathbf{\Lambda}} = \begin{bmatrix} e^{\Lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\Lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\Lambda_K} \end{bmatrix} \quad (38)$$

Appendix II. References

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APPENDIX III. NOTATION

The following symbols are used in this paper:

$\mathbf{A}^i(t)$ = damage-state vector during seismic event i;
a_{10} = 10% upper fractile of lifetime maximum peak ground acceleration;
C_p = pth damage state (cell);
$\mathbf{c}$ = damping matrix;
$\mathbf{D}^i(t)$ = normalized damage-state vector during seismic event i;
$\mathbf{d}$ = vector of influence coefficients;
$F(w)$, $F_\tau(w)$ = event and lifetime distributions of peak ground acceleration;
$\mathbf{F}^i$ = vector function representing restoring forces;
$G(\omega)$ = one-sided spectral density;
G_0 = one-sided spectral intensity;
$\mathbf{g}^i$ = vector functional representing restoring forces;
$\mathbf{I}$ = K-dimensional identity matrix;
K = dimension of transition matrix; total number of discrete states;
$\mathbf{k}_{nh}^i$, $\mathbf{k}_h^i$ = nonhysteretic and hysteretic parts of stiffness matrix;
$\mathbf{m}$ = mass matrix;
$N(\tau)$ = number of seismic events during lifetime τ;
n = dimension of damage-state vector $\mathbf{D}^i(t)$ or $\mathbf{A}^i(t)$; $n_c \times n_p$;
n_c = number of critical components;
n_p = number of parameters of restoring force at each critical cross section;
$\mathbf{P}(i)$ = vector (row) of damage probabilities following seismic event i;
$\mathbf{P}(\tau)$ = vector (row) of damage probabilities during lifetime τ;
T_s = deterministic strong motion duration;
$\mathbf{T}(i)$ = one-step transition matrix for seismic event i;
t, t^i = time and random duration of ith seismic event;
$\mathbf{U}$ = real square matrix;
$W^i(t)$ = ith seismic event with strong motion duration t^i;
$\mathbf{X}^i(t)$, $\dot{\mathbf{X}}^i(t)$ = Generalized displacement and velocity vector during seismic event i;
$\mathbf{Z}^i(t)$ = vector of hysteretic variables during seismic event i;
α, β, γ, μ, δ_A = parameters of Bouc-Wen restoring force model;
δ_{ij} = Kronecker delta;
$\varepsilon(t)$ = dissipated hysteretic energy until time t;
ζ, ω_0 = damping ratio and initial natural frequency of an oscillator;
$\boldsymbol{\theta}^i(t)$ = response state vector during seismic event i; $[\mathbf{X}^i(t), \dot{\mathbf{X}}^i(t), \mathbf{Z}^i(t)]^T$;
Λ = eigenvalue of $\mathbf{U}$;
$\boldsymbol{\Lambda}$ = diag$(\Lambda_1, \Lambda_1, \ldots, \Lambda_K)$;
λ, λ_A, λ_B = mean rate of earthquake occurrence;
λ_i = ith spectral moment;
$\mu_{\mathcal{A}}$ = mean first passage time with uncertain initial state;
$\mu_{\mathcal{A}}(p)$ = mean first passage time with deterministic initial state C_p;

$\mathbf{\Phi}, \mathbf{\Psi}$ = matrix of right and left eigenvectors;
$\boldsymbol{\phi}^{(i)}, \boldsymbol{\psi}^{(i)}$ = ith right and ith left eigenvectors;
$\bar{\omega}$ = bandwidth of spectral density of white noise;
$\mathcal{A}_i$ = ith damage set; $\cup_{15-2i}^{16} C_p$;
$\mathcal{C}^n$ = n-dimensional complex vector space;
$\mathcal{D}$ = domain of $\mathbf{D}^i$;
$\mathcal{L}(\mathfrak{R}^n \times \mathfrak{R}^n)$ = set of linear mapping from $\mathfrak{R}^n$ to $\mathfrak{R}^n$; and
$\mathfrak{R}^n$ = n-dimensional real vector space.