A New Computational Method for Probabilistic Elastic-Plastic Fracture Analysis

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This paper presents a polynomial dimensional decomposition method for calculating the probability distributions of random crack-driving forces commonly encountered in elastic-plastic fracture analysis of ductile solids. The method involves a hierarchical decomposition of a multivariate function in terms of variables with increasing dimensions, a broad range of orthonormal polynomial bases consistent with the probability measure for Fourier-polynomial expansion of component functions, and an innovative dimension-reduction integration for calculating the expansion coefficients. Unlike the previous development, the new decomposition does not require sample points, yet it generates a convergent sequence of lower-variate estimates of the probability distributions of crack-driving forces. Numerical results, including the probability of fracture initiation of a through-walled-cracked pipe, indicate that the decomposition method developed provides accurate, convergent, and computationally efficient estimates of the probabilistic characteristics of the J-integral. [DOI: 10.1115/1.4000159]

Keywords: J-integral, stochastic fracture, polynomial dimensional decomposition, Fourier-polynomial expansion, pipe

1 Introduction

Probabilistic fracture mechanics (PFM) accounts for both mechanistic and statistical aspects of a crack-driving force, leading to the probabilistic characteristics of fracture initiation and growth of an existing crack, real or postulated, in an engineering structure. The most common methods employed in PFM are the first- and second-order reliability methods (FORM/SORM) [1,2] and simulation or sampling methods [3,4]. FORM/SORM are approximate methods and are based on linear or quadratic approximation of a performance function at the most probable point, which in turn requires first- or second-order response sensitivities or gradients. When response sensitivities are not available or a sensitivity analysis is computationally intensive, FORM/SORM are ineffective. In addition, for highly nonlinear performance functions, which exist in many fracture problems, FORM/SORM may produce inadequate reliability estimates. In contrast, sampling methods generally require a large number of simulations to calculate low failure probability and are impractical when each simulation involves expensive finite element calculations. Consequently, sampling methods have been traditionally employed as a last resort or for benchmarking approximate solutions.

Recently, the author developed a dimensional decomposition method for reliability analysis of general cracked structures subject to random loads, material properties, and crack geometry [5]. However, the existing decomposition method requires a reference point, commonly assumed to be the mean value of the random input and sample points surrounding that reference point. Based on these sample points, deterministic calculations of crack-driving forces, either exactly or numerically, are conducted to generate Lagrange interpolations of various component functions embedded in the decomposition. There are two weaknesses in this procedure. First, the decomposition constructed above depends on the selected reference point, which, if improperly selected, can spoil the approximation. Second, and more importantly, the sample points are vaguely selected with no strict guidelines. If an input variable is strictly positive or strictly negative, or follows a probability density with compact support, the resultant sample points may fall outside the physical domain; in this case, existing decomposition methods may produce unrealistic sample properties of a random crack-driving force. Therefore, alternative means of approximating the component functions by dropping the sample points altogether are desirable.

This paper presents a polynomial dimensional decomposition method for calculating the probability distributions of the J-integral frequently encountered in elastic-plastic fracture mechanics. The method is based on (1) a hierarchical decomposition of a multivariate function in terms of variables with increasing dimensions, (2) a broad range of orthonormal polynomial bases consistent with the probability measure for Fourier-polynomial expansion of component functions, and (3) an innovative dimension-reduction integration for calculating the expansion coefficients. Section 2 defines the random parameters and discusses their impact on propagating uncertainties to the J-integral. Section 3 describes the polynomial decomposition method, comprising multivariate function decomposition, Fourier-polynomial expansion, and dimension-reduction integration, and then discusses computational effort. Two numerical examples illustrate the accuracy, convergence, and computational efficiency of the proposed method in Sec. 4. Finally, the conclusions are drawn in Sec. 5.

2 Random Parameters and Fracture Response

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-field on Ω, and $P:\mathcal{F}\rightarrow[0,1]$ is a probability measure, and $\mathbb{R}^N$ be an N-dimensional real vector space. Defined on the probability space $(\Omega,\mathcal{F},P)$, let $\boldsymbol{X}=\{X_1,\cdots,X_N\}^T$ denote an N-dimensional input random vector, which characterizes statistical uncertainties of all relevant input parameters, including loads, material properties, and geometry. For example, if the crack length $2a$, tensile properties E, α_0, and m_0, and external loads $F_1,\cdots,F_M$ are stochastic variables in nonlinear fracture analysis of a homogeneous solid, then $\boldsymbol{X}=\{2a,E,\alpha_0,m_0,F_1,\cdots,F_M\}^T$. The input random vector $\boldsymbol{X}\in\mathbb{R}^N$ must be characterized by its joint probability density function

Contributed by the Pressure Vessel and Piping Division of ASME for publication in the JOURNAL OF PRESSURE VESSEL TECHNOLOGY. Manuscript received December 26, 2008; final manuscript received June 6, 2009; published online October 1, 2009. Assoc. Editor: G. E. Otto Widera.

$f_{\boldsymbol{X}}(\boldsymbol{x})$. However, in most practical applications, complete information required to derive the joint probability density may not be available. In this work, the ith random variable X_i on the probability triple $(\Omega_i, \mathcal{F}_i, P_i)$ is assumed to be independent with known marginal probability density $f_i(x_i)$, so that $f_{\boldsymbol{X}}(\boldsymbol{x}) = \Pi_{i=1}^{i=N} f_i(x_i)$.

Consider the J-integral for a crack tip in an elastic-plastic solid that can be calculated using standard finite element analysis (FEA) for a given input. Suppose that a failure is defined when the crack propagation is initiated at that crack tip, i.e., when $J > J_{Ic}$, where J_{Ic} is a relevant mode-I fracture toughness of the material measured in terms of the J-integral. This requirement cannot be satisfied with certainty since J is dependent on the input vector $\boldsymbol{X}$, which is random, and J_{Ic} itself may be a random variable. Hence, the performance of the cracked structure should be evaluated by the reliability or its complement, the probability of failure P_F, defined as the multifold integral

$$P_F := P[J(\boldsymbol{X}) > J_{Ic}(\boldsymbol{X})] := \int_{J_{Ic}(\boldsymbol{x}) - J(\boldsymbol{x}) < 0} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{1}$$

where $\boldsymbol{x}$ is a realization of $\boldsymbol{X}$. The evaluation of the multidimensional integral in Eq. (1), either analytically or numerically, is not possible because N is large, $\boldsymbol{X}$ is generally non-Gaussian, and $J(\boldsymbol{x})$ is a highly nonlinear function of $\boldsymbol{x}$. Crude Monte Carlo simulation is impractical for calculating small failure probabilities since the evaluation of $J(\boldsymbol{x})$ entails expensive FEA.

Equation (1) represents the probability of initiation of crack growth, which provides a conservative estimate of structural performance. A less conservative evaluation requires calculating the probability when crack growth, if occurs, is unstable. The latter probability, known as the probability of fracture instability, is more difficult to compute since it must be obtained by incorporating automatic crack growth simulation in a stochastic fracture mechanics analysis. In this paper, all probabilistic calculations are limited to fracture initiation, although any performance function describing input-output system behavior is applicable.

3 Polynomial Dimensional Decomposition

Let $J(\boldsymbol{X})$, a real-valued, measurable transformation on $(\Omega, \mathcal{F})$, define a relevant crack-driving force for a generic elastic-plastic fracture problem. In general, the multivariate function $J: \mathbb{R}^N \to \mathbb{R}$ is implicit, is not analytically available, and can only be viewed as a high-dimensional input-output mapping, where the evaluation of the output function J for a given input $\boldsymbol{x}$ requires expensive FEA. Therefore, methods employed in stochastic analysis must be capable of generating accurate probabilistic characteristics of $J(\boldsymbol{X})$ with an acceptably small number of output function evaluations.

3.1 Multivariate Function Decomposition. Consider a function $J(\boldsymbol{x})$ that depends on $\boldsymbol{x} = \{x_1, \cdots, x_N\}^T \in \mathbb{R}^N$. The dimensional decomposition of $J(\boldsymbol{x})$, also known as analysis of variance [6] or high-dimensional model representation [7,8], represents a finite, hierarchical, convergent expansion of [9]

$$J(\boldsymbol{x}) = J_0 + \sum_{i=1}^{N} J_i(x_i) + \sum_{i_1, i_2 = 1; i_1 < i_2}^{N} J_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1, \cdots, i_S = 1; i_1 < \cdots < i_S}^{N} J_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S}) + \cdots + J_{12 \cdots N}(x_1, \cdots, x_N) \tag{2}$$

in terms of input variables with increasing dimensions, where J_0 is a constant, $J_i(x_i)$ is a univariate component function representing individual contribution to $J(\boldsymbol{x})$ by input variable x_i acting alone, $J_{i_1 i_2}(x_{i_1}, x_{i_2})$ is a bivariate component function describing the cooperative influence of two input variables J_{i_1} and J_{i_2}, $J_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S})$ is an S-variate component function quantifying the cooperative effects of S input variables $x_{i_1}, \cdots, x_{i_S}$, and so on. The last term in Eq. (2) represents any residual dependence of all input variables cooperatively locked together to affect the output function J. If

$$\widetilde{J}_S(\boldsymbol{x}) = J_0 + \sum_{i=1}^{N} J_i(x_i) + \sum_{i_1, i_2 = 1; i_1 < i_2}^{N} J_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1, \cdots, i_S = 1; i_1 < \cdots < i_S}^{N} J_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S}) \tag{3}$$

represents a general S-variate approximation of $J(\boldsymbol{x})$, the univariate $(S=1)$ and bivariate $(S=2)$ approximations, $\widetilde{J}_1(\boldsymbol{x})$ and $\widetilde{J}_2(\boldsymbol{x})$, respectively, provide the two- and three-term approximants of the finite decomposition in Eq. (2). Similarly, trivariate, quadrivariate, and other higher-variate approximations can be derived by appropriately selecting the value of S. The fundamental conjecture underlying this decomposition is that component functions arising in the function decomposition will exhibit insignificant S-variate effects cooperatively when $S \to N$, leading to useful lower-variate approximations of $J(\boldsymbol{x})$. When $S=N$, $\widetilde{J}_S(\boldsymbol{x})$ converges to the exact function $J(\boldsymbol{x})$. In other words, Eq. (3) generates a hierarchical and convergent sequence of approximations of $J(\boldsymbol{x})$.

3.2 Fourier-Polynomial Expansion. Let $\mathcal{L}_2(\Omega_i, \mathcal{F}_i, P_i)$ be a Hilbert space that is equipped with a set of complete orthonormal bases $\{\psi_{ij}(x_i);\ j = 0, 1, \cdots\}$, which is consistent with the probability measure of X_i. For example, classical orthonormal polynomials, including Hermite, Legendre, and Jacobi polynomials, can be used when X_i follows Gaussian, uniform, and Beta probability distributions, respectively [10]. Defined on the product probability triple $(\times_{k=1}^{k=S} \Omega_{i_k}, \times_{k=1}^{k=S} \mathcal{F}_{i_k}, \times_{k=1}^{k=S} P_{i_k})$, denote the space of square integrable S-variate component functions of J by

$$\mathcal{L}_2(\times_{k=1}^{k=S} \Omega_{i_k}, \times_{k=1}^{k=S} \mathcal{F}_{i_k}, \times_{k=1}^{k=S} P_{i_k}) := \left\{ J_{i_1 \cdots i_S}(X_{i_1}, \cdots, X_{i_S}) : \int_{\mathbb{R}^S} J_{i_1 \cdots i_S}^2(x_{i_1}, \cdots, x_{i_S}) \times \prod_{k=1}^{S} f_{i_k}(x_{i_k}) dx_{i_k} < \infty \right\} \tag{4}$$

which is also a Hilbert space. Since the joint probability density of $\{X_{i_1}, \cdots, X_{i_S}\}^T$ is separable (independent), the tensor product $\{\Pi_{k=1}^{S} \psi_{i_k j_k}(x_{i_k})\}$ constitutes an orthonormal polynomial basis in $\mathcal{L}_2(\times_{k=1}^{k=S} \Omega_{i_k}, \times_{k=1}^{k=S} \mathcal{F}_{i_k}, \times_{k=1}^{k=S} P_{i_k})$. Therefore, there exists a Fourier-polynomial expansion

$$J_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S}) = \sum_{j_S = 1}^{\infty} \cdots \sum_{j_1 = 1}^{\infty} C_{i_1 \cdots i_S j_1 \cdots j_S} \prod_{k=1}^{S} \psi_{i_k j_k}(x_{i_k}) \tag{5}$$

with

$$C_{i_1 \cdots i_S j_1 \cdots j_S} = \int_{\mathbb{R}^S} J_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S}) \prod_{k=1}^{S} \psi_{i_k j_k}(x_{i_k}) f_{i_k}(x_{i_k}) dx_{i_k} \tag{6}$$

representing the expansion coefficient for the S-variate component function. By minimizing an error functional associated with a given $J(\boldsymbol{x})$ and the joint probability density of $\{X_{i_1}, \cdots, X_{i_S}\}^T$, the coefficients J_0 and $C_{i_1 \cdots i_S j_1 \cdots j_S}$ can be expressed by N-dimensional integrals [9]

$$J_0 = \int_{\mathbb{R}^N} J(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{7}$$

and

$$C_{i_1\cdots i_S j_1\cdots j_S} = \int_{\mathbb{R}^N} J(\boldsymbol{x}) \prod_{k=1}^{S} \psi_{i_k j_k}(x_{i_k}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{8}$$

Since the right side of Eq. (5) is an infinite series, it must be truncated, say, by m-terms in each variable, yielding a Fourier-polynomial approximation

$$J_{i_1\cdots i_S}(x_{i_1},\cdots,x_{i_S}) \cong \sum_{j_S=1}^{m} \cdots \sum_{j_1=1}^{m} C_{i_1\cdots i_S j_1\cdots j_S} \prod_{k=1}^{S} \psi_{i_k j_k}(x_{i_k}) \tag{9}$$

which approaches $J_{i_1\cdots i_S}(x_{i_1},\cdots,x_{i_S})$ in Eq. (5) in the mean square sense as $m\to\infty$.

The Fourier-polynomial approximation is valid for any finite-dimensional Hilbert space $\mathcal{L}_2(\times_{k=1}^{k=S}\Omega_{i_k}, \times_{k=1}^{k=S}\mathcal{F}_{i_k}, \times_{k=1}^{k=S}P_{i_k})$ with $1 \le S \le N$. In other words, Eq. (9) can represent all component functions of the multivariate function decomposition in Eq. (2). In particular, when $S=1$ and 2, Eq. (9) reduces to

$$J_i(x_i) \cong \sum_{j=1}^{m} \alpha_{ij}\psi_{ij}(x_i) \tag{10}$$

and

$$J_{i_1 i_2}(x_{i_1},x_{i_2}) \cong \sum_{j_2=1}^{m}\sum_{j_1=1}^{m} \beta_{i_1 i_2 j_1 j_2}\psi_{i_1 j_1}(x_{i_1})\psi_{i_2 j_2}(x_{i_2}) \tag{11}$$

where

$$\alpha_{ij} := \int_{\mathbb{R}^N} J(\boldsymbol{x})\psi_{ij}(x_i) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{12}$$

and

$$\beta_{i_1 i_2 j_1 j_2} := \int_{\mathbb{R}^N} J(\boldsymbol{x})\psi_{i_1 j_1}(x_{i_1})\psi_{i_2 j_2}(x_{i_2}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{13}$$

are the corresponding expansion coefficients. Applying Eqs. (9)–(13) into an S-variate approximation of Eq. (2) yields

$$\begin{aligned}\tilde{J}_S(\boldsymbol{X}) \cong J_0 &+ \sum_{i=1}^{N}\sum_{j=1}^{m} \alpha_{ij}\psi_{ij}(X_i) \\ &+ \sum_{i_1,i_2=1;i_1<i_2}^{N}\sum_{j_2=1}^{m}\sum_{j_1=1}^{m} \beta_{i_1 i_2 j_1 j_2}\psi_{i_1 j_1}(X_{i_1})\psi_{i_2 j_2}(X_{i_2}) + \cdots \\ &+ \sum_{i_1,\cdots,i_S=1;i_1<\cdots<i_S}^{N}\sum_{j_S=1}^{m}\cdots\sum_{j_1=1}^{m} C_{i_1\cdots i_S j_1\cdots j_S}\prod_{k=1}^{S}\psi_{i_k j_k}(X_{i_k})\end{aligned} \tag{14}$$

which, for $S=N$, converges to $J(\boldsymbol{X})$ in the mean square sense as $m\to\infty$. Once the embedded coefficients J_0, α_{ij}, $\beta_{i_1 i_2 j_1 j_2}$, and $C_{i_1\cdots i_S j_1\cdots j_S}$ are calculated, as described in Sec. 3.3, Eq. (14) furnishes an approximate but explicit map $\tilde{J}_S:\mathbb{R}^N\to\mathbb{R}$ that can be viewed as a surrogate of the exact map $J:\mathbb{R}^N\to\mathbb{R}$, which describes the input-output relation from a complex fracture mechanics simulation. Therefore, any probabilistic characteristic of $J(\boldsymbol{X})$, including its statistical moments and probability distribution, can be easily estimated by performing Monte Carlo simulation of $\tilde{J}_S(\boldsymbol{X})$ rather than of $J(\boldsymbol{X})$. The simulation of $\tilde{J}_S(\boldsymbol{X})$, which entails evaluation of simple analytical functions, can be performed for an arbitrarily large sample size. In contrast, the simulation of $J(\boldsymbol{X})$, referred to as crude Monte Carlo simulation in this paper, requires expensive numerical calculations and can therefore be prohibitive when estimating small probabilities.

3.3 Dimension-Reduction Integration for Calculating Expansion Coefficients. The determination of the expansion coefficients, which involve N-dimensional integrals over $\mathbb{R}^N$, is computationally prohibitive when N is large. Instead, a dimension-reduction integration, presented as follows, can be applied to estimate the coefficients efficiently.

Let $\boldsymbol{c}=\{c_1,\cdots,c_N\}^T$ be the mean value of $\boldsymbol{X}$ and $J(c_1,\cdots,c_{k_1-1},x_{k_1},c_{k_1+1},\cdots,c_{k_{R-k}-1},x_{k_{R-k}},c_{k_{R-k}+1},\cdots,c_N)$ represent an $(R-k)$th dimensional component function of $J(\boldsymbol{x})$, where $S\le R<N$ and $k=0,\cdots,R$. For example, when $R=1$, the zero-dimensional component function, which is constant, is $J(\boldsymbol{c})$, and the one-dimensional component functions are $J(x_1,c_2,\cdots,c_N)$, $J(c_1,x_2,\cdots,c_N)$, …, $J(c_1,c_2,\cdots,x_N)$. Using the multivariate function theorem of Xu and Rahman [11], it can be shown that a special R-variate approximation of $J(\boldsymbol{x})$, defined by

$$\begin{aligned}\hat{J}_R(\boldsymbol{x}) := \sum_{k=0}^{R}(-1)^k\binom{N-R+k-1}{k} &\times \sum_{k_1,\cdots,k_{R-k}=1;k_1<\cdots<k_{R-k}}^{N} \\ &\times J(c_1,\cdots,c_{k_1-1},x_{k_1},c_{k_1+1},\cdots,c_{k_{R-k}-1},x_{k_{R-k}},c_{k_{R-k}+1},\cdots,c_N)\end{aligned} \tag{15}$$

consists of all terms of the Taylor series of $J(\boldsymbol{x})$ that have less than or equal to R variables. The expanded form of Eq. (15), when compared with the Taylor expansion of $J(\boldsymbol{x})$, reveals that the residual error in $\hat{J}_R(\boldsymbol{x})$ includes terms of dimensions $R+1$ and higher. All higher-order R- and lower-variate terms of $J(\boldsymbol{x})$ are included in Eq. (15), which should therefore generally provide a higher-order approximation of a multivariate function than equations derived from first- or second-order Taylor expansions. Therefore, for $R<N$, an N-dimensional integral can be efficiently estimated by at most R-dimensional integrations, if the contributions from terms of dimensions $R+1$ and higher are negligible.

Substituting $J(\boldsymbol{x})$ in Eqs. (7) and (8) by $\hat{J}_R(\boldsymbol{x})$, the coefficients can be estimated from

$$J_0 \cong \sum_{k=0}^{R}(-1)^k\binom{N-R+k-1}{k} \times \sum_{k_1,\cdots,k_{R-k}=1;k_1<\cdots<k_{R-k}}^{N} \times \int_{\mathbb{R}^{R-k}} J(c_1,\cdots,c_{k_1-1},x_{k_1},c_{k_1+1},\cdots,c_{k_{R-k}-1},x_{k_{R-k}},c_{k_{R-k}+1},\cdots,c_N)\prod_{s=1}^{R-k} f_{k_s}(x_{k_s})dx_{k_s} \tag{16}$$

and

$$\begin{aligned}C_{i_1\cdots i_S j_1\cdots j_S} \cong \sum_{k=0}^{R}(-1)^k\binom{N-R+k-1}{k} &\times \sum_{k_1,\cdots,k_{R-k}=1;k_1<\cdots<k_{R-k}}^{N} \times \int_{\mathbb{R}^{R-k}} J(c_1,\cdots,c_{k_1-1},x_{k_1},c_{k_1+1},\cdots,c_{k_{R-k}-1},x_{k_{R-k}},c_{k_{R-k}+1},\cdots,c_N) \\ &\times \prod_{s=1}^{S}\psi_{i_s j_s}(x_{i_s})\prod_{s=1}^{R-k} f_{k_s}(x_{k_s})dx_{k_s}\end{aligned} \tag{17}$$

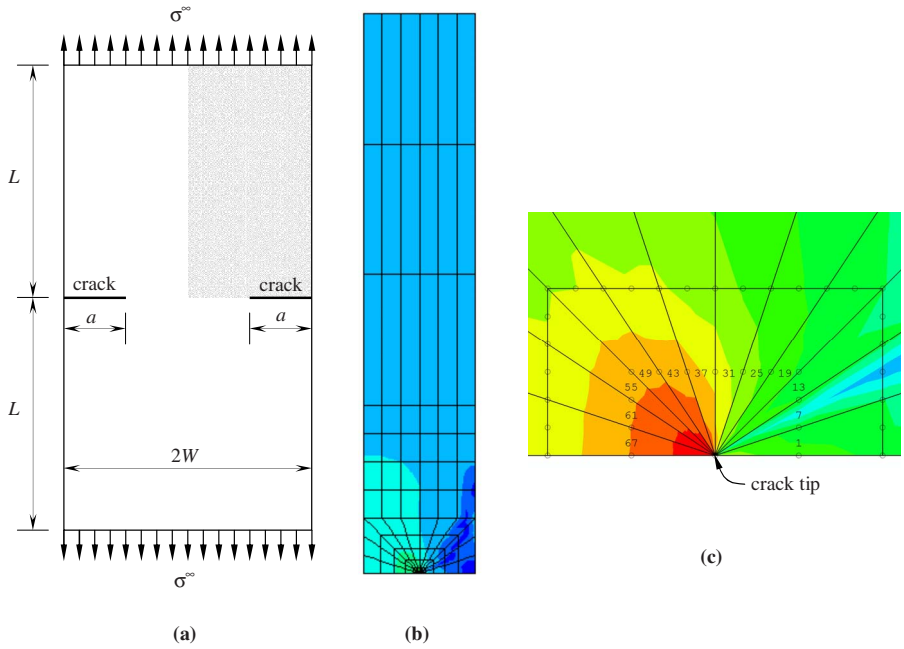

Fig. 1 A DE(T) specimen: (*a*) geometry and loads; (*b*) finite element mesh at mean crack length; and (*c*) singular elements at the crack tip

which require evaluating at most R-dimensional integrals. The proposed equations (Eqs. (16) and (17)) are substantially simpler and more efficient than performing one N-dimensional integration, as in Eqs. (7) and (8), particularly when $R \ll N$. Hence, the computational effort in calculating the coefficients is significantly lowered using the dimension-reduction integration. When $R=1$ or 2, Eqs. (16) and (17) involve one- and at most two-dimensional integrations, respectively. Nonetheless, numerical integration is still required for a general function J. The integration nodes and associated weights, which depend on the probability distribution of X_i, can be obtained from Gauss quadrature rules. In performing the dimension-reduction integration, the value of R should be selected in such a way that it is either equal to or greater than the value of S, which defines the truncation of Eq. (2). Then all expansion coefficients of S- or lower-variate approximations of $J(\boldsymbol{x})$ will have nontrivial solutions [9].

3.4 Computational Effort. The S-variate approximation in the polynomial decomposition method requires evaluation of the deterministic coefficients J_0 and $C_{i_1 \cdots i_S j_1 \cdots j_S}$. If these coefficients are estimated by at most R-dimensional $(R \geq S \geq 1)$ numerical integration with an n-point quadrature rule in Eqs. (16) and (17), the following deterministic responses (FEA) are required: $J(\boldsymbol{c})$, $J(c_1, \cdots, c_{k_1-1}, x_{k_1}^{(j_1)}, c_{k_1+1}, \cdots, c_{k_R-1}, x_{k_R}^{(j_R)}, c_{k_R+1}, \cdots, c_N)$ for $k_1, \cdots, k_R = 1, \cdots, N$ and $j_1, \cdots, j_R = 1, \cdots, n$, where the superscripts on variables indicate the corresponding integration points. Therefore, the total cost for an S-variate polynomial dimensional decomposition entails a maximum of $\Sigma_{k=0}^{k=R} \binom{N}{R-k} n^{R-k}$ J-integral evaluations and is, therefore, polynomic with respect to N or n.

If the integration points include a common point in each coordinate x_i, as in Sec. 4, the numbers of J-integral evaluations reduce to $\Sigma_{k=0}^{k=R} \binom{N}{R-k} (n-1)^{R-k}$. In the latter case, for example, the univariate $(S=R=1)$ and bivariate $(S=R=2)$ approximations require $(n-1)N+1$ and $N(N-1)(n-1)^2/2+(n-1)N+1$ J-integral evaluations, respectively.

4 Numerical Examples

Two numerical examples involving elastic-plastic fracture mechanics analysis of cracked structures are presented to illustrate the polyomial dimensional decomposition method. Whenever possible, comparisons have been made with crude Monte Carlo simulation to evaluate the accuracy and efficiency of the proposed method. For the non-Gaussian random input, all original random variables were transformed into standard Gaussian random variables, employing Hermite orthonormal polynomials as bases and the Gauss–Hermite quadrature rule for calculating the expansion coefficients. The expansion coefficients were estimated by dimension-reduction integration with $R=S$, so that an S-variate decomposition method requires at most S-variate numerical integration. In Example 1, the sample sizes for crude Monte Carlo simulation and the embedded Monte Carlo simulation of the decomposition method are 5000 and 10^6, respectively. The sample size in the decomposition method varies from 10^6 to 10^7 in Example 2. The polynomial order m varies, depending on the example, but in all cases the number of integration points $n=m+1$.

4.1 Example 1: A Double-Edged-Notched Tension Specimen. Consider a double-edged-notched tension (DE(T)) specimen with width $2W=1.016$ m (40 in.), length $2L=5.08$ m (200 in.), and random crack length a, subject to a random far-field tensile stress σ^∞, as shown in Fig. 1(*a*). The nonlinear-elastic constitutive equation under small-displacement condition is the well-known Ramberg–Osgood relation [12]

$$\epsilon_{ij} = \frac{1+\nu}{E} s_{ij} + \frac{1-2\nu}{3E} \sigma_{kk} \delta_{ij} + \frac{3}{2E} \alpha_0 \left(\frac{\sigma_e}{\sigma_0} \right)^{m_0-1} s_{ij} \qquad (18)$$

where σ_{ij} and ϵ_{ij} are the stress and strain components, respectively, E is the Young's modulus, ν is the Poisson's ratio, σ_0 is the reference stress, α_0 is a dimensionless material coefficient, m_0 is the strain hardening exponent, δ_{ij} is the Kronecker delta, $s_{ij} := \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ is the deviatoric stress, and $\sigma_e := \sqrt{(3/2) s_{ij} s_{ij}}$ is the von Mises equivalent stress. For nonlinear-elastic cracked structures, the J-integral uniquely defines the asymptotic crack tip stress and strain fields [12]. The crack length a, material constants E, α_0, and m_0, and far-field stress σ^∞ were treated as five independent random variables with their statistical properties listed in Table 1. The remaining deterministic parameters are as follows: $\sigma_0=154.78$ MPa (22.450 psi) and $\nu=0.3$. Due to the double-symmetry in the DE(T) problem, Fig. 1(*b*) shows a finite element

Table 1 Statistical properties of random input for DE(T) specimen

Random variable	Mean	Coefficient of variation	Probability distribution
Crack length a (cm) (in.)	25.4 (10)	0.1	Lognormal
Elastic modulus E (GPa) (psi)	206.83 (30×10^6)	0.05	Gaussian
Ramberg–Osgood coefficient α_0	10	0.1	Lognormal
Ramberg–Osgood exponent m_0	5	0.1	Lognormal
Far-field tensile stress σ^∞, (MPa) (psi)	124.1 (18,000)	0.05	Gaussian

mesh at mean crack length of the quarter-cylinder model. A total of 114 elements and 393 nodes were used in the mesh. Both plane stress and plane strain conditions were studied. Second-order elements from the ABAQUS (version 6.8) [13] element library were employed. Focused singular elements were deployed in the vicinity of the crack tip (Fig. 1(*c*)). A 2×2 Gaussian integration rule was employed in the FEA.

Since the J-integral, in general, is nonpolynomial, a convergence study with respect to S and m is required to calculate its probability characteristics accurately. Figures 2(*a*) and 2(*b*) present the probability densities of $J(\boldsymbol{X})$ in a plane stress condition by the proposed univariate ($S=1$) and bivariate ($S=2$) polynomial decomposition methods, respectively, for $m=3$, 4, and 5. The decomposition methods, which entail explicit forms of univariate or bivariate approximation in Eq. (14), permit inexpensive calculation of the J-integral by sidestepping additional FEA. Hence, an arbitrarily large sample size of the embedded Monte Carlo, such as 10^6 in this particular example, was selected to obtain the probability densities of J by the decomposition methods. Figures 2(*a*)

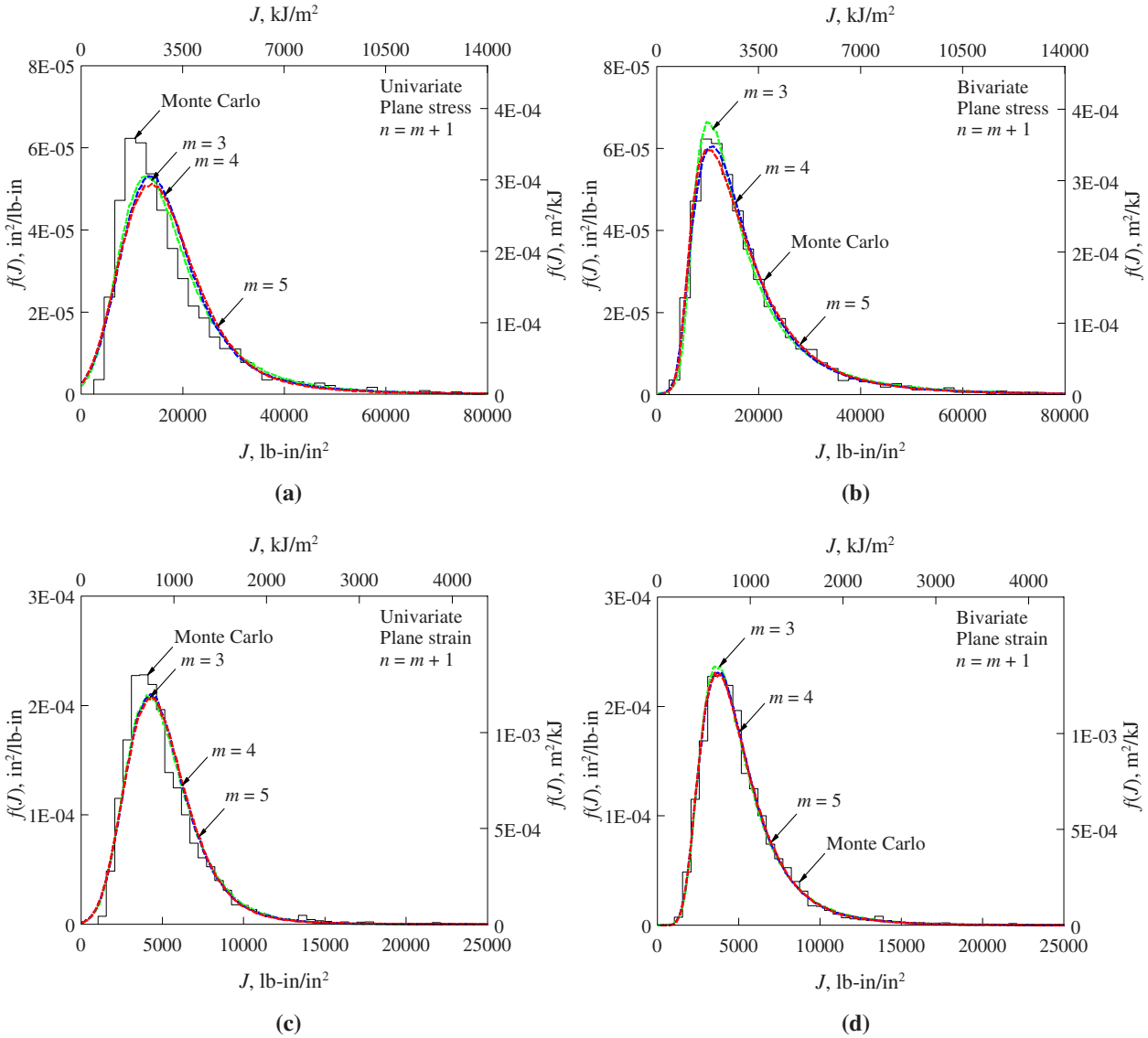

Fig. 2 Probability density of the *J*-integral for a DE(T) specimen: (*a*) univariate solution for plane stress; (*b*) bivariate solution for plane stress; (*c*) univariate solution for plane strain; and (*d*) bivariate solution for plane strain

Table 2 Numbers of FEA by decomposition methods for DE(T) specimen ($N=5$)

m	n	Univariate method[a]	Bivariate method[b]
3	4	16	106
4	5	21	181
5	6	26	276

[a] $(n-1)N+1$
[b] $N(N-1)(n-1)^2/2+(n-1)N+1$

and 2(b) also include corresponding histograms generated by crude Monte Carlo simulation from 5000 FEA (samples), providing benchmark solutions. The probability densities from both decomposition methods converge when m reaches four or five, but the Monte Carlo results are better predicted by the bivariate method than by the univariate method. The errors in the univariate method are primarily due to the absence of cooperative terms in the univariate approximation. Similar probabilistic analyses under a plane strain condition, leading to the results in Figs. 2(c) and 2(d), reveal the same qualitative trend, except that the J-integral values for plane strain are much lower than that in plane stress [5].

Table 2 compares the numbers of FEA required by the proposed decomposition methods for either plane stress or plane strain condition when $3 \leq m \leq 5$ and $n=m+1$. The computational effort by the univariate or bivariate method increases linearly or quadratically with respect to m or n. Clearly, the univariate method is inexpensive, but it is not as accurate as the bivariate method. Nonetheless, both versions of the decomposition method are far less expensive than crude Monte Carlo simulation. Note that the sample size of 5000 FEA in the Monte Carlo simulation was deemed adequate for providing rough estimates of the histograms. A significantly larger sample size will be required for obtaining their tail characteristics, which is not practical for an FEA-based probabilistic analysis.

4.2 Example 2: A Through-Walled-Cracked Pipe. The final example involves a circumferential, through-walled-cracked (TWC) pipe, which is subjected to a four-point bending, as shown in Fig. 3(a). The pipe has a midthickness radius $R_m=50.8$ mm, a wall thickness $t=5.08$ mm, and a symmetrically centered through-wall crack with the normalized crack angle $\theta/\pi=1/8$. The outer span $L_o=1.5$ m and the inner span $L_i=0.6$ m. The cross-sectional geometry at the cracked section is shown in Fig. 3(b). The pipe material is an ASTM Type 304 stainless steel, which follows the Ramberg–Osgood constitutive law defined by Eq. (18). Table 3 lists the means, coefficients of variation, and probability distributions of the tensile parameters (E, α_0, m_0), four-point bending load (F), and fracture toughness (J_{Ic}). All random variables are statistically independent. Also, $\sigma_0=154.78$ MPa and $\nu=0.3$. A finite element mesh of the quarter-

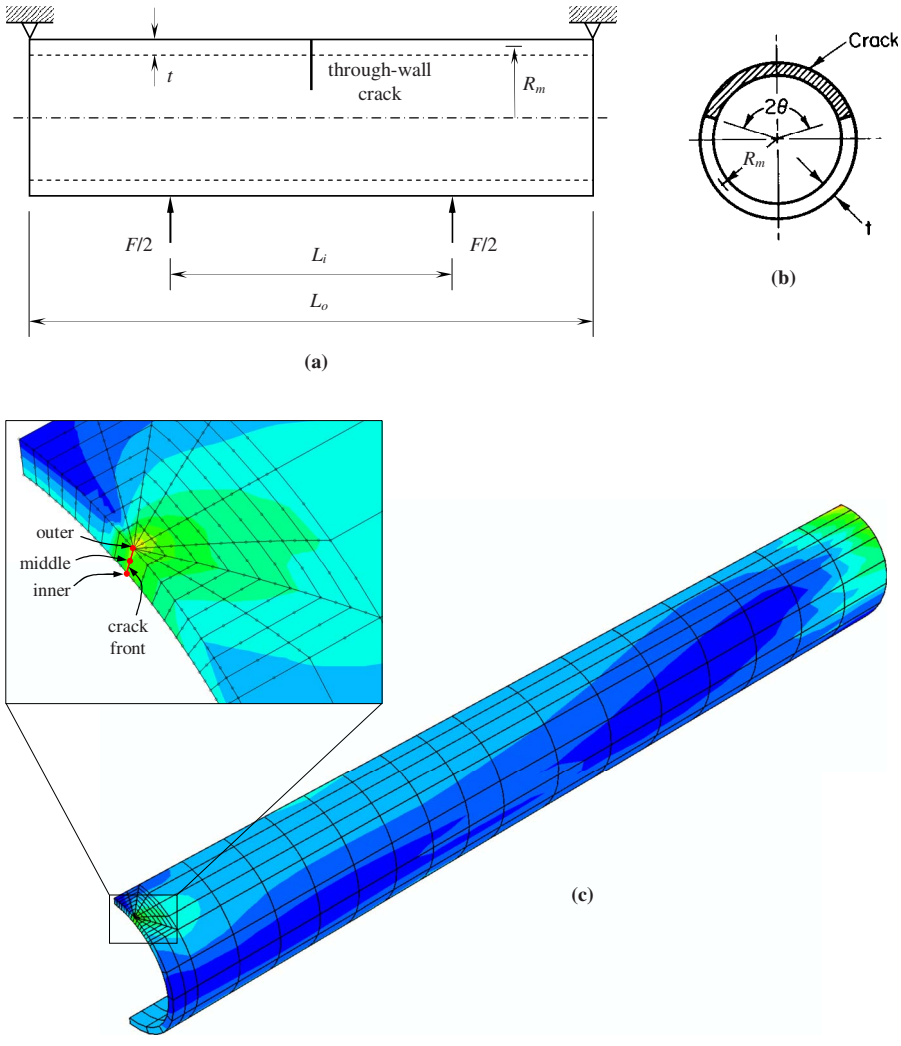

Fig. 3 A TWC pipe under four-point bending: (*a*) geometry and loads; (*b*) cracked cross section; and (*c*) finite element mesh

Table 3 Statistical properties of random input for through-walled-cracked pipe

Random variable	Mean	Coefficient of variation	Probability distribution
Elastic modulus E (GPa)	182.7	0.1	Gaussian
Ramberg–Osgood coefficient α_0	8.073	0.439	Lognormal
Ramberg–Osgood exponent m_0	3.8	0.146	Lognormal
Four-point bending load F (kN)	μ_F[a]	0.1	Gaussian
Initiation toughness J_{Ic} (kJ/m^2)	1242.6	0.47	Lognormal

[a]μ_F varies as 23.2 kN, 25.6 kN, 28 kN, 30.4 kN, 33.6 kN, 36 kN, 38.4 kN, 41.6 kN, 45.6 kN, 56 kN, and 64 kN.

cylinder model, consisting of 236 elements and 1805 nodes, is shown in Fig. 3(*c*). Twenty noded isoparametric solid elements from the ABAQUS library [13] were used, with focused singular elements at the crack tip.

The objective of this example is to calculate the probability of fracture initiation of the TWC pipe, defined by Eq. (1), where $J(\boldsymbol{X})$ is calculated at the midthickness level. The problem was solved for 11 distinct values of the mean applied load μ_F as follows: 23.2 kN, 25.6 kN, 28 kN, 30.4 kN, 33.6 kN, 36 kN, 38.4 kN, 41.6 kN, 45.6 kN, 56 kN, and 64 kN, where the standard deviation $\sigma_F=0.1\mu_F$ in all load cases. The smaller mean values were selected to produce low probabilities, posing a significant challenge to any computational method. The proposed univariate ($S=R=1$) and bivariate ($S=R=2$) decomposition methods, involving $m=4$, $n=5$, and 10^6-10^7 samples (depending on P_F) of the embedded Monte Carlo, were employed to calculate the respective failure probabilities, which are presented in Fig. 4 as scatter plots. Since the J-integral depends solely on the first four random variables in Table 3, only $(5-1)4+1=17$ and $4(4-1)(5-1)^2/2+(5-1)4+1=113$ ABAQUS-aided FEA were required by the univariate and bivariate methods, respectively. Due to expensive FEA, crude Monte Carlo simulation was infeasible to verify the low probabilities in this example. Instead, a J-estimation based Monte Carlo simulation [14] (solid line) involving 10^4-10^7 samples (depending on P_F) was performed, the results of which are displayed in Fig. 4. The proposed decomposition methods, particularly the bivariate version, provide excellent estimates of the probability of fracture initiation of the TWC pipe examined. It is worth noting that any uncertainty or error, if it exists, in the J-estimation scheme and its impact on the failure probability were not examined in this study.

The decomposition method is not only accurate, but also computationally efficient. For instance, when $\mu_F=23.2$ kN, the bivariate decomposition method yields a failure probability estimate of 5.2×10^{-6}, requiring only 113 FEA. In contrast, crude Monte Carlo simulation, if it can be performed, will require $10/5.2\times10^{-6}\cong2\times10^6$ FEA to calculate this low probability with a 30% coefficient of variation in the estimator. The computational savings by the proposed method for calculating low probabilities is obvious.

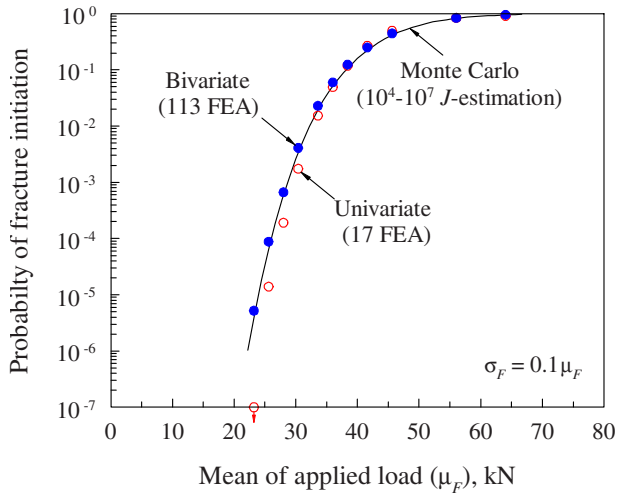

Fig. 4 Probability of fracture initiation of a TWC pipe

In both examples, the expansion coefficients for univariate and bivariate approximations were obtained by calculating the J-integrals at selected deterministic input defined by the integration points. Therefore, the proposed method is nonintrusive, as it can be easily adapted to solving complex, stochastic fracture problems requiring external commercial codes. The method developed is expected to solve practical problems both accurately and economically.

5 Conclusions

A polynomial dimensional decomposition method was developed for calculating the probability distributions of random crack-driving forces frequently encountered in elastic-plastic fracture mechanics. The method is based on a hierarchical decomposition of a multivariate function in terms of variables with increasing dimensions, a broad range of orthonormal polynomial bases consistent with the probability measure for Fourier-polynomial expansion of component functions, and an innovative dimension-reduction integration for calculating the expansion coefficients. Compared with the previous development, the new decomposition method does not require sample points around the mean input to approximate the component functions. Instead, orthogonal polynomial basis functions in the Hilbert space, such as Hermite polynomials, were employed, yielding the Fourier-polynomial expansion of the component functions. Due to nonintrusive evaluation of the expansion coefficients, the method can be easily adapted to solving complex, stochastic fracture problems requiring external deterministic codes.

Two elastic-plastic fracture mechanics problems were examined to evaluate the probability distributions of the J-integrals, including the reliability of a through-walled-cracked pipe. The results indicate that the polynomial dimensional decomposition method developed, particularly the bivariate version, provides accurate and convergent estimates of the distributions of the crack-driving forces. The computational effort by the univariate method varies linearly with respect to the number of random variables or the number of integration points; therefore, the univariate method is economical. In contrast, the bivariate method, which is generally superior to the univariate method, demands a quadratic cost scaling, making the former method more expensive than the latter method. Nonetheless, both versions of the decomposition method are significantly more economical than crude Monte Carlo simulation.

Acknowledgment

The author would like to acknowledge financial support from the U.S. National Science Foundation under Grant Nos. CMS-0409463 and CMMI-0653279.

Nomenclature

a = length of cracks in DE(T) specimen
c = mean of $\boldsymbol{X}$
$f_i(x_i)$ = marginal probability density of X_i
$f_X(x)$ = joint probability density of $\boldsymbol{X}$
m = degree of orthonormal polynomial basis
m_0 = Ramberg–Osgood material exponent
n = number of Gauss quadrature points
t = pipe thickness
x_i, $\boldsymbol{x}$ = realizations of X_i and $\boldsymbol{X}$
$C_{i_1\cdots i_S j_1\cdots j_S}$ = coefficient for S-variate component function of J
E = Young's modulus
F = applied load on pipe
J = J-integral
J_{Ic} = mode-I fracture toughness at plane strain
J_0 = first coefficient of polynomial decomposition of J
$J_{i_1\cdots i_S}$ = S-variate component function of J
$\tilde{J}_S(\boldsymbol{X})$ = S-variate polynomial approximation of J
$\hat{J}_R(\boldsymbol{X})$ = R-variate approximation of J
$2L$ = length of DE(T) specimen
L_i, L_o = inner and outer spans of pipe
N = number of random variables
P_i, P = probability measures of X_i and $\boldsymbol{X}$
R = positive integer ranging from S to N
R_m = mean radius of pipe
$\mathbb{R}^N$ = N-dimensional real vector space
S = positive integer ranging from 1 to N
$2W$ = width of DE(T) specimen
$\boldsymbol{X}$ = N-dimensional input random vector
X_i = ith input random variable
α_0 = Ramberg–Osgood material coefficient
α_{ij} = coefficient for univariate component function of J
$\beta_{i_1 i_2 j_1 j_2}$ = coefficient for bivariate component function of J
ϵ_{ij} = strain component
μ_F = mean of pipe load
ν = Poisson's ratio
θ = half of crack angle in pipe
σ_e = von Mises equivalent stress
σ_F = standard deviation of pipe load
σ_0 = Ramberg–Osgood reference stress
σ_{ij} = stress component
σ^∞ = far-field tensile stress on DE(T) specimen
$\psi_{ij}(x_i)$ = jth univariate polynomial basis for ith variable
Ω_i, Ω = sample spaces of X_i and $\boldsymbol{X}$
$\mathcal{F}_i$, $\mathcal{F}$ = σ-fields on Ω_i and Ω
$\mathcal{L}_2$ = Hilbert space

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