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Stochastic design optimization accounting for structural and distributional design variables

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Abstract

Purpose – This paper aims to present a new method, named as augmented polynomial dimensional decomposition (PDD) method, for robust design optimization (RDO) and reliability-based design optimization (RBDO) subject to mixed design variables comprising both distributional and structural design variables.

Design/methodology/approach – The method involves a new augmented PDD of a high-dimensional stochastic response for statistical moments and reliability analyses; an integration of the augmented PDD, score functions, and finite-difference approximation for calculating the sensitivities of the first two moments and the failure probability with respect to distributional and structural design variables; and standard gradient-based optimization algorithms.

Findings – New closed-form formulae are presented for the design sensitivities of moments that are simultaneously determined along with the moments. A finite-difference approximation integrated with the embedded Monte Carlo simulation of the augmented PDD is put forward for design sensitivities of the failure probability.

Originality/value – In conjunction with the multi-point, single-step design process, the new method provides an efficient means to solve a general stochastic design problem entailing mixed design variables with a large design space. Numerical results, including a three-hole bracket design, indicate that the proposed methods provide accurate and computationally efficient sensitivity estimates and optimal solutions for RDO and RBDO problems.

Keywords Augmented polynomial dimensional decomposition, Distributional design variables, Stochastic design optimization, Structural design variables

Paper type Research paper

1. Introduction

Robust design optimization (RDO) (Taguchi, 1993; Chen *et al.*, 1996; Du and Chen, 2000; Mourelatos and Liang, 2006; Zaman *et al.*, 2011; Park *et al.*, 2006) and reliability-based design optimization (RBDO) (Enevoldsen and Sørensen, 1994; Kuschel and Rackwitz, 1997; Tu *et al.*, 1999; Chiralaksanakul and Mahadevan, 2005; Agarwal and Renaud, 2006; Liang *et al.*, 2007; Du and Chen, 2004; Rahman and Wei, 2008) are two important prototypes for solving engineering design problems in the presence of uncertainty, as manifested by probabilistic descriptions of the objective and constraint functions. Intended for reducing the variability of the system performance, RDO minimizes the propagation of input uncertainty to output responses of interest, leading to an insensitive design. In contrast, RBDO aims to find an optimal design with low probabilities of failure corresponding to

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some critical failure mechanisms. With new formulations and methods appearing almost every year, RDO and RBDO, in conjunction with finite-element analysis (FEA), are becoming increasingly relevant and perhaps necessary for design of aerospace, civil, microelectronics and automotive structures and systems (Gu *et al.*, 2013; Sun *et al.*, 2018; Sun *et al.*, 2017; Sun *et al.*, 2017; Zhang *et al.*, 2018).

In engineering design, the design variables can be grouped into two principal classes:

(1) distributional design variables; and

(2) structural design variables.

A distributional design variable can be any distribution parameter or a statistic – for instance, the mean and standard deviation – of one or more random variables describing the performance function of a complex system. In contrast, a structural design variable can be any deterministic parameter of the performance function. For solving a general RDO/RBDO problem, not only the distributional design variables but also the structural design variables should be considered. A design problem simultaneously accounting for both classes of design variables is referred to as the mixed design variable problem in this paper. However, much of the existing research, whether in conjunction with RDO or RBDO, focuses strictly on one of the two classes of design variables. For example, the existing design optimization methods, such as the Taylor series or perturbation expansions (Huang and Du, 2007), the point estimate method (Huang and Du, 2007), polynomial chaos expansion (Wang and Kim, 2006), the tensor-product quadrature rule (Lee *et al.*, 2009), meta-model and kriging (Sun *et al.*, 2018; Zhao *et al.*, 2011; Sun *et al.*, 2011) and dimension-reduction methods (Lee *et al.*, 2009; Lee *et al.*, 2008) for RDO, and the first-order reliability method (FORM) or FORM-based methods (Enevoldsen and Sørensen, 1994; Kuschel and Rackwitz, 1997; Tu *et al.*, 1999; Chiralaksanakul and Mahadevan, 2005; Agarwal and Renaud, 2006; Liang *et al.*, 2007; Du and Chen, 2004; Rahman and Wei, 2008) and decomposition-based methods (Rahman and Wei, 2008; Lee *et al.*, 2008; Lee *et al.*, 2012) for RBDO, are all concentrated on solely distributional design variables. More recently, the polynomial dimensional decomposition (PDD) (Rahman, 2008; Rahman, 2009), derived from the ANOVA dimensional decomposition (Efron and Stein, 1981), was developed to furnish accurate RDO/RBDO solutions to high-dimensional problems. The associated RDO/RBDO algorithms (Ren and Rahman, 2013; Ren *et al.*, 2016) are based on PDD-based stochastic analysis (Rahman, 2009; Rahman and Ren, 2014), which integrates PDD and score function to determine stochastic design sensitivities concurrently from a single stochastic simulation or analysis. The algorithms also facilitate a multi-point, single-step design process, affording the ability to solve industrial-scale design problems. However, these relatively newer methods are also limited to solving RDO/RBDO problems involving distributional design variables only. Indeed, there is a lack of unified frameworks for tackling stochastic design optimization problems in the presence of both distributional and structural design variables. Therefore, the work described in this paper delves into a general stochastic design optimization involving mixed design variables.

This paper presents a new method for RDO and RBDO involving both distributional and structural design variables. The method comprises:

- a new augmented PDD of a high-dimensional stochastic response for statistical moment and reliability analyses;
- new formulations for design sensitivity analysis of the first two moments, which integrate not only the score functions but also the derivatives of orthonormal basis functions for the sensitivity with respect to structural design variables;

- finite-difference approximations integrating the augmented PDD for calculating the sensitivities of the failure probability with respect to both distributional and structural design variables; and
- standard gradient-based optimization algorithms, encompassing a multi-point, single-step design process.

Section 2 formally defines general RDO and RBDO problems involving mixed design variables, including their concomitant mathematical statements. Section 3 introduces the augmented PDD and its truncation in terms of both input random variables and new random variables affiliated with the distributional and structural design variables. The section also explains how the truncated augmented PDD leads to stochastic analysis consisting of analytical formulae for evaluating the first two moments and the embedded Monte Carlo simulation (MCS) for reliability analysis. Section 4 demonstrates that the effort required to calculate statistical moments or failure probability also delivers their design sensitivities. Section 5 introduces a multi-point, single-step iterative scheme for RDO and RBDO and elucidates how the stochastic analysis and design sensitivities are integrated with a gradient-based optimization algorithm. Section 6 presents four numerical examples involving mathematical functions or solid-mechanics problems and contrasts the accuracy and computational efforts of the proposed methods for sensitivity analysis of moments and reliability as well as solutions of two RDO/RBDO problems, all entailing mixed design variables. Finally, the conclusions are drawn in Section 7.

2. Design under uncertainty

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}_0^+$ represent the sets of positive integer (natural), non-negative integer, real and non-negative real numbers, respectively. For $k \in \mathbb{N}$, denote by $\mathbb{R}^k$ the k-dimensional Euclidean space and by $\mathbb{N}_0^k$ the k-dimensional multi-index space. These standard notations will be used throughout the forthcoming sections.

Consider a measurable space $(\boldsymbol{\Omega}_{\mathbf{d}}, \mathcal{F}_{\mathbf{d}})$, where $\boldsymbol{\Omega}_{\mathbf{d}}$ is a sample space and $\mathcal{F}_{\mathbf{d}}$ is a σ-field on $\boldsymbol{\Omega}_{\mathbf{d}}$. For $M \in \mathbb{N}$ and $N \in \mathbb{N}$, let $\mathbf{d}_T = (\mathbf{d}, \mathbf{s}) = \left(d_1, \ldots, d_{M_d}, s_1, \ldots, s_{M_s}\right)^T \in \mathcal{D}$ be an $\mathbb{R}^M$-valued design vector with non-empty closed set $\mathcal{D} \subseteq \mathbb{R}^M$, where $M_d, M_s \in \mathbb{N}$ and $M = M_d + M_s$, and let $\mathbf{X} := (X_1, \ldots, X_N)^T : (\boldsymbol{\Omega}_{\mathbf{d}}, \mathcal{F}_{\mathbf{d}}) \to \left(\mathbb{R}^N, \mathcal{B}^N\right)$ be an $\mathbb{R}^N$-valued input random vector with $\mathcal{B}^N$ representing the Borel σ-field on $\mathbb{R}^N$, describing the statistical uncertainties in loads, material properties and geometry of a complex mechanical system. The design variables are grouped into two major classes:

(1) distributional design vector $\mathbf{d}$ with dimensionality M_d; and

(2) structural design vector $\mathbf{s}$ with dimensionality M_s.

A distributional design variable d_k, $k = 1, \ldots, M_d$, can be any distribution parameter or a statistic – for instance, the mean and standard deviation – of one or more random variables. A structural design variable s_p, $p = 1, \ldots, M_s$, can be any deterministic parameter of a performance function. Defined over $(\boldsymbol{\Omega}_{\mathbf{d}}, \mathcal{F}_{\mathbf{d}})$, let $\{P_{\mathbf{d}} : \mathcal{F} \to [0, 1]\}$ be a family of probability measures. The probability law of $\mathbf{X}$ is completely defined by a family of the joint probability density functions (PDF) $\{f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}),\ \mathbf{x} \in \mathbb{R}^N,\ \mathbf{d} \in \mathcal{D}\}$ that are associated with corresponding probability measures $\left\{P_{\mathbf{d}}, \mathbf{d} \in \mathbb{R}^{M_d}\right\}$, so that the probability triple $(\boldsymbol{\Omega}_{\mathbf{d}}, \mathcal{F}_{\mathbf{d}}, P_{\mathbf{d}})$ of $\mathbf{X}$ depends on $\mathbf{d}$.

Let $y_l(\mathbf{X}; \mathbf{d}, \mathbf{s})$, $l = 0,1,2, \ldots, K$, be a collection of $K + 1$ real-valued, square-integrable, measurable transformations on $(\boldsymbol{\Omega}_{\mathbf{d}}, \mathcal{F}_{\mathbf{d}})$, describing relevant geometry (e.g., length, area, volume, mass) and performance functions of a complex system. The function

$y_l : (\mathbb{R}^N, \mathcal{B}^N) \to (\mathbb{R}, \mathcal{B})$ in general is not only an explicit function of distributional and structural design variables $\mathbf{d}$ and $\mathbf{s}$ but also implicitly depends on distributional design variables $\mathbf{d}$ via the probability law of $\mathbf{X}$. There exist two prominent variants of design optimization under uncertainty, described as follows.

2.1 Robust design optimization

The mathematical formulation of a general RDO problem involving an objective function $c_0 : \mathbb{R}^M \to \mathbb{R}$ and constraint functions $c_l : \mathbb{R}^M \to \mathbb{R}$, where $l = 1, \ldots, K$ and $1 \leq K < \infty$, requires one to:

$$\begin{aligned}
&\min_{(\mathbf{d},\mathbf{s})\in\mathcal{D}\subseteq\mathbb{R}^M} \; c_0(\mathbf{d},\mathbf{s}) := w_1 \frac{\mathbb{E}_{\mathbf{d}}[y_0(\mathbf{X};\mathbf{d},\mathbf{s})]}{\mu_0^*} + w_2 \frac{\sqrt{\mathrm{var}_{\mathbf{d}}[y_0(\mathbf{X};\mathbf{d},\mathbf{s})]}}{\sigma_0^*}, \\
&\text{subject to } \; c_l(\mathbf{d},\mathbf{s}) := \alpha_l \sqrt{\mathrm{var}_{\mathbf{d}}[y_l(\mathbf{X};\mathbf{d},\mathbf{s})]} - \mathbb{E}_{\mathbf{d}}[y_l(\mathbf{X};\mathbf{d},\mathbf{s})] \leq 0, \\
&\qquad l = 1, \ldots, K, \\
&d_{k,L} \leq d_k \leq d_{k,U}, \; k = 1, \ldots, M_d, \\
&s_{p,L} \leq s_p \leq s_{p,U}, \; p = 1, \ldots, M_s,
\end{aligned} \tag{1}$$

where $\mathbb{E}_{\mathbf{d}}[y_l(\mathbf{X};\mathbf{d},\mathbf{s})] := \int_{\mathbb{R}^N} y_l(\mathbf{x};\mathbf{d},\mathbf{s}) f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x}$ is the mean of $y_l(\mathbf{X};\mathbf{d}, \mathbf{s})$ with $\mathbb{E}_{\mathbf{d}}$ denoting the expectation operator with respect to the probability measure $f_{\mathrm{x}}(\mathbf{x};\mathbf{d})d\mathbf{x}$ of $\mathbf{X}$; $\mathrm{var}_{\mathbf{d}}[y_l(\mathbf{X};\mathbf{d},\mathbf{s})] := \mathbb{E}_{\mathbf{d}}\left[\{y_l(\mathbf{X};\mathbf{d},\mathbf{s}) - \mathbb{E}_{\mathbf{d}}[y_l(\mathbf{X};\mathbf{d},\mathbf{s})]\}^2\right]$ is the variance of $y_l(\mathbf{X};\mathbf{d}, \mathbf{s})$; $w_1 \in \mathbb{R}_0^+$ and $w_2 \in \mathbb{R}_0^+$ are two non-negative, real-valued weights, satisfying $w_1 + w_2 = 1$; $\mu_0^* \in \mathbb{R}\backslash\{0\}$ and $\sigma_0^* \in \mathbb{R}_0^+\backslash\{0\}$ are two non-zero, real-valued scaling factors; $\alpha_l \in \mathbb{R}_0^+, l = 0, 1, \ldots, K$, are non-negative, real-valued constants associated with the probabilities of constraint satisfaction; $d_{k,L}$ and $d_{k,U}$ are the lower and upper bounds, respectively, of d_k; and $s_{p,L}$ and $s_{p,U}$ are the lower and upper bounds, respectively, of s_p.

In equation (1), $c_0(\mathbf{d}, \mathbf{s})$ describes the objective robustness, and $c_l(\mathbf{d}, \mathbf{s})$, $l = 1, \ldots, K$, describe the feasibility robustness of a given design. Evaluations of both objective robustness and feasibility robustness, involving the first two moments of various responses, are required for solving RDO problems, consequently demanding statistical moment analysis.

2.2 Reliability-based design optimization

The mathematical formulation of a general RBDO problem involving an objective function $c_0 : \mathbb{R}^M \to \mathbb{R}$ and constraint functions $c_l : \mathbb{R}^M \to \mathbb{R}$, where $l = 1, \ldots, K$ and $1 \leq K < \infty$, requires one to:

$$\begin{aligned}
&\min_{(\mathbf{d},\mathbf{s})\in\mathcal{D}\subseteq\mathbb{R}^M} \; c_0(\mathbf{d},\mathbf{s}) \\
&\text{subject to } \; c_l(\mathbf{d},\mathbf{s}) := P_{\mathbf{d}}\left[\mathbf{X} \in \Omega_{F,l}(\mathbf{d},\mathbf{s})\right] - p_l \leq 0, \; l = 1, \ldots, K, \\
&d_{k,L} \leq d_k \leq d_{k,U}, \; k = 1, \ldots, M_d, \\
&s_{p,L} \leq s_p \leq s_{p,U}, \; p = 1, \ldots, M_s,
\end{aligned} \tag{2}$$

where $\Omega_{F,l}(\mathbf{d}, \mathbf{s}) \subseteq \Omega$ is the lth failure set that, in general, may depend on $\mathbf{d}$ and $\mathbf{s}$, and $0 \leq p_l \leq 1, l = 1, \ldots, K$, are target failure probabilities.

In equation (2), the objective function c_0 (**d**, **s**) is commonly prescribed as a deterministic function of **d** and **s**, describing relevant system geometry, such as area, volume and mass. In contrast, the constraint functions *cl (d, s)*, $l = 1,2, \ldots, K$, are generally more complicated than the objective function. Depending on the failure domain $\Omega_{F,l}$ (**d**, **s**), a component or a system reliability analysis can be envisioned. For component reliability analysis, the failure domain is often adequately described by a single performance function y_l (**X**;**d**, **s**), for instance, $\Omega_{F,l}$ (**d**, **s**):= {x: y_l(**x**;**d**, **s**) < 0}, whereas multiple, interdependent performance functions $y_{l,i}$ (**x**;**d**, **s**), $i = 1,2, \ldots$, are required for system reliability analysis, leading, for example, to $\Omega_{F,l}$ (**d**, **s**):= {**x**: $\cup_i y_{l,i}$ (**x**;**d**, **s**) < 0} and $\Omega_{F,l}$ (**d**, **s**):= {**x**: $\cap_i y_{l,i}$ (**x**;**d**, **s**) < 0} for series and parallel systems, respectively.

The RDO and RBDO problems described by equations (1) or (2) entail mixed design variables, and, therefore, they constitute more general stochastic design problems than those studied in the past (Taguchi, 1993; Rahman and Wei, 2008; Huang and Du, 2007; Lee *et al.*, 2009; Lee *et al.*, 2008; Lee *et al.*, 2012; Ren and Rahman, 2013; Ren *et al.*, 2016). Solving such an RDO or RBDO problem using gradient-based optimization algorithms mandates not only statistical moment and reliability analyses but also the gradients of moments and failure probability with respect to both distributional and structural design variables. The focus of this work is to solve a general high-dimensional RDO or RBDO problem described by equations (1) or (2) for arbitrary square-integrable functions y_l (**X**;**d**, **s**), $l = 0,1,2, \ldots, K$, and arbitrary probability distributions of **X**, provided that a few regularity conditions are met.

3. Stochastic analysis

3.1 Augmented polynomial dimensional decomposition

Consider two additional measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$, where Ω_1 and Ω_2 are two sample spaces and $\mathcal{F}_1$ and $\mathcal{F}_2$ are two σ-fields on Ω_1 and Ω_2, respectively. Let $\mathbf{D} := \left(D_1, \ldots, D_{M_d}\right)^T : (\Omega_1, \mathcal{F}_1) \to \left(\mathbb{R}^{M_d}, \mathcal{B}^{M_d}\right)$ and $\mathbf{S} := \left(S_1, \ldots, S_{M_s}\right)^T : (\Omega_2, \mathcal{F}_2) \to \left(\mathbb{R}^{M_s}, \mathcal{B}^{M_s}\right)$ be two affiliated random vectors with $\mathcal{B}^{M_d}$ and $\mathcal{B}^{M_s}$ representing the Borel σ-fields on $\mathbb{R}^{M_d}$ and $\mathbb{R}^{M_s}$, respectively. The fictitious probability laws of **D** and **S** are completely defined by selecting two families of the joint PDFs $\{f_{\mathbf{D}}(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}), \mathbf{d} \in \mathbb{R}^{M_d}\}$ and $\{f_{\mathbf{S}}(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}), \mathbf{s} \in \mathbb{R}^{M_s}\}$ with probability measures P_1 and P_2, and corresponding mean vectors $\mathbb{E}_1[\mathbf{D}] = \boldsymbol{\mu}_{\mathbf{D}}$ and $\mathbb{E}_2[\mathbf{S}] = \boldsymbol{\mu}_{\mathbf{S}}$, respectively.

Introduce an augmented measurable space $(\Omega_A, \mathcal{F}_A)$, where $\Omega_A := \Omega_{\mathbf{d}} \times \Omega_1 \times \Omega_2$ is the augmented sample space and $\mathcal{F}_A = \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_1 \times \mathcal{F}_2$ is the corresponding σ-field on Ω_A. Let $y(\mathbf{X};\mathbf{d}, \mathbf{s}) := y(X_1, \ldots, X_N; \mathbf{d}, \mathbf{s})$ represent any one of the random functions y_l, $l = 0,1, \ldots, K$, introduced in Section 2. Then $y(\mathbf{X};\mathbf{D}, \mathbf{S}) := y(X_1, \ldots, X_N; \mathbf{D}, \mathbf{S})$, obtained by simply replacing deterministic vectors **d** and **s** with random vectors **D** and **S** in $y(\mathbf{X};\mathbf{d}, \mathbf{s})$, has the same functional form of $y(\mathbf{X};\mathbf{d}, \mathbf{s})$. Let $\mathcal{L}_2(\Omega_A, \mathcal{F}_A, P_A)$ represent a Hilbert space of square-integrable functions $y(\mathbf{X};\mathbf{d}, \mathbf{s})$ with respect to the probability measure $P_A = P_{\mathbf{d}} \times P_1 \times P_2$ supported on $\mathbb{R}^{N+M}$. Assuming independent coordinates, the joint PDFs of **X**, **D**, and **S** are expressed by the products:

$$f_A(\mathbf{x}, \mathbf{d}, \mathbf{s}) = f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}) f_{\mathbf{D}}(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}) f_{\mathbf{S}}(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}) = \prod_{i=1}^{N} f_{X_i}(x_i; \mathbf{d}) \prod_{k=1}^{M_d} f_{D_k}(d_k; \boldsymbol{\mu}_{\mathbf{D}}) \prod_{p=1}^{M_s} f_{S_p}(s_p; \boldsymbol{\mu}_{\mathbf{S}}), \tag{3}$$

of marginal PDFs $f_{X_i} : \mathbb{R} \to \mathbb{R}_0^+$ of X_i, $f_{D_k} : \mathbb{R} \to \mathbb{R}_0^+$ of D_k, and $f_{S_p} : \mathbb{R} \to \mathbb{R}_0^+$ of S_p, $i = 1, \ldots, N$, $k = 1, \ldots, M_d$, and $p = 1, \ldots, M_s$. Then, for three given subsets $u \subseteq \{1, \ldots, N\}$, $v \subseteq \{1, \ldots, M_d\}$, and $w \subseteq \{1, \ldots, M_s\}$:

$$f_{uvw}(\mathbf{x}_u, \mathbf{d}_v, \mathbf{s}_w) := \prod_{q=1}^{|u|} f_{X_{i_q}}(x_{i_q}; \mathbf{d}) \prod_{r=1}^{|v|} f_{D_{k_r}}(d_{k_r}; \boldsymbol{\mu}_\mathbf{D}) \prod_{t=1}^{|w|} f_{S_{q_t}}(s_{q_t}; \boldsymbol{\mu}_\mathbf{S}) \tag{4}$$

defines the marginal density function of the subvector $(X_{i_1}, \ldots, X_{i_{|u|}}, D_{k_1}, \ldots, D_{k_{|v|}}, S_{p_1}, \ldots, S_{p_{|w|}})^T$, where $|\cdot|$ denotes cardinality.

Let $\{\psi_{i_q j_q}(X_{i_q}; \mathbf{d}); j_q = 0, 1, \ldots\}$, $\{\phi_{k_r l_r}(D_{k_r}; \boldsymbol{\mu}_\mathbf{D}); l_r = 0, 1, \ldots\}$ and $\{\varphi_{p_t n_t}(S_{p_t}; \boldsymbol{\mu}_\mathbf{S}); n_t = 0, 1, \ldots\}$ be three sets of univariate orthonormal polynomial basis functions in the Hilbert spaces $\mathcal{L}_2(\boldsymbol{\Omega}_{i_q,\mathbf{d}}, \mathcal{F}_{i_q,\mathbf{d}}, P_{i_q,\mathbf{d}})$, $\mathcal{L}_2(\boldsymbol{\Omega}_{k_r,1}, \mathcal{F}_{k_r,1}, P_{k_r,1})$ and $\mathcal{L}_2(\boldsymbol{\Omega}_{p_t,2}, \mathcal{F}_{p_t,2}, P_{p_t,2})$, respectively, which are consistent with the probability measures $P_{i_q,\mathbf{d}}$, $P_{k_r,1}$ and $P_{q_t,2}$, respectively, where $i_q = 1, \ldots, N$, $k_r = 1, \ldots, \mathrm{M_d}$, and $q_t = 1, \ldots, \mathrm{M_s}$. For given $\emptyset \neq u = \{i_1, \ldots, i_{|u|}\} \subseteq \{1, \ldots, N\}$, $\emptyset \neq v = \{k_1, \ldots, k_{|v|}\} \subseteq \{1, \ldots, M_d\}$ and $\emptyset \neq w = \{p_1, \ldots, p_{|w|}\} \subseteq \{1, \ldots, M_s\}$, define three associated multi-indices $\mathbf{j}_{|u|} = (j_1, \ldots, j_{|u|}) \in \mathbb{N}_0^{|u|}$, $\mathbf{l}_{|v|} = (l_1, \ldots, l_{|v|}) \in \mathbb{N}_0^{|v|}$ and $\mathbf{n}_{|w|} = (n_1, \ldots, n_{|w|}) \in \mathbb{N}_0^{|w|}$. Denote the product polynomials by:

$$\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d}) = \begin{cases} 1 & u = \emptyset, \\ \prod_{q=1}^{|u|} \psi_{i_q j_q}(X_{i_p}; \mathbf{d}) & \emptyset \neq u = \{i_1, \ldots, i_{|u|}\} \subseteq \{1, \ldots, N\}, \end{cases} \tag{5}$$

$$\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_\mathbf{D}) = \begin{cases} 1 & v = \emptyset, \\ \prod_{r=1}^{|v|} \phi_{k_r l_r}(d_{k_r}; \boldsymbol{\mu}_\mathbf{D}) & \emptyset \neq v = \{k_1, \ldots, k_{|v|}\} \subseteq \{1, \ldots, M_d\}, \end{cases} \tag{6}$$

and

$$\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_\mathbf{S}) = \begin{cases} 1 & w = \emptyset, \\ \prod_{t=1}^{|w|} \varphi_{p_t n_t}(s_{p_t}; \boldsymbol{\mu}_\mathbf{S}) & \emptyset \neq w = \{p_1, \ldots, p_{|w|}\} \subseteq \{1, \ldots, M_s\}, \end{cases} \tag{7}$$

which form three orthonormal basis functions in $\mathcal{L}_2\left(\times_{q=1}^{|u|}\boldsymbol{\Omega}_{i_q,\mathbf{d}}, \times_{q=1}^{|u|}\mathcal{F}_{i_q,\mathbf{d}}, \times_{q=1}^{|u|}P_{i_q,\mathbf{d}}\right)$, $\mathcal{L}_2\left(\times_{r=1}^{|v|}\boldsymbol{\Omega}_{k_r,1}, \times_{r=1}^{|v|}\mathcal{F}_{k_r,1}, \times_{r=1}^{|v|}P_{k_r,1}\right)$ and $\mathcal{L}_2\left(\times_{t=1}^{|w|}\boldsymbol{\Omega}_{p_t,2}, \times_{t=1}^{|w|}\mathcal{F}_{p_t,2}, \times_{t=1}^{|w|}P_{p_t,2}\right)$, respectively. As the PDF of the subvector $(X_{i_1}, \ldots, X_{i_{|u|}}, D_{k_1}, \ldots, D_{k_{|v|}}, S_{p_1}, \ldots, S_{p_{|w|}})^T$ is separable (independent), the product polynomial:

$$\psi_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{X}_u, \mathbf{D}_v, \mathbf{S}_w) := \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d})\, \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_\mathbf{D})\, \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_\mathbf{S}) \tag{8}$$

is consistent with the PDF f_{uvw} $(\mathbf{x}_u, \mathbf{d}_v, \mathbf{s}_w)$ and constitutes an orthonormal basis in $\mathcal{L}_2(\times_{q=1}^{|u|} \boldsymbol{\Omega}_{i_q,\mathbf{d}} \times_{r=1}^{|v|} \boldsymbol{\Omega}_{k_r,1} \times_{t=1}^{|w|} \boldsymbol{\Omega}_{p_t,2}, \times_{q=1}^{|u|}\mathcal{F}_{i_q,\mathbf{d}} \times_{r=1}^{|v|} \mathcal{F}_{k_r,1} \times_{t=1}^{|w|} \mathcal{F}_{p_t,2}, \times_{q=1}^{|u|}P_{i_q,\mathbf{d}} \times_{r=1}^{|v|} P_{k_r,1} \times_{t=1}^{|w|} P_{p_t,2})$.

The augmented PDD of a square-integrable function y represents a hierarchical expansion:

$$y(\mathbf{X};\mathbf{d},\mathbf{s}) = y_{\varnothing}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) + \sum_{\substack{u\subseteq\{1,\ldots,N\}, v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\}, |u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}, \mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}}$$
$$C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}})\,\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}})\,\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}}). \tag{9}$$

in terms of a set of random multivariate orthonormal polynomials of input variables with increasing dimensions, where:

$$y_{\varnothing}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) := \int_{\mathbb{R}^{N+M}} y(\mathbf{x};\mathbf{d},\mathbf{s}) f_A(\mathbf{x},\mathbf{d},\mathbf{s})\,d\mathbf{x}\,d\mathbf{d}\,d\mathbf{s} \tag{10}$$

and

$$\begin{aligned}&C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) := \int_{\mathbb{R}^{N+M}} y(\mathbf{x};\mathbf{d},\mathbf{s})\,\psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u;\mathbf{d})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}})\\ &\times\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}}) f_A(\mathbf{x},\mathbf{d},\mathbf{s})\,d\mathbf{x}\,d\mathbf{d}\,d\mathbf{s},\\ &u\subseteq\{1,\ldots,N\}, v\subseteq\{1,\ldots,M_d\}, w\subseteq\{1,\ldots,M_s\}, |u|+|v|+|w|\geq 1,\\ &\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}, \mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}, j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0\end{aligned} \tag{11}$$

are various expansion coefficients. The inner sum of equation (9) precludes $j_1,\ldots,j_{|u|}\neq 0$, $l_1,\ldots,l_{|v|}\neq 0$, and $n_1,\ldots,n_{|w|}\neq 0$, that is, the individual degree of involved variables cannot be *zero* as $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$, $\phi_{v\mathbf{l}_{|v|}}(\mathbf{D}_v;\boldsymbol{\mu}_{\mathbf{D}})$ and $\varphi_{w\mathbf{n}_{|w|}}(\mathbf{S}_w;\boldsymbol{\mu}_{\mathbf{S}})$ are *zero*-mean strictly $|u|$-variate, $|v|$-variate and $|w|$-variate functions, respectively. Derived from the ANOVA dimensional decomposition (Efron and Stein, 1981), equation (9) provides an exact representation because it includes all main and interactive effects of input and affiliated variables. For instance, $|u|+|v|+|w|=0$ corresponds to the constant component function y_{ϕ}, representing the mean effect of y; $|u|+|v|+|w|=1$ leads to the univariate component functions, describing the main effects of input and affiliated variables; and $|u|+|v|+|w|=S$, $1<S\leq N+M$, results in the S-variate component functions, facilitating the interaction among at most S input and affiliated variables. The augmented PDD expansion in equation (9) can be used to reproduce the function $y(\mathbf{X};\mathbf{d},\mathbf{s})$ by simply replacing the random vectors $\mathbf{D}$ and $\mathbf{S}$ in equation (9) with deterministic vectors $\mathbf{d}$ and $\mathbf{s}$, that is:

$$y(\mathbf{X};\mathbf{d},\mathbf{s}) = y_{\varnothing}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) + \sum_{\substack{u\subseteq\{1,\ldots,N\}, v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\}, |u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}, \mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}}$$
$$C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}})\,\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}})\,\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}}). \tag{12}$$

3.2 Truncated augmented polynomial dimensional decomposition approximation

The augmented PDD in equations (9) is grounded on a fundamental conjecture known to be true in many real-world applications: given a high-dimensional function y, its $(|u|+|v|+|w|)$-variate component functions decay rapidly with respect to $|u|+|v|+|w|=0$, leading to accurate lower-variate approximations of y. Furthermore, the largest order of polynomials in each variable can be restricted to a finite integer. Indeed, given the integers $0 \leq S < N$ and $1 \leq m < \infty$ for all $1 \leq |u| + |v| + |w| \leq S$ and the ∞-norms $1 \leq ||\mathbf{j}_{|u|}||_\infty := \max (j_1, \ldots, j_{|u|}) \leq m$, $1 \leq ||\mathbf{l}_{|v|}||_\infty := \max (l_1, \ldots, l_{|v|}) \leq m$ and $1 \leq ||\mathbf{n}_{|w|}||_\infty := \max (n_1, \ldots, n_{|w|}) \leq m$, the truncated augmented PDD:

$$\tilde{y}_{S,m}(\mathbf{X}; \mathbf{D}, \mathbf{S}) = y_\varnothing(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{S}) + \sum_{\substack{u \subseteq \{1,\ldots,N\}, v \subseteq \{1,\ldots,M_d\} \\ w \subseteq \{1,\ldots,M_s\}, 1 \leq |u|+|v|+|w| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \mathbf{l}_{|v|} \in \mathbb{N}_0^{|v|}, \mathbf{n}_{|w|} \in \mathbb{N}_0^{|w|} \\ ||\mathbf{j}_{|u|}||_\infty, ||\mathbf{l}_{|v|}||_\infty, ||\mathbf{n}_{|w|}||_\infty \leq m \\ j_1, \ldots, j_{|u|}, l_1, \ldots, l_{|v|}, n_1, \ldots, n_{|w|} \neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{S}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d}) \phi_{v\mathbf{l}_{|v|}}(\mathbf{D}_v; \boldsymbol{\mu}_\mathbf{D}) \varphi_{w\mathbf{n}_{|w|}}(\mathbf{S}_w; \boldsymbol{\mu}_\mathbf{S}), \tag{13}$$

leads to the S-variate, mth-order augmented PDD approximation, which for $S > 0$ includes interactive effects of at most S input and affiliated variables, on y. It is elementary to show that when $S \to N + M$ and/or $m \to \infty$, $\tilde{y}_{S,m}$ converges to y in the mean-square sense, generating a hierarchical and convergent sequence of approximations of y. The truncation parameters S and m depend on the dimensional structure and nonlinearity of a stochastic response. The higher the values of S and m, the higher the accuracy, and also the computational cost that is endowed with an Sth-order polynomial computational complexity. Simply replacing the random vectors $\mathbf{D}$ and $\mathbf{S}$ in equation (13) with deterministic vectors $\mathbf{d}$ and $\mathbf{s}$ renders an S-variate, mth-order augmented PDD approximation:

$$\tilde{y}_{S,m}(\mathbf{X}; \mathbf{d}, \mathbf{s}) = y_\varnothing(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{S}) + \sum_{\substack{u \subseteq \{1,\ldots,N\}, v \subseteq \{1,\ldots,M_d\} \\ w \subseteq \{1,\ldots,M_s\}, 1 \leq |u|+|v|+|w| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \mathbf{l}_{|v|} \in \mathbb{N}_0^{|v|}, \mathbf{n}_{|w|} \in \mathbb{N}_0^{|w|} \\ ||\mathbf{j}_{|u|}||_\infty, ||\mathbf{l}_{|v|}||_\infty, ||\mathbf{n}_{|w|}||_\infty \leq m \\ j_1, \ldots, j_{|u|}, l_1, \ldots, l_{|v|}, n_1, \ldots, n_{|w|} \neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{S}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d}) \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_\mathbf{D}) \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_\mathbf{S}) \tag{14}$$

of the original function $y(\mathbf{X};\mathbf{d}, \mathbf{s})$. The S-variate, mth-order augmented PDD approximation will be referred to as *truncated augmented PDD approximation* in this paper. It is worth to note that the truncation parameters S and m depend on the dimensional structure and nonlinearity of a stochastic response. In the case that the dimensional hierarchy or nonlinearity is not known *a priori*, an adaptive-sparse approach (Ren *et al.*, 2016; Yadav and Rahman, 2014) is suggested to determine the truncation parameters adaptively and automatically.

3.3 Statistical moment analysis

Let $m^{(r)}(\mathbf{d}, \mathbf{s}) := \mathbb{E}_\mathbf{d}[y^r(\mathbf{X}; \mathbf{d}, \mathbf{s})]$, if it exists, define the raw moment of y of order r, where $r \in \mathbb{N}$. Given an S-variate, mth-order PDD approximation $\tilde{y}_{S,m}(\mathbf{X}; \mathbf{d}, \mathbf{s})$ of $y(\mathbf{X};\mathbf{d}, \mathbf{s})$, let $\tilde{m}^{(r)}_{S,m}(\mathbf{d}, \mathbf{s}) := \mathbb{E}_\mathbf{d}\left[\tilde{y}^r_{S,m}(\mathbf{X}; \mathbf{d}, \mathbf{s})\right]$ define the raw moment of $\tilde{y}_{S,m}$ of order r. The following paragraphs describe the explicit formulae or analytical expressions for calculating the first two moments by the PDD approximation.

Applying the expectation operator $\mathbb{E}_\mathbf{d}$ on $\tilde{y}_{S,m}(\mathbf{X}; \mathbf{d}, \mathbf{s})$ and $\tilde{y}^2_{S,m}(\mathbf{X}; \mathbf{d}, \mathbf{s})$, and recognizing the *zero*-mean and orthonormal properties of orthonormal basis, the first and second moments of the S-variate, mth-order augmented PDD approximation are:

$$\tilde{m}_{S,m}^{(1)}(\mathbf{d},\mathbf{s}) := \mathbb{E}_{\mathbf{d}}\left[\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right] = y_{\emptyset}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) + \sum_{\substack{u\subseteq\{1,\ldots,N\},v\subseteq\{1,\ldots,M_d\},w\subseteq\{1,\ldots,M_s\}\\ |u|=0,1\le|u|+|v|+|w|\le S}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ \|\mathbf{j}_{|u|}\|_\infty,\|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}})\,\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}}) \tag{15}$$

and

$$\tilde{m}_{S,m}^{(2)}(\mathbf{d},\mathbf{s}) := \mathbb{E}_{\mathbf{d}}\left[\tilde{y}_{S,m}^{2}(\mathbf{X};\mathbf{d},\mathbf{s})\right] = \left[\tilde{m}_{S,m}^{(1)}(\mathbf{d},\mathbf{s})\right]^2 + \sum_{\substack{u\subseteq\{1,\ldots,N\}\\ 1\le|u|\le S}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\|\mathbf{j}_{|u|}\|_\infty\le m\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|},S,m}^2(\mathbf{d},\mathbf{s}), \tag{16}$$

respectively, where the second moment involves new expansion coefficients:

$$E_{u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s}) = \sum_{\substack{v\subseteq\{1,\ldots,M_d\},w\subseteq\{1,\ldots,M_s\}\\ 1\le|u|+|v|+|w|\le S}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|},\|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}})\,\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}}), \tag{17}$$

via restructuring:

$$\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s}) = \tilde{m}_{S,m}^{(1)}(\mathbf{d},\mathbf{s}) + \sum_{\substack{u\subseteq\{1,\ldots,N\}\\ 1\le|u|\le S}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}^{|u|},\|\mathbf{j}_{|u|}\|_\infty\le m\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s})\,\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d}) \tag{18}$$

in terms of $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$. Clearly, the approximate moments in equations (15) and (16) approach the exact moments:

$$m^{(1)}(\mathbf{d},\mathbf{s}) := \mathbb{E}_{\mathbf{d}}\left[y(\mathbf{X})\right] = y_{\emptyset}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) + \sum_{\substack{u\subseteq\{1,\ldots,N\},v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\},|u|=0,|u|+|v|+|w|\ge 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}})\,\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}}) \tag{19}$$

and

$$m^{(2)}(\mathbf{d},\mathbf{s}) := \mathbb{E}_{\mathbf{d}}\left[y^2(\mathbf{X};\mathbf{d},\mathbf{s})\right] = \left[m^{(1)}(\mathbf{d},\mathbf{s})\right]^2 + \sum_{\substack{u\subseteq\{1,\ldots,N\}\\|u|\geq 1}}\ \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\ldots,j_{|u|}\neq 0}} E^2_{u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}) \tag{20}$$

of y when $S \to N+M$ and $m \to \infty$, where:

$$E_{u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}) = \sum_{\substack{v\subseteq\{1,\ldots,M_d\},w\subseteq\{1,\ldots,M_s\}\\|u|+|v|+|w|\geq 1}}\ \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}})$$
$$\times\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}})\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}}) \tag{21}$$

is again derived from restructuring equation (9) in terms of $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$, that is:

$$y(\mathbf{X};\mathbf{d},\mathbf{s}) = m^{(1)}(\mathbf{d},\mathbf{s}) + \sum_{\substack{u\subseteq\{1,\ldots,N\}\\|u|\geq 1}}\ \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s})\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d}). \tag{22}$$

The mean-square convergence of $\tilde{y}_{S,m}$ is guaranteed as y, and its component functions are all members of the associated Hilbert spaces. In other words, the mean and variance of $\tilde{y}_{S,m}$ are also convergent.

3.4 Reliability analysis

A fundamental problem in reliability analysis entails calculation of the failure probability:

$$P_F(\mathbf{d},\mathbf{s}) := P_{\mathbf{d}}[\mathbf{X}\in\Omega_F(\mathbf{d},\mathbf{s})] = \int_{\mathbb{R}^N} I_{\Omega_F}(\mathbf{x};\mathbf{d},\mathbf{s})f_{\mathbf{X}}(\mathbf{x};\mathbf{d})d\mathbf{x} =: \mathbb{E}_{\mathbf{d}}\left[I_{\Omega_F}(\mathbf{X};\mathbf{d},\mathbf{s})\right], \tag{23}$$

where $\Omega_F(\mathbf{d},\mathbf{s})$ is the failure set and $I_{\Omega F}(\mathbf{x};\mathbf{d},\mathbf{s})$ is the associated indicator function, which is equal to *one* when $\mathbf{x}\in\Omega_F$ (d, s) and *zero* otherwise. In this subsection, the augmented PDD method for reliability analysis, which exploits the augmented PDD approximation for MCS, is elucidated.

Depending on component or system reliability analysis, let $\tilde{\Omega}_{F,S,m} := \{\mathbf{x} : \tilde{y}_{S,m}(\mathbf{x};\mathbf{d},\mathbf{s}) < 0\}$ or $\tilde{\Omega}_{F,S,m} := \{\mathbf{x} : \cup_i\tilde{y}_{i,S,m}(\mathbf{x};\mathbf{d},\mathbf{s}) < 0\}$ or $\tilde{\Omega}_{F,S,m} := \{\mathbf{x} : \cap_i\tilde{y}_{i,S,m}(\mathbf{x};\mathbf{d},\mathbf{s}) < 0\}$ be an approximate failure set as a result of S-variate, mth-order PDD approximations $\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})$ of $y(\mathbf{X};\mathbf{d},\mathbf{s})$ or $\tilde{y}_{i,S,m}(\mathbf{X};\mathbf{d},\mathbf{s})$ of $y_i(\mathbf{X};\mathbf{d},\mathbf{s})$. Then the augmented PDD estimate of the failure probability $P_F(\mathbf{d},\mathbf{s})$ is:

$$\tilde{P}_{F,S,m}(\mathbf{d},\mathbf{s}) = \mathbb{E}_{\mathbf{d}}\left[I_{\tilde{\Omega}_{F,S,m}}(\mathbf{X};\mathbf{d},\mathbf{s})\right] = \lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L} I_{\tilde{\Omega}_{F,S,m}}\left(\mathbf{x}^{(l)};\mathbf{d},\mathbf{s}\right), \tag{24}$$

where L is the sample size, $\mathbf{x}^{(l)}$ is the lth realization of $\mathbf{X}$, and $I_{\tilde{\Omega}_{F,S,m}}(\mathbf{x};\mathbf{d},\mathbf{s})$ is another indicator function, which is equal to *one* when $\mathbf{x}\in\tilde{\Omega}_{F,S,m}$ and *zero* otherwise.

Note that the simulation of the augmented PDD approximation in equation (24) should not be confused with the crude MCS commonly used for producing benchmark results. The crude MCS, which requires numerical calculations of $y(\mathbf{x}^{(l)};\mathbf{d},\mathbf{s})$ or $y_i(\mathbf{x}^{(l)};\mathbf{d},\mathbf{s})$ for input samples $\mathbf{x}^{(l)}$, $l = 1, \ldots, L$, can be expensive or even prohibitive, particularly when the sample size L needs to be very large for estimating small failure probabilities. In contrast, the MCS embedded in PDD requires evaluations of simple analytical functions that stem from an S-variate, mth-order approximation $\tilde{y}_{S,m}\left(\mathbf{x}^{(l)};\mathbf{d},\mathbf{s}\right)$ or $\tilde{y}_{i,S,m}\left(\mathbf{x}^{(l)};\mathbf{d},\mathbf{s}\right)$. Therefore, an arbitrarily large sample size can be accommodated in the augmented PDD method.

3.5 Expansion coefficients

The determination of augmented PDD expansion coefficients y_ϕ $(\mathbf{d})$ and $C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}$ $(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$ is vitally important for moment and reliability analysis, including their design sensitivities. As defined in Equations (10) and (11), the coefficients involve various $N+M$-dimensional integrals over $\mathbb{R}^{N+M}$. For large $N + M$, a multivariate numerical integration employing an $N + M$ -dimensional tensor product of a univariate quadrature formula is computationally prohibitive and is, therefore, ruled out. An attractive alternative approach entails dimension-reduction integration, which was originally developed by Xu and Rahman (Xu and Rahman, 2004) for high-dimensional numerical integration. For calculating y_ϕ and $C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}$, this is accomplished by replacing the $N+M$ -variate function y in equations (10) and (11) with an R-variate truncation of the referential dimensional decomposition (RDD) at a chosen reference point, where $R \leq N+M$. The result is a reduced integration scheme, requiring evaluations of at most R-dimensional integrals, described as follows.

Let $\mathbf{c} = (c_1,\ldots,c_N)^T \in \mathbb{R}^N$, $\mathbf{c}' = \left(c_1^{'},\ldots,c_{M_d}^{'}\right)^T \in \mathbb{R}^{M_d}$ and $\mathbf{c}'' = \left(c_1^{''},\ldots,c_{M_s}^{''}\right)$ $T \in \mathbb{R}^{M_s}$, which are commonly adopted as the means of $\mathbf{X}$, $\mathbf{D}$ and $\mathbf{S}$, respectively, be the reference points. Let $y\left(\mathbf{x}_{u_1},\mathbf{d}_{v_1},\mathbf{s}_{w_1},\mathbf{c}_{-u_1},\mathbf{c}^{'}_{-v_1},\mathbf{c}^{''}_{-w_1}\right)$ represent an $(|u_1| + |v_1| + |w_1|)$-variate RDD component function of $y(\mathbf{x},\mathbf{d},\mathbf{s})$, where $u_1 \subseteq \{1,\ldots,N\}$, $v_1 \subseteq \{1,\ldots,M_d\}$, and $w_1 \subseteq \{1,\ldots,M_s\}$. Given a positive integer $S \leq R \leq N+M$, when $y(\mathbf{x},\mathbf{d},\mathbf{s})$ in equations (10) and(11) is replaced with its R-variate RDD approximation, the coefficients $y\phi$ $(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{s})$ and $C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$ are estimated from (Xu and Rahman, 2004):

$$y_\varnothing(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S}) \cong \sum_{i=0}^{R}(-1)^i\binom{N+M-R+i-1}{i}\sum_{\substack{u_1\subseteq\{1,\ldots,N\},v_1\subseteq\{1,\ldots,M_d\}\\ w_1\subseteq\{1,\ldots,M_s\},|u_1|+|v_1|+|w_1|=R-i}}$$

$$\int_{\mathbb{R}^{|u_1|+|v_1|+|w_1|}} y\left(\mathbf{x}_{u_1},\mathbf{d}_{v_1},\mathbf{s}_{w_1},\mathbf{c}_{-u_1},\mathbf{c}^{'}_{-v_1},\mathbf{c}^{''}_{-w_1}\right) f_{u_1v_1w_1}(\mathbf{x}_{u_1},\mathbf{d}_{v_1},\mathbf{s}_{w_1})d\mathbf{x}_{u_1}d\mathbf{d}_{v_1}d\mathbf{s}_{w_1} \tag{25}$$

and

$$C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S}) \cong \sum_{i=0}^{R}(-1)^i\binom{N+M-R+i-1}{i}$$

$$\sum_{\substack{u_1\subseteq\{1,\ldots,N\},v_1\subseteq\{1,\ldots,M_d\},w_1\subseteq\{1,\ldots,M_s\}\\ |u_1|+|v_1|+|w_1|=R-i,u\subseteq u_1,v\subseteq v_1,w\subseteq w_1}} \int_{\mathbb{R}^{|u_1|+|v_1|+|w_1|}} y\left(\mathbf{x}_{u_1},\mathbf{d}_{v_1},\mathbf{s}_{w_1},\mathbf{c}_{-u_1},\mathbf{c}^{'}_{-v_1},\mathbf{c}^{''}_{-w_1}\right)$$

$$\times\,\psi_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{x}_u,\mathbf{d}_v,\mathbf{s}_w) f_{u_1v_1w_1}(\mathbf{x}_{u_1},\mathbf{d}_{v_1},\mathbf{s}_{w_1})d\mathbf{x}_{u_1}d\mathbf{d}_{v_1}d\mathbf{s}_{w_1}, \tag{26}$$

respectively, requiring evaluation of at most R-dimensional integrals. The reduced integration facilitates calculation of the coefficients approaching their exact values as $R \to N + M$ and is significantly more efficient than performing one $(N + M)$-dimensional integration, particularly when $R \ll N + M$. Hence, the computational effort is significantly lowered using the dimension-reduction integration. For instance, when $R = 1$ or 2, equations (25) and (26) involve one-, or at most, two-dimensional integrations, respectively. For a general function y, numerical integrations are required for performing various low-dimensional integrations in equations (25) and (26). Refer Xu and Rahman (2004) for further details.

3.6 Computational expense

The S-variate, mth-order augmented PDD approximation requires evaluations of $\sum_{k=0}^{k=S}\binom{N+M}{k} m^k$ expansion coefficients, including y_ϕ $(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{s})$. If these coefficients are estimated by dimension-reduction integration with $R = S < N + M$ and, therefore, involve at most an S-dimensional tensor product of an n-point univariate quadrature rule depending on m, then the total cost for the S-variate, mth-order approximation entails a maximum of $\sum_{k=0}^{k=S}\binom{N+M}{k} n^k(m)$ function evaluations. If the integration points include a common point in each coordinate – a special case of symmetric input PDFs and odd values of n– the number of function evaluations reduces to $\sum_{k=0}^{k=S}\binom{N+M}{k}(n(m)-1)^k$. Nonetheless, the computational complexity of the S-variate augmented PDD approximation is an Sth-order polynomial with respect to the number of random variables or integration points. Therefore, augmented PDD with dimension-reduction integration of the expansion coefficients alleviates the curse of dimensionality to an extent determined by S.

4. Design sensitivity analysis

When solving RDO and RBDO problems using gradient-based optimization algorithms, at least the first-order sensitivities of the first two moments of $y(\mathbf{X};\mathbf{d}, \mathbf{s})$ and the failure probability with respect to each distributional and structural design variable are required. In this section, a new method involving the augmented PDD, score functions and finite-difference approximation is presented. For such sensitivity analysis, the following regularity conditions are assumed:

(1) The design variables $d_k \in \mathcal{D}_k \subset \mathbb{R}$, $k = 1, \ldots, M_d$ and $s_p \in \mathcal{S}_p \subset \mathbb{R}$, $p = 1, \ldots, M_s$, where $\mathcal{D}_k$ and $\mathcal{S}_p$ are open intervals of $\mathbb{R}$.

(2) The PDF $F_\mathrm{x}(\mathbf{x};\mathbf{d})$ of $\mathbf{X}$ is continuous. In addition, the partial derivative $\partial f_\mathrm{x}(x;d)/\partial d_\mathrm{k}$, $k = 1, \ldots, M_d$, exists and is finite for all $\mathbf{x} \in \mathbb{R}^N$ and $d_k \in \mathcal{D}_k$. Furthermore, the statistical moments of y and failure probability are differentiable functions of $\mathbf{d} \in \mathbb{R}^{M_d}$.

(3) The performance function $y(\mathbf{x};\mathbf{d}, \mathbf{s})$ is continuous. In addition, the partial derivative $\partial y(x;d, s)/\partial s_p$, $p = 1, \ldots, M_s$, exists and is finite for all $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{d} \in \mathbb{R}^{M_d}$, and $s_p \in \mathcal{S}_p$. Furthermore, the statistical moments of y and failure probability are differentiable functions of $\mathbf{s} \in \mathbb{R}^{M_s}$.

(4) There exists a Lebesgue integrable dominating function $z(\mathbf{x})$ such that:

$$\left| y^r(\mathbf{x};\mathbf{d},\mathbf{s}) \frac{\partial f_{\mathbf{X}}(\mathbf{x};\mathbf{d})}{\partial d_k} \right| \le z(\mathbf{x}), \qquad \left| I_{\Omega_F}(\mathbf{x};\mathbf{d},\mathbf{s}) \frac{\partial f_{\mathbf{X}}(\mathbf{x};\mathbf{d})}{\partial d_k} \right| \le z(\mathbf{x}), \tag{27}$$
$$r = 1, 2, \;\; k = 1, \ldots, M_d.$$

4.1 Sensitivity of moments

Suppose that the first-order derivative of a moment $m^{(r)}(\mathbf{d}, \mathbf{s})$, where $r = 1, 2$, of a generic stochastic response $y(\mathbf{X};\mathbf{d}, \mathbf{s})$ with respect to a distributional design variable d_k, $1 \le k \le M_d$, or with respect to a structural design variable s_p, $1 \le p \le M_s$ is sought. Taking a partial derivative of the moment with respect to d_k and then applying the Lebesgue dominated convergence theorem [39], which permits the differential and integral operators to be interchanged, yields the sensitivity:

$$\begin{aligned}
\frac{\partial m^{(r)}(\mathbf{d},\mathbf{s})}{\partial d_k} &:= \frac{\partial \mathbb{E}_{\mathbf{d}}[y^r(\mathbf{X};\mathbf{d},\mathbf{s})]}{\partial d_k} = \frac{\partial}{\partial d_k} \int_{\mathbb{R}^N} y^r(\mathbf{x};\mathbf{d},\mathbf{s}) f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x} \\
&= \int_{\mathbb{R}^N} y(\mathbf{x};\mathbf{d},\mathbf{s}) \frac{\partial \ln f(\mathbf{x};\mathbf{d})}{\partial d_k} f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x} + \int_{\mathbb{R}^N} \frac{\partial y^r(\mathbf{x};\mathbf{d},\mathbf{s})}{\partial d_k} f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x} \\
&=: \mathbb{E}_{\mathbf{d}}\left[y^r(\mathbf{X};\mathbf{d},\mathbf{s}) s_{d_k}^{(1)}(\mathbf{X};\mathbf{d}) \right] + \mathbb{E}_{\mathbf{d}}\left[\frac{\partial y^r(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k} \right]
\end{aligned} \tag{28}$$

with respect to the distributional design variables, provided that $f_{\mathrm{x}}(\mathbf{x};\mathbf{d}) > 0$. In the last line of equation (28), $s_{d_k}^{(1)}(\mathbf{X};\mathbf{d}) := \partial \ln f_{\mathbf{X}}(\mathbf{X};\mathbf{d})/\partial d_k$ is known as the first-order score function for the design variable d_k (Rahman, 2009; Rubinstein and Shapiro, 1993). Compared with the existing sensitivity analysis (Rahman, 2009; Rahman and Ren, 2014), the second term, $\mathbb{E}_{\mathbf{d}}\left[\partial y^r(\mathbf{X};\mathbf{d},\mathbf{s})/\partial d_k\right]$, appears due to the permissible explicit dependence of y on the distributional design variables.

The evaluation of score functions, $s_{d_k}^{(1)}(\mathbf{X};\mathbf{d})$, $k = 1, \ldots, M$, requires differentiating only the PDF of $\mathbf{X}$. Therefore, the resulting score functions can be determined easily and, in many cases, analytically – for instance, when $\mathbf{X}$ follows classical probability distributions (Rahman, 2009). If the density function of $\mathbf{X}$ is arbitrarily prescribed, the score functions can be calculated numerically, yet inexpensively, as no evaluation of the performance function is involved. When $\mathbf{X}$ comprises independent variables, as assumed here, $\ln f_{\mathbf{X}}(\mathbf{X};\mathbf{d}) = \sum_{i=1}^{i=N} \ln f_{X_i}(x_i;\mathbf{d})$ is a sum of N univariate log-density (marginal) functions of random variables. Hence, in general, the score function for the kth design variable, expressed by:

$$s_{d_k}^{(1)}(\mathbf{X};\mathbf{d}) = \sum_{i=1}^{N} \frac{\partial \ln f_{X_i}(X_i;\mathbf{d})}{\partial d_k} = \sum_{i=1}^{N} s_{ki}(X_i;\mathbf{d}), \tag{29}$$

is also a sum of univariate functions $s_{ki}(X_i;\mathbf{d}) := \partial \ln f_{X_i}(X_i;\mathbf{d})/\partial d_k$, $i = 1, \ldots, N$, which are the derivatives of log-density (marginal) functions. If d_k is a distribution parameter of a single random variable X_{i_k}, then the score function reduces to $s_{d_k}^{(1)}(\mathbf{X};\mathbf{d}) = \partial \ln f_{X_{i_k}}(X_{i_k};\mathbf{d})/\partial d_k =: s_{ki_k}(X_{i_k};\mathbf{d})$, the derivative of the log-density (marginal) function of X_{i_k}, which remains a univariate function. Nonetheless, combining equations (28) and (29), the sensitivity is obtained as:

$$\frac{\partial m^{(r)}(\mathbf{d},\mathbf{s})}{\partial d_k} = \sum_{i=1}^{N} \mathbb{E}_{\mathbf{d}}[y^r(\mathbf{X};\mathbf{d},\mathbf{s}) s_{ki}(X_i;\mathbf{d})] + \mathbb{E}_{\mathbf{d}}\left[\frac{\partial y^r(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k}\right]. \tag{30}$$

Similarly, taking a partial derivative of the moment with respect to s_p yields the sensitivity:

$$\begin{aligned}\frac{\partial m^{(r)}(\mathbf{d},\mathbf{s})}{\partial s_p} &:= \frac{\partial \mathbb{E}_{\mathbf{d}}[y^r(\mathbf{X};\mathbf{d},\mathbf{s})]}{\partial s_p} = \frac{\partial}{\partial d_k}\int_{\mathbb{R}^N} y^r(\mathbf{x};\mathbf{d},\mathbf{s}) f_{\mathbf{X}}(\mathbf{x};\mathbf{d})d\mathbf{x} \\ &= \int_{\mathbb{R}^N} \frac{\partial y^r(\mathbf{x};\mathbf{d},\mathbf{s})}{\partial s_p} f_{\mathbf{X}}(\mathbf{x};\mathbf{d})d\mathbf{x} =: \mathbb{E}_{\mathbf{d}}\left[\frac{\partial y^r(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial s_p}\right]\end{aligned} \tag{31}$$

with respect to the structural design variables, involving only one term because the PDF $f_{\mathbf{X}}(\mathbf{x};\mathbf{d})$ does not depend on $\mathbf{s}$. In general, these sensitivities are not available analytically, as the moments are not either. Nonetheless, the moments and their sensitivities, whether in conjunction with the distributional or structural design variables, have both been formulated as expectations of stochastic quantities with respect to the same probability measure, facilitating their concurrent evaluations in a single stochastic simulation or analysis.

Given an S-variate, mth-order augmented PDD approximation $\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})$ of $y(\mathbf{X};\mathbf{d},\mathbf{s})$, let $\partial\tilde{m}^{(r)}_{S,m}(\mathbf{d},\mathbf{s})/\partial d_k$ and $\partial\tilde{m}^{(r)}_{S,m}(\mathbf{d},\mathbf{s})/\partial s_p$ define the concomitant approximations of moment sensitivities. The following subsections describe the explicit formulae or analytical expressions for calculating the moments by augmented PDD approximations for $r = 1, 2$.

4.1.1 Sensitivity of the first moment. Setting $r = 1$ in equations (30) and (31), the sensitivities of the first moment are:

$$\frac{\partial m^{(1)}(\mathbf{d},\mathbf{s})}{\partial d_k} = \sum_{i=1}^{N} \mathbb{E}_{\mathbf{d}}[y(\mathbf{X};\mathbf{d},\mathbf{s}) s_{ki}(X_i;\mathbf{d})] + \mathbb{E}_{\mathbf{d}}\left[\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k}\right] \tag{32}$$

and

$$\frac{\partial m^{(1)}(\mathbf{d},\mathbf{s})}{\partial s_p} = \mathbb{E}_{\mathbf{d}}\left[\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial s_p}\right], \tag{33}$$

where $k = 1, \ldots, M_d$ and $p = 1, \ldots, M_s$.

For independent coordinates of $\mathbf{X}$, consider the Fourier-polynomial expansion of the kth log-density derivative function:

$$s_{ki}(X_i;\mathbf{d}) = s_{ki,\varnothing}(\mathbf{d}) + \sum_{j=1}^{\infty} D_{k,ij}(\mathbf{d})\psi_{ij}(X_i;\mathbf{d}), \tag{34}$$

Consisting of its own expansion coefficients:

$$s_{ki,\varnothing}(\mathbf{d}) := \int_{\mathbb{R}} s_{ki}(x_i;\mathbf{d}) f_{X_i}(x_i;\mathbf{d})dx_i \tag{35}$$

and

$$D_{k,ij}(\mathbf{d}) := \int_{\mathbb{R}} s_{ki}(x_i;\mathbf{d})\psi_{ij}(x_i;\mathbf{d}) f_{X_i}(x_i;\mathbf{d})dx_i. \tag{36}$$

The expansion is valid if s_{ki} is square integrable with respect to the probability measure of X_i. When blended with the PDD approximation, the score function leads to analytical or closed-form expressions of the exact or approximate sensitivities as follows.

4.1.1.1 Exact sensitivities. Restructuring equation (12) as:

$$y(\mathbf{X};\mathbf{d},\mathbf{s}) = m^{(1)}(\mathbf{d},\mathbf{s}) + \sum_{u=\{i\}\subset\{1,\ldots,N\}} \sum_{\substack{j\in\mathbb{N}_0 \\ j\neq 0}} C_{ij}(\mathbf{d},\mathbf{s})\psi_{ij}(X_i;\mathbf{d})$$
$$+ \sum_{\substack{u\subseteq\{1,\ldots,N\},v\subseteq\{1,\ldots,M_d\} \\ w\subseteq\{1,\ldots,M_s\},|u|>1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|} \\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$$
$$\times \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}), \tag{37}$$

where

$$C_{ij}(\mathbf{d},\mathbf{s}) = \sum_{\substack{u=\{i\},v\subseteq\{1,\ldots,M_d\} \\ w\subseteq\{1,\ldots,M_s\}}} \sum_{\substack{\mathbf{j}_{|u|}=j\in\mathbb{N}_0,\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|} \\ j,l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$$
$$\times\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}), \tag{38}$$

and using equations (34) and (37), the product appearing on the right side of equation (32) expands to:

$$y(\mathbf{X};\mathbf{d},\mathbf{s})s_{ki}(X_i;\mathbf{d}) = \Bigg(m^{(1)}(\mathbf{d},\mathbf{s}) + \sum_{u=\{i\}\subset\{1,\ldots,N\}} \sum_{\substack{j\in\mathbb{N}_0 \\ j\neq 0}} C_{ij}(\mathbf{d},\mathbf{s})\psi_{ij}(X_i;\mathbf{d})$$
$$+ \sum_{\substack{u\subseteq\{1,\ldots,N\},v\subseteq\{1,\ldots,M_d\} \\ w\subseteq\{1,\ldots,M_s\},|u|>1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|} \\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$$
$$\times\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S})\Bigg)\left(s_{ki,\varnothing}(\mathbf{d}) + \sum_{j=1}^{\infty} D_{k,ij}(\mathbf{d})\psi_{ij}(X_i;\mathbf{d})\right), \tag{39}$$

encountering the same orthonormal polynomial bases that are consistent with the probability measure $f_\mathrm{X}(\mathbf{x};\mathbf{d})d\mathbf{x}$. Taking the expectation of equations (39), aided by the *zero*-mean and orthonormal properties of orthonormal basis, leads to:

$$\mathbb{E}_\mathbf{d}[y(\mathbf{X};\mathbf{d},\mathbf{s})s_{ki}(X_i;\mathbf{d})] = m^{(1)}(\mathbf{d},\mathbf{s})s_{ki,\varnothing} + \sum_{j=1}^{\infty} D_{k,ij}(\mathbf{d})C_{ij}(\mathbf{d},\mathbf{s}). \tag{40}$$

In equation (12), the PDD coefficients $y_\phi(\mathbf{d},\ \boldsymbol{\mu}_\mathbf{D},\ \boldsymbol{\mu}_\mathbf{S})$ and $C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$ and the polynomial basis $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$ are written as functions involving $\mathbf{d}$; however, they should be treated as constants when seeking the derivatives of $y(\mathbf{X};\mathbf{d},\ \mathbf{s})$ with respect to $\mathbf{d}$. Therefore, the term ∂y(X;d, s)/∂d$_\mathrm{k}$ can be written as:

$$\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k} = \sum_{\substack{u\subseteq\{1,\ldots,N\},\,k\in v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\},\,|u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\,\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$$

$$\times\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\frac{\partial\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k}\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}). \tag{41}$$

Applying the expectation operator $\mathbb{E}_\mathbf{d}$ on ∂y(X;d, s)/∂d_k and recognizing again the *zero*-mean and orthonormal properties of orthonormal basis, leads to:

$$\mathbb{E}_\mathbf{d}\left[\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k}\right] = \sum_{\substack{u=\emptyset,\,k\in v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\},\,|u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\,\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\frac{\partial\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k}\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}). \tag{42}$$

Similarly, applying the expectation operator $\mathbb{E}_\mathbf{d}$ on ∂y(X;d, s)/∂s_p and recognizing the *zero*-mean and orthonormal properties of orthonormal basis, leads to:

$$\mathbb{E}_\mathbf{d}\left[\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial s_p}\right] = \sum_{\substack{u=\emptyset,\,v\subseteq\{1,\ldots,M_d\}\\ p\in w\subseteq\{1,\ldots,M_s\},\,|u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\,\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\frac{\partial\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S})}{\partial s_p}. \tag{43}$$

Thus, the sensitivities of the first moment are:

$$\frac{\partial m^{(1)}(\mathbf{d},\mathbf{s})}{\partial d_k} = \sum_{i=1}^{N}\left[m^{(1)}(\mathbf{d},\mathbf{s})s_{ki,\emptyset}+\sum_{j=1}^{\infty}D_{k,ij}(\mathbf{d})C_{ij}(\mathbf{d},\mathbf{s})\right] + \sum_{\substack{u=\emptyset,\,k\in v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\},\,|u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\,\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\frac{\partial\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k}\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}) \tag{44}$$

and

$$\frac{\partial m^{(1)}(\mathbf{d},\mathbf{s})}{\partial s_p} = \sum_{\substack{u=\emptyset, v\subseteq\{1,\ldots,M_d\} \\ p\in w\subseteq\{1,\ldots,M_s\}, |u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}, \mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|} \\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\frac{\partial\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S})}{\partial s_p}, \tag{45}$$

representing closed-form expressions of the sensitivities in terms of the augmented PDD or Fourier-polynomial expansion coefficients of the response or log-density derivative functions.

4.1.1.2 Approximate sensitivities. When $y(\mathbf{X};\mathbf{d},\mathbf{s})$ and $s_{ki}(X_i;\mathbf{d})$ are replaced with their S-variate, mth-order augmented PDD and m' th-order Fourier-polynomial approximations, respectively, the resultant sensitivity equations, expressed by:

$$\frac{\partial\tilde{m}^{(1)}_{S,m}(\mathbf{d},\mathbf{s})}{\partial d_k} := \frac{\partial\mathbb{E}_\mathbf{d}\left[\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}{\partial d_k} = \sum_{i=1}^{N}\left[\tilde{m}^{(1)}_{S,m}(\mathbf{d},\mathbf{s})s_{ki,\emptyset} + \sum_{j=1}^{m_{\min}} D_{k,ij}(\mathbf{d}) \times C_{ij,S,m}(\mathbf{d},\mathbf{s})\right]$$
$$+ \sum_{\substack{u=\emptyset, k\in v\subseteq\{1,\ldots,M_d\} \\ w\subseteq\{1,\ldots,M_s\}, 1\leq|u|+|v|+|w|\leq S}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}, \mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|} \\ \|\mathbf{j}_{|u|}\|_\infty, \|\mathbf{l}_{|v|}\|_\infty, \|\mathbf{n}_{|w|}\|_\infty\leq m \\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\frac{\partial\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k}\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}) \tag{46}$$

and

$$\frac{\partial\tilde{m}^{(1)}_{S,m}(\mathbf{d},\mathbf{s})}{\partial s_p} := \frac{\partial\mathbb{E}_\mathbf{d}\left[\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}{\partial s_p} = \sum_{\substack{u=\emptyset, v\subseteq\{1,\ldots,M_d\} \\ p\in w\subseteq\{1,\ldots,M_s\}, 1\leq|u|+|v|+|w|\leq S}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}, \mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|} \\ \|\mathbf{j}_{|u|}\|_\infty, \|\mathbf{l}_{|v|}\|_\infty, \|\mathbf{n}_{|w|}\|_\infty\leq m \\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\frac{\partial\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S})}{\partial s_p}, \tag{47}$$

where $m_{\min}$: = min(m, m'), and

$$C_{ij,S,m}(\mathbf{d},\mathbf{s}) = \sum_{\substack{u=\{i\}, v\subseteq\{1,\ldots,M_d\} \\ w\subseteq\{1,\ldots,M_s\}, |u|+|v|+|w|\leq S}} \sum_{\substack{\mathbf{j}_{|u|}=j\in\mathbb{N}_0, \mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|} \\ \|\mathbf{j}_{|u|}\|_\infty, \|\mathbf{l}_{|v|}\|_\infty, \|\mathbf{n}_{|w|}\|_\infty\leq m \\ j,l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\,\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\,\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}) \tag{48}$$

become approximate, relying on the truncation parameters S, m and m' in general. It is elementary to show that the approximate sensitivities of the first moment, at appropriate limits, converge to the exact sensitivities when $S \to N+M$, $m \to \infty$, and $m' \to \infty$.

4.1.2 Sensitivity of the second moment. Setting $r = 2$ in equations (30) and (31), the sensitivities of the second moment are:

$$\frac{\partial m^{(2)}(\mathbf{d},\mathbf{s})}{\partial d_k} = \sum_{i=1}^{N} \mathbb{E}_{\mathbf{d}}\left[y^2(\mathbf{X};\mathbf{d},\mathbf{s}) s_{ki}(X_i;\mathbf{d})\right] + 2\mathbb{E}_{\mathbf{d}}\left[y(\mathbf{X};\mathbf{d},\mathbf{s})\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k}\right] \tag{49}$$

and

$$\frac{\partial m^{(2)}(\mathbf{d},\mathbf{s})}{\partial s_p} = 2\mathbb{E}_{\mathbf{d}}\left[y(\mathbf{X};\mathbf{d},\mathbf{s})\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial s_p}\right], \tag{50}$$

where $k = 1, \ldots, M_d$ and $p = 1, \ldots, M_s$.

4.1.2.1 Exact sensitivities. Using equations (34) and (37), the first term, $\mathbb{E}_{\mathbf{d}}\left[y^2(\mathbf{X};\mathbf{d},\mathbf{s}) s_{ki}(X_i;\mathbf{d})\right]$, on the right hand side of equation (49), aided by the *zero*-mean and orthonormal properties of orthonormal basis, can be expressed by:

$$\mathbb{E}_{\mathbf{d}}\left[y^2(\mathbf{X};\mathbf{d},\mathbf{s}) s_{ki}(X_i;\mathbf{d})\right] = m^{(2)}(\mathbf{d},\mathbf{s}) s_{ki,\varnothing} + 2m^{(1)}(\mathbf{d},\mathbf{s}) \sum_{j=1}^{\infty} C_{ij}(\mathbf{d},\mathbf{s}) D_{k,ij}(\mathbf{d}) + T_{ki}, \tag{51}$$

where

$$\begin{aligned} T_{ki} = & \sum_{\substack{u_1 \subseteq \{1,\ldots,N\},\, v_1 \subseteq \{1,\ldots,M_d\} \\ w_1 \subseteq \{1,\ldots,M_s\},\, |u_1|+|v_1|+|w_1| \ge 1}} \; \sum_{\substack{u_2 \subseteq \{1,\ldots,N\},\, v_2 \subseteq \{1,\ldots,M_d\} \\ w_2 \subseteq \{1,\ldots,M_s\},\, |u_2|+|v_2|+|w_2| \ge 1}} \\ & \sum_{\substack{\mathbf{j}_{|u_1|} \in \mathbb{N}_0^{|u_1|},\, \mathbf{l}_{|v_1|} \in \mathbb{N}_0^{|v_1|},\, \mathbf{n}_{|w_1|} \in \mathbb{N}_0^{|w_1|} \\ j_1,\ldots,j_{|u_1|},\, l_1,\ldots,l_{|v_1|},\, n_1,\ldots,n_{|w_1|} \neq 0}} \; \sum_{\substack{\mathbf{j}'_{|u_2|} \in \mathbb{N}_0^{|u_2|},\, \mathbf{l}'_{|v_2|} \in \mathbb{N}_0^{|v_2|},\, \mathbf{n}'_{|w_2|} \in \mathbb{N}_0^{|w_2|} \\ j'_1,\ldots,j'_{|u_2|},\, l'_1,\ldots,l'_{|v_2|},\, n'_1,\ldots,n'_{|w_2|} \neq 0}} \\ & \sum_{q=1}^{\infty} C_{u_1 v_1 w_1 \mathbf{j}_{|u_1|} \mathbf{l}_{|v_1|} \mathbf{n}_{|w_1|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) C_{u_2 v_2 w_2 \mathbf{j}'_{|u_2|} \mathbf{l}'_{|v_2|} \mathbf{n}'_{|w_2|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) D_{k,iq}(\mathbf{d}) \\ & \times \mathbb{E}_{\mathbf{d}}\Big[\psi_{u_1 \mathbf{j}_{|u_1|}}(\mathbf{X}_{u_1};\mathbf{d})\, \phi_{v_1 \mathbf{l}_{|v_1|}}(\mathbf{D}_{v_1};\boldsymbol{\mu}_{\mathbf{D}})\, \varphi_{w_1 \mathbf{n}_{|w_1|}}(\mathbf{S}_{w_1};\boldsymbol{\mu}_{\mathbf{S}})\, \psi_{u_2 \mathbf{j}'_{|u_2|}}(\mathbf{X}_{u_2};\mathbf{d})\, \phi_{v_2 \mathbf{l}'_{|v_2|}} \\ & \times (\mathbf{D}_{v_2};\boldsymbol{\mu}_{\mathbf{D}})\, \varphi_{w_2 \mathbf{n}'_{|w_2|}}(\mathbf{S}_{w_2};\boldsymbol{\mu}_{\mathbf{S}})\, \psi_{iq}(X_i;\mathbf{d})\Big], \end{aligned} \tag{52}$$

requiring expectations of various products of three random orthonormal polynomials as discussed in previous works (Ren and Rahman, 2013; Rahman and Ren, 2014).

The evaluation of the second term, $\mathbb{E}_{\mathbf{d}}\left[y(\mathbf{X};\mathbf{d},\mathbf{s})\partial y(\mathbf{X};\mathbf{d},\mathbf{s})/\partial d_k\right]$, on the right hand side of equation (49) requires restructuring equations (41) as:

$$\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k} = y_{d_k,\varnothing}(\mathbf{d},\mathbf{s}) + \sum_{\substack{u \subseteq \{1,\ldots,N\} \\ |u| \ge 1}} \; \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|} \\ j_1,\ldots,j_{|u|} \neq 0}} F_{d_k,u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s})\, \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d}), \tag{53}$$

where

$$y_{d_k,\varnothing}(\mathbf{d},\mathbf{s}) = \sum_{\substack{u=\varnothing, k\in v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\}, |u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$$
$$\times \frac{\partial \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k} \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}), \tag{54}$$

and

$$F_{d_k,u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}) = \sum_{\substack{k\in v\subseteq\{1,\ldots,M_d\}, w\subseteq\{1,\ldots,M_s\}\\ |u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$$
$$\times \frac{\partial \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k} \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}). \tag{55}$$

Hence, from equations (22) and (53) and using the orthonormal properties of $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$,

$$\mathbb{E}_\mathbf{d}\left[y(\mathbf{X};\mathbf{d},\mathbf{s})\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial d_k}\right] = m^{(1)}(\mathbf{d},\mathbf{s})y_{d_k,\varnothing}(\mathbf{d},\mathbf{s})$$
$$+ \sum_{\substack{u\subseteq\{1,\ldots,N\}\\ |u|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s})F_{d_k,u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}). \tag{56}$$

Similarly, the term $\mathbb{E}_\mathbf{d}\left[y(\mathbf{X};\mathbf{d},\mathbf{s})\partial y(\mathbf{X};\mathbf{d},\mathbf{s})/\partial s_p\right]$ on the right hand side of equation (50) can be analytically derived as:

$$\mathbb{E}_\mathbf{d}\left[y(\mathbf{X};\mathbf{d},\mathbf{s})\frac{\partial y(\mathbf{X};\mathbf{d},\mathbf{s})}{\partial s_p}\right] = m^{(1)}(\mathbf{d},\mathbf{s})y_{s_p,\varnothing}(\mathbf{d},\mathbf{s})$$
$$+ \sum_{\substack{u\subseteq\{1,\ldots,N\}\\ |u|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s})G_{s_p,u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}), \tag{57}$$

where

$$y_{s_p,\varnothing}(\mathbf{d},\mathbf{s}) = \sum_{\substack{u=\varnothing, v\subseteq\{1,\ldots,M_d\}\\ p\in w\subseteq\{1,\ldots,M_s\}, |u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|}, \mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})$$
$$\times \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D}) \frac{\partial \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S})}{\partial s_p}, \tag{58}$$

and

$$G_{s_p,u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}) = \sum_{\substack{v\subseteq\{1,\ldots,M_d\},p\in w\subseteq\{1,\ldots,M_s\}\\ |u|+|v|+|w|\geq 1}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) \times \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_{\mathbf{D}}) \frac{\partial\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}})}{\partial s_p}. \tag{59}$$

Thus, the sensitivities of the second moment are:

$$\frac{\partial m^{(2)}(\mathbf{d},\mathbf{s})}{\partial d_k} = \sum_{i=1}^{N}\left[m^{(2)}(\mathbf{d},\mathbf{s})s_{ki,\varnothing} + 2m^{(1)}(\mathbf{d},\mathbf{s})\sum_{j=1}^{\infty}C_{ij}(\mathbf{d},\mathbf{s})D_{k,ij}(\mathbf{d}) + T_{ki}\right] + m^{(1)}(\mathbf{d},\mathbf{s})y_{d_k,\varnothing}(\mathbf{d},\mathbf{s}) + \sum_{\substack{u\subseteq\{1,\ldots,N\}\\ |u|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s})F_{d_k,u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}) \tag{60}$$

and

$$\frac{\partial m^{(2)}(\mathbf{d},\mathbf{s})}{\partial s_p} = m^{(1)}(\mathbf{d},\mathbf{s})y_{s_p,\varnothing}(\mathbf{d},\mathbf{s}) + \sum_{\substack{u\subseteq\{1,\ldots,N\}\\ |u|\geq 1}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}) \times G_{s_p,u\mathbf{j}_{|u|}}(\mathbf{d},\mathbf{s}), \tag{61}$$

representing closed-form expressions of the sensitivities in terms of the augmented PDD or Fourier-polynomial expansion coefficients of the response or log-density derivative functions.

4.1.2.2 Approximate sensitivities. When $y(\mathbf{X};\mathbf{d},\mathbf{s})$ and $s_{ki}(X_i;\mathbf{d})$ are replaced by their S-variate, mth-order augmented PDD and m' th-order Fourier-polynomial approximations, respectively, the resultant sensitivity equations, expressed by:

$$\begin{aligned}\frac{\partial\tilde{m}^{(2)}_{S,m}(\mathbf{d},\mathbf{s})}{\partial d_k} &:= \frac{\partial\mathbb{E}_{\mathbf{d}}\left[\tilde{y}^2_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}{\partial d_k}\\ &= \sum_{i=1}^{N}\left[\tilde{m}^{(2)}_{S,m}(\mathbf{d},\mathbf{s})s_{ki,\varnothing} + 2\tilde{m}^{(1)}_{S,m}(\mathbf{d},\mathbf{s})\times\sum_{j=1}^{m_{\min}}C_{ij,S,m}(\mathbf{d},\mathbf{s})D_{k,ij}(\mathbf{d}) + \tilde{T}_{ki,m,m'}\right]\\ &\quad + \tilde{m}^{(1)}_{S,m}(\mathbf{d},\mathbf{s})\tilde{y}_{d_k,\varnothing,S,m}(\mathbf{d},\mathbf{s})\\ &\quad + \sum_{u\subseteq\{1,\ldots,N\}1\leq|u|\leq S}\ \sum_{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\|\mathbf{j}_{|u|}\|_\infty\leq m j_1,\ldots,j_{|u|}\neq 0} E_{u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s})F_{d_k,u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s})\end{aligned} \tag{62}$$

and

$$\frac{\partial \tilde{m}^{(2)}_{S,m}(\mathbf{d},\mathbf{s})}{\partial s_p} := \frac{\partial \mathbb{E}_{\mathbf{d}}\left[\tilde{y}^2_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}{\partial s_p} = \tilde{m}^{(1)}_{S,m}(\mathbf{d},\mathbf{s})\tilde{y}_{s_p,\varnothing,S,m}(\mathbf{d},\mathbf{s}) \sum_{\substack{u\subseteq\{1,\ldots,N\}\\ 1\le|u|\le S}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\ \|\mathbf{j}_{|u|}\|_\infty\le m\\ j_1,\ldots,j_{|u|}\neq 0}} E_{u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s})G_{s_p,u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s}), \tag{63}$$

where $m_{\min} := \min(m, m')$:

$$\tilde{y}_{d_k,\varnothing,S,m}(\mathbf{d},\mathbf{s}) = \sum_{\substack{u=\varnothing,\,k\in v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\}\\ 1\le|u|+|v|+|w|\le S}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ \|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S}) \times \frac{\partial \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k}\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}), \tag{64}$$

$$\tilde{y}_{s_p,\varnothing,S,m}(\mathbf{d},\mathbf{s}) = \sum_{\substack{u=\varnothing,\,v\subseteq\{1,\ldots,M_d\}\\ p\in w\subseteq\{1,\ldots,M_s\}\\ 1\le|u|+|v|+|w|\le S}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ \|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S}) \times \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\frac{\partial \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S})}{\partial s_p}, \tag{65}$$

$$F_{d_k,u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s}) = \sum_{\substack{k\in v\subseteq\{1,\ldots,M_d\},\,w\subseteq\{1,\ldots,M_s\}\\ 1\le|u|+|v|+|w|\le S}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ \|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S}) \times \frac{\partial \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})}{\partial d_k}\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S}), \tag{66}$$

$$\tilde{T}_{ki,m,m'} = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\sum_{j_3=1}^{m'}\tilde{C}_{i_1j_1}(\mathbf{d},\mathbf{s})\tilde{C}_{i_2j_2}(\mathbf{d},\mathbf{s})D_{k,ij_3}(\mathbf{d}) \times \mathbb{E}_{\mathbf{d}}\left[\psi_{i_1j_1}(X_{i_1};\mathbf{d})\,\psi_{i_2j_2}(X_{i_2};\mathbf{d})\,\psi_{ij_3}(X_i;\mathbf{d})\right], \tag{67}$$

$$G_{s_p,u\mathbf{j}_{|u|},S,m}(\mathbf{d},\mathbf{s}) = \sum_{\substack{v\subseteq\{1,\ldots,M_d\},\,p\in w\subseteq\{1,\ldots,M_s\}\\ 1\le|u|+|v|+|w|\le S}} \sum_{\substack{\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\,\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ \|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S}) \times \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\frac{\partial \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_\mathbf{S})}{\partial s_p}. \tag{68}$$

become approximate, relying on the truncation parameters S, m, and m' in general. It is elementary to show that the approximate sensitivities of the second moment also converge, to the exact sensitivities when $S \to N + M$, $m \to \infty$, and $m' \to \infty$.

4.2 Sensitivity of failure probability

Taking a partial derivative of the augmented PDD estimate of the failure probability in equation (24) with respect to d_k, $k = 1, \ldots, M_d$ or s_p, $p = 1, \ldots, M_s$, produces:

$$\frac{\partial \tilde{P}_{F,S,m}(\mathbf{d},\mathbf{s})}{\partial d_k} := \frac{\partial \mathbb{E}_{\mathbf{d}}\left[I_{\tilde{\Omega}_{F,S,m}}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}{\partial d_k} \tag{69}$$

or

$$\frac{\partial \tilde{P}_{F,S,m}(\mathbf{d},\mathbf{s})}{\partial s_p} := \frac{\partial \mathbb{E}_{\mathbf{d}}\left[I_{\tilde{\Omega}_{F,S,m}}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}{\partial s_p}, \tag{70}$$

where $I_{\tilde{\Omega}_{F,S,m}}(\mathbf{x};\mathbf{d},\mathbf{s})$ is the augmented PDD-generated indicator function, which is equal to *one* when $\mathbf{x} \in \mathbf{\Omega}_{F,S,m}$ and *zero* otherwise. As $I_{\tilde{\Omega}_{F,S,m}}(\mathbf{x};\mathbf{d},\mathbf{s})$ depends on the design vectors $\mathbf{d}$ and $\mathbf{s}$ and their corresponding derivatives are infinite, the Lebesgue dominated convergence theorem is not applicable. Hence, the PDD-MCS method developed in previous works (Ren *et al.*, 2016; Rahman, 2009; Rahman and Ren, 2014) for the reliability sensitivity of performance functions involving solely distributional design variables cannot be applied. The following finite-difference formulae, utilizing the augmented PDD expansion of the response function $y(\mathbf{X};\mathbf{d},\mathbf{s})$, are proposed to evaluate the sensitivity of reliability.

Assume that the design sensitivities at the design point $(\mathbf{d}, \mathbf{s})$ are sought. Let the small perturbations of the finite-difference approximation be Δd_k and Δs_p for the kth component of $\mathbf{d}$ and the pth component of $\mathbf{s}$, respectively, where $k = 1, \ldots, M_d$ and $p = 1, \ldots, M_s$. For the forward finite-difference approximation, the corresponding perturbed design vectors are $\mathbf{d} + \Delta d_k \cdot \mathbf{e}_k$ and $\mathbf{s} + \Delta s_p \cdot \mathbf{e}_p$, respectively, where $\mathbf{e}_k$ is the M_d-dimensional basis vector, in which the kth component is *one* and other components are *zeros*; similarly, $\mathbf{e}_p$ is the M_s-dimensional basis vector, in which the pth component is *one* and other components are *zeros*. Then, equation (14) induces two additional approximate response functions:

$$\begin{aligned}\tilde{y}_{S,m}(\mathbf{X};\mathbf{d}+\Delta d_k\cdot\mathbf{e}_k,\mathbf{s}) = y_{\varnothing}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}}) + \sum_{\substack{u\subseteq\{1,\ldots,N\},v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\},1\le|u|+|v|+|w|\le S}} \\ \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ \|\mathbf{j}_{|u|}\|_\infty,\|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{D}},\boldsymbol{\mu}_{\mathbf{S}})\,\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d}) \\ \times\,\phi_{v\mathbf{l}_{|v|}}\left((\mathbf{d}+\Delta d_k\cdot\mathbf{e}_k)_v;\boldsymbol{\mu}_{\mathbf{D}}\right)\varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w;\boldsymbol{\mu}_{\mathbf{S}})\end{aligned} \tag{71}$$

and

$$\tilde{y}_{S,m}\left(\mathbf{X};\mathbf{d},\mathbf{s}+\Delta s_p\cdot\mathbf{e}_p\right)=y_{i,\emptyset}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})+\sum_{\substack{u\subseteq\{1,\ldots,N\},v\subseteq\{1,\ldots,M_d\}\\ w\subseteq\{1,\ldots,M_s\},1\le|u|+|v|+|w|\le S}}\sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\mathbf{l}_{|v|}\in\mathbb{N}_0^{|v|},\mathbf{n}_{|w|}\in\mathbb{N}_0^{|w|}\\ \|\mathbf{j}_{|u|}\|_\infty,\|\mathbf{l}_{|v|}\|_\infty,\|\mathbf{n}_{|w|}\|_\infty\le m\\ j_1,\ldots,j_{|u|},l_1,\ldots,l_{|v|},n_1,\ldots,n_{|w|}\neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d},\boldsymbol{\mu}_\mathbf{D},\boldsymbol{\mu}_\mathbf{S})\,\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$$
$$\times\phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v;\boldsymbol{\mu}_\mathbf{D})\,\varphi_{w\mathbf{n}_{|w|}}\left((\mathbf{s}+\Delta s_p\cdot\mathbf{e}_p)_w;\boldsymbol{\mu}_\mathbf{S}\right), \tag{72}$$

owing to two finite-difference perturbations. The sensitivity of the probability of failure with respect to d_k by the forward finite-difference approximation is:

$$\begin{aligned}\frac{\partial\tilde{P}_{F,S,m}(\mathbf{d},\mathbf{s})}{\partial d_k}&=\lim_{\Delta d_k\to 0}\frac{1}{\Delta d_k}\left[\int_{\mathbb{R}^N}I_{\tilde{\Omega}_{F,S,m,\Delta d}}(\mathbf{X};\mathbf{d}+\Delta d_k\cdot\mathbf{e}_k,\mathbf{s})\right.\\&\left.\quad\times f_\mathbf{X}(\mathbf{x};\mathbf{d}+\Delta d_k\cdot\mathbf{e}_k)\mathrm{d}\mathbf{x}-\int_{\mathbb{R}^N}I_{\tilde{\Omega}_{F,S,m}}(\mathbf{X};\mathbf{d},\mathbf{s})f_\mathbf{X}(\mathbf{x};\mathbf{d})\mathrm{d}\mathbf{x}\right]\\&=\lim_{\Delta d_k\to 0}\frac{1}{\Delta d_k}\lim_{L\to\infty}\frac{1}{L}\left[\sum_{l_1=1}^{L}I_{\tilde{\Omega}_{F,S,m,\Delta d}}\left(\mathbf{x}^{(l_1)};\mathbf{d}+\Delta d_k\cdot\mathbf{e}_k,\mathbf{s}\right)\right.\\&\left.\quad-\sum_{l_2=1}^{L}I_{\tilde{\Omega}_{F,S,m}}\left(\mathbf{x}^{(l_2)};\mathbf{d},\mathbf{s}\right)\right],\quad k=1,\ldots,M_d,\end{aligned} \tag{73}$$

where $\tilde{\Omega}_{F,S,m,\Delta d}$ and $\tilde{\Omega}_{F,S,m}$ are failure domains determined by $\tilde{y}_{S,m}(\mathbf{X};\mathbf{d}+\Delta d_k\cdot\mathbf{e}_k,\mathbf{s})$ and $\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})$, respectively; L is the sample size; $\mathbf{x}^{(l_1)}$ is the l_1th realization of $\mathbf{X}$ with respect to PDF $f_\mathrm{X}(\mathbf{x};\mathbf{d}+\Delta d_k\cdot\mathbf{e}_k)$; and $\mathbf{x}^{(l_2)}$ is the l_2th realization of $\mathbf{X}$ with respect to PDF $f_\mathrm{X}(\mathbf{x};\mathbf{d})$.

Similarly, the sensitivity of the probability of failure with respect to s_p by finite-difference approximation is:

$$\begin{aligned}\frac{\partial\tilde{P}_{F,S,m}(\mathbf{d},\mathbf{s})}{\partial s_p}&=\lim_{\Delta s_p\to 0}\frac{1}{\Delta s_p}\left[\int_{\mathbb{R}^N}I_{\tilde{\Omega}_{F,S,m,\Delta s}}(\mathbf{X};\mathbf{d},\mathbf{s}+\Delta s_p\cdot\mathbf{e}_p)f_\mathbf{X}(\mathbf{x};\mathbf{d})\mathrm{d}\mathbf{x}\right.\\&\left.\quad-\int_{\mathbb{R}^N}I_{\tilde{\Omega}_{F,S,m}}(\mathbf{X};\mathbf{d},\mathbf{s})f_\mathbf{X}(\mathbf{x};\mathbf{d})\mathrm{d}\mathbf{x}\right]\\&=\lim_{\Delta s_p\to 0}\frac{1}{\Delta s_p}\lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L}\left[I_{\tilde{\Omega}_{F,S,m,\Delta s}}\left(\mathbf{x}^{(l)};\mathbf{d},\mathbf{s}+\Delta s_p\cdot\mathbf{e}_p\right)\right.\\&\left.\quad-I_{\tilde{\Omega}_{F,S,m}}\left(\mathbf{x}^{(l)};\mathbf{d},\mathbf{s}\right)\right],\quad p=1,\ldots,M_s,\end{aligned} \tag{74}$$

where $\tilde{\Omega}_{F,S,m,\Delta s}$ and $\tilde{\Omega}_{F,S,m}$ are failure domains determined by $\tilde{y}_{S,m}\left(\mathbf{X};\mathbf{d},\mathbf{s}+\Delta s_p\cdot\mathbf{e}_p\right)$ and $\tilde{y}_{S,m}(\mathbf{X};\mathbf{d},\mathbf{s})$, respectively; L is the sample size; and $\mathbf{x}^{(l)}$ is the lth realization of $\mathbf{X}$ with respect to PDF $f_\mathrm{X}(\mathbf{x};\mathbf{d})$.

It is essential to note that two additional approximate response functions in equations (71) and (72) are derived from the existing augmented PDD approximation used in equation (24) for reliability analysis, requiring no additional original function evaluations. Therefore, the reliability and its sensitivities have both been formulated as embedded MCS based on the same PDD expansion, facilitating their concurrent evaluations in a single stochastic simulation or analysis. In addition, it is important to note that fictitious distributions assigned to structural and distributional design variables are only for the purpose of incorporating those variables into PDD expansions and are involved in uncertainty quantification of response functions and associated sensitivity analysis.

5. Proposed optimization method

The augmented PDD approximations described in the preceding sections provide a means to evaluate the objective and constraint functions, including their design sensitivities, from a single stochastic analysis. An integration of reliability analysis, design sensitivity analysis, and a suitable optimization algorithm should render a convergent solution of the RDO and RBDO problems in equations (1) and (2). However, new stochastic and design sensitivity analyses, entailing re-calculations of the augmented PDD expansion coefficients, are needed at every design iteration. Therefore, a straightforward integration is expensive, depending on the cost of evaluating the objective and constraint functions and the requisite number of design iterations. In this section, a multi-point design process (Ren and Rahman, 2013; Ren *et al.*, 2016; Toropov *et al.*, 1993), where a series of single-step, augmented PDD approximations are built on a local subregion of the design space, is presented for solving the RDO and RBDO problems.

5.1 Multipoint approximation

Let

$$\mathcal{D} = \times_{k=1}^{k=M_d} \left[d_{k,L}, d_{k,U}\right] \times_{p=1}^{p=M_s} \left[s_{p,L}, s_{p,U}\right] \subseteq \mathbb{R}^M \tag{75}$$

be a rectangular domain, representing the design space of the RDO and RBDO problems defined by equation (1) or equation (2). For scalar variables $0 < \beta_{d,k}^{(q)} \leq 1$, $0 < \beta_{s,p}^{(q)} \leq 1$ and an initial design vector $\mathbf{d}_0^{(q)} = \left(d_{1,0}^{(q)}, \ldots, d_{M_d,0}^{(q)}, s_{1,0}^{(q)}, \ldots, s_{M_s,0}^{(q)}\right)$, the subset:

$$\begin{aligned}\mathcal{D}^{(q)} = {} & \times_{k=1}^{k=M_d} \left[d_{k,0}^{(q)} - \beta_{d,k}^{(q)}\left(d_{k,U} - d_{k,L}\right)/2, d_{k,0}^{(q)} + \beta_{d,k}^{(q)}\left(d_{k,U} - d_{k,L}\right)/2\right] \\ & \times_{p=1}^{p=M_s} \left[s_{p,0}^{(q)} - \beta_{s,p}^{(q)}\left(s_{p,U} - s_{p,L}\right)/2, s_{p,0}^{(q)} + \beta_{s,p}^{(q)}\left(s_{p,U} - s_{p,L}\right)/2\right] \\ & \subseteq \mathcal{D} \subseteq \mathbb{R}^M\end{aligned} \tag{76}$$

defines the qth subregion for q = 1,2, ... Using the multipoint approximation (Ren and Rahman, 2013; Ren *et al.*, 2016; Toropov *et al.*, 1993), the original RDO and RBDO problems in equations (1) and (2) are exchanged with a succession of simpler subproblems, as follows.

5.1.1 RDO.

$$\begin{aligned}
&\min_{(\mathbf{d},\mathbf{s})\in\mathcal{D}^{(q)}\subseteq\mathcal{D}}\ \tilde{c}^{(q)}_{0,S,m}(\mathbf{d},\mathbf{s}) := w_1\frac{\mathbb{E}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{0,S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}{\mu^*_0} + w_2\frac{\sqrt{\mathrm{var}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{0,S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right]}}{\sigma^*_0},\\
&\text{subject to}\ \tilde{c}^{(q)}_{l,S,m}(\mathbf{d},\mathbf{s}) := \alpha_l\sqrt{\mathrm{var}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{l,S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right]} - \mathbb{E}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{l,S,m}(\mathbf{X};\mathbf{d},\mathbf{s})\right] \le 0,\\
&\quad l=1,\ldots,K,\\
&d^{(q)}_{k,0} - \beta^{(q)}_{d,k}(d_{k,U}-d_{k,L})/2 \le d_k \le d^{(q)}_{k,0} + \beta^{(q)}_{d,k}(d_{k,U}-d_{k,L})/2,\\
&\quad k=1,\ldots,M_d,\\
&s^{(q)}_{p,0} - \beta^{(q)}_{s,p}(s_{p,U}-s_{p,L})/2 \le s_p \le s^{(q)}_{p,0} + \beta^{(q)}_{s,p}(s_{p,U}-s_{p,L})/2,\\
&\quad p=1,\ldots,M_s.
\end{aligned} \tag{77}$$

5.1.2 RBDO.

$$\begin{aligned}
&\min_{(\mathbf{d},\mathbf{s})\in\mathcal{D}^{(q)}\subseteq\mathcal{D}}\ \tilde{c}^{(q)}_{0,S,m}(\mathbf{d},\mathbf{s}),\\
&\text{subject to}\ \tilde{c}^{(q)}_{l,S,m}(\mathbf{d},\mathbf{s}) := P_{\mathbf{d}}\left[\mathbf{X}\in\tilde{\mathbf{\Omega}}^{(q)}_{F,l,S,m}(\mathbf{d},\mathbf{s})\right] - p_l \le 0\\
&\quad l=1,\ldots,K,\\
&d^{(q)}_{k,0} - \beta^{(q)}_{d,k}(d_{k,U}-d_{k,L})/2 \le d_k \le d^{(q)}_{k,0} + \beta^{(q)}_{d,k}(d_{k,U}-d_{k,L})/2,\\
&\quad k=1,\ldots,M_d,\\
&s^{(q)}_{p,0} - \beta^{(q)}_{s,p}(s_{p,U}-s_{p,L})/2 \le s_p \le s^{(q)}_{p,0} + \beta^{(q)}_{s,p}(s_{p,U}-s_{p,L})/2,\\
&\quad p=1,\ldots,M_s.
\end{aligned} \tag{78}$$

In equations (77) and (78), $\tilde{c}^{(q)}_{0,S,m}, \tilde{y}^{(q)}_{0,S,m}, \tilde{c}^{(q)}_{l,S,m}, \tilde{y}^{(q)}_{l,S,m}$ and $\tilde{\mathbf{\Omega}}^{(q)}_{F,l,S,m}$, $l=1,2,\ldots,K$, are local S-variate, mth-order augmented PDD approximations of c_0, y_0, c_l, y_l and $\mathbf{\Omega}_{F,l}$, respectively, at iteration q, where $\tilde{\mathbf{\Omega}}^{(q)}_{F,l,S,m}$ is defined using local augmented PDD approximations of $\tilde{y}^{(q)}_{l,S,m}$ of y_l; and $d^{(q)}_{k,0} - \beta^{(q)}_{k}(d_{k,U}-d_{k,L})/2$, $d^{(q)}_{k,0} + \beta^{(q)}_{k}(d_{k,U}-d_{k,L})/2$, $s^{(q)}_{p,0} - \beta^{(q)}_{s,p}(s_{p,U}-s_{p,L})/2$, and $s^{(q)}_{p,0} + \beta^{(q)}_{s,p}(s_{p,U}-s_{p,L})/2$, also known as the move limits, are the lower and upper bounds, respectively, of the associated coordinate of subregion $\mathcal{D}^{(q)}$. Hence, the original objective and constraint functions are replaced with those derived locally from respective augmented PDD approximations. As the augmented PDD approximations are mean-square convergent (Rahman, 2008; Rahman, 2009), they also converge in probability and in distribution. Therefore, given a subregion $\mathcal{D}^{(q)}$, the solution of the associated RDO and RBDO subproblems also converges when $S\to N+M$, $m\to\infty$, and $m'\to\infty$.

5.2 Single-step procedure

The single-step procedure is motivated on solving each RDO or RBDO subproblem in Equations (77) or (78) from a single stochastic analysis by sidestepping the need to recalculate the PDD expansion coefficients at every design iteration. It subsumes two important assumptions: an S-variate, mth-order augmented PDD approximation $\tilde{y}_{S,m}$ of y at the initial design is acceptable for all possible designs in the subregion; and the expansion coefficients for one design, derived from those generated for another design, are accurate.

Consider a change of the probability measure of $(\mathbf{X}, \mathbf{D}, \mathbf{S})$ from $f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}) f_{\mathbf{D}}(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}) f_{\mathbf{S}}(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}) d\mathbf{x} d\mathbf{d} d\mathbf{s}$ to $f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}') f_{\mathbf{D}}(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}') f_{\mathbf{S}}(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}') d\mathbf{x} d\mathbf{d} d\mathbf{s}$, where $(\mathbf{d}, \mathbf{s})$ and $(\mathbf{d}', \mathbf{s}')$ are two arbitrary design vectors corresponding to old and new designs, respectively, and $\boldsymbol{\mu}_{\mathbf{D}}'$ and $\boldsymbol{\mu}_{\mathbf{S}}'$ are new mean vectors for corresponding affiliated random vector, respectively. Let $\{\psi_{i_q j_q}(X_{i_q}; \mathbf{d}'); j_q = 0, 1, \ldots\}$, $\{\phi_{k_r l_r}(D_{k_r}; \boldsymbol{\mu}_{\mathbf{D}}'); l_r = 0, 1, \ldots\}$ and $\{\varphi_{p_t n_t}(S_{p_t}; \boldsymbol{\mu}_{\mathbf{S}}'); n_t = 0, 1, \ldots\}$ be three sets of new orthonormal polynomial basis functions consistent with the marginal probability measures $f_{X_{i_q}}(x_{i_q}; \mathbf{d}') dx_{i_q}$ of X_i, $f_{D_{k_r}}(d_{k_r}; \boldsymbol{\mu}_{\mathbf{D}}') dd_{k_r}$ of D_{k_r} and $f_{S_{p_t}}(s_{p_t}; \boldsymbol{\mu}_{\mathbf{S}}') ds_{p_t}$ of S_{p_t}, respectively, producing new product polynomials:

$$\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d}') \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_{\mathbf{D}}') \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_{\mathbf{S}}') = \prod_{q=1}^{|u|} \psi_{i_q j_q}(X_{i_q}; \mathbf{d}') \times \prod_{r=1}^{|v|} \phi_{k_r l_r}(d_{k_r}; \boldsymbol{\mu}_{\mathbf{D}}') \prod_{t=1}^{|w|} \varphi_{p_t n_t}(s_{p_t}; \boldsymbol{\mu}_{\mathbf{S}}'), \tag{79}$$

where $\emptyset \neq u \subseteq \{1, \ldots, N\}$, $\emptyset \neq v \subseteq \{1, \ldots, M_d\}$ and $\emptyset \neq w \subseteq \{1, \ldots, M_s\}$. Assume that the expansion coefficients, $y_\emptyset(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}})$ and $C_{u\mathbf{j}_{|u|}}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}})$, for the old design have been calculated already. Then, the expansion coefficients for the new design are determined from:

$$y_\emptyset(\mathbf{d}', \boldsymbol{\mu}_{\mathbf{D}}', \boldsymbol{\mu}_{\mathbf{S}}') = \int_{\mathbb{R}^{N+M}} \Bigg[\sum_{\substack{u \subseteq \{1,\ldots,N\}, v \subseteq \{1,\ldots,M_d\} \\ w \subseteq \{1,\ldots,M_s\}, |u|+|v|+|w| \geq 1}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \mathbf{l}_{|v|} \in \mathbb{N}_0^{|v|}, \mathbf{n}_{|w|} \in \mathbb{N}_0^{|w|} \\ j_1, \ldots, j_{|u|}, l_1, \ldots, l_{|v|}, n_1, \ldots, n_{|w|} \neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u; \mathbf{d}) \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_{\mathbf{D}}) \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_{\mathbf{S}}) + y_\emptyset(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}}) \Bigg] \times f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}') f_{\mathbf{D}}(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}') f_{\mathbf{S}}(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}') d\mathbf{x} d\mathbf{d} d\mathbf{s} \tag{80}$$

and

$$C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}', \boldsymbol{\mu}_{\mathbf{D}}', \boldsymbol{\mu}_{\mathbf{S}}') = \int_{\mathbb{R}^{N+M}} \Bigg[\sum_{\substack{u \subseteq \{1,\ldots,N\}, v \subseteq \{1,\ldots,M_d\} \\ w \subseteq \{1,\ldots,M_s\}, |u|+|v|+|w| \geq 1}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \mathbf{l}_{|v|} \in \mathbb{N}_0^{|v|}, \mathbf{n}_{|w|} \in \mathbb{N}_0^{|w|} \\ j_1, \ldots, j_{|u|}, l_1, \ldots, l_{|v|}, n_1, \ldots, n_{|w|} \neq 0}} C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u; \mathbf{d}) \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_{\mathbf{D}}) \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_{\mathbf{S}}) + y_\emptyset(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}}) \Bigg] \times \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u; \mathbf{d}') \phi_{v\mathbf{n}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_{\mathbf{D}}') \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_{\mathbf{S}}') f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}') f_{\mathbf{D}}(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}') f_{\mathbf{S}}(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}') d\mathbf{x} d\mathbf{d} d\mathbf{s} \tag{81}$$

by recycling the old expansion coefficients and using orthonormal polynomials associated with both designs. The relationship between the old and new coefficients, described by equations (80) and (81), is exact and is obtained by replacing y in equations (10) and (11) with the right side of equation (9). However, in practice, when the S-variate, m-th order augmented PDD approximation [equation (13)] is used to replace y in equations (10) and (11), then the new expansion coefficients:

$$y_{\varnothing}\left(\mathbf{d}', \boldsymbol{\mu}_{\mathbf{D}}', \boldsymbol{\mu}_{\mathbf{S}}'\right) = \int_{\mathbb{R}^{N+M}} \left[\sum_{\substack{u \subseteq \{1,\ldots,N\}, v \subseteq \{1,\ldots,M_d\} \\ w \subseteq \{1,\ldots,M_s\}, 1 \le |u|+|v|+|w| \le S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \mathbf{l}_{|v|} \in \mathbb{N}_0^{|v|}, \mathbf{n}_{|w|} \in \mathbb{N}_0^{|w|} \\ \|\mathbf{j}_{|u|}\|_\infty, \|\mathbf{l}_{|v|}\|_\infty, \|\mathbf{n}_{|w|}\|_\infty \le m \\ j_1,\ldots,j_{|u|}, l_1,\ldots,l_{|v|}, n_1,\ldots,n_{|w|} \neq 0}} \right.$$

$$\left. \times C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u; \mathbf{d}) \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_{\mathbf{D}}) \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_{\mathbf{S}}) + y_{\varnothing}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}}) \right]$$
$$\times f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}') f_{\mathbf{D}}\left(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}'\right) f_{\mathbf{S}}\left(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}'\right) d\mathbf{x}\, d\mathbf{d}\, d\mathbf{s} \tag{82}$$

and

$$C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}\left(\mathbf{d}', \boldsymbol{\mu}_{\mathbf{D}}', \boldsymbol{\mu}_{\mathbf{S}}'\right) = \int_{\mathbb{R}^{N+M}} \left[\sum_{\substack{u \subseteq \{1,\ldots,N\}, v \subseteq \{1,\ldots,M_d\} \\ w \subseteq \{1,\ldots,M_s\}, 1 \le |u|+|v|+|w| \le S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \mathbf{l}_{|v|} \in \mathbb{N}_0^{|v|}, \mathbf{n}_{|w|} \in \mathbb{N}_0^{|w|} \\ \|\mathbf{j}_{|u|}\|_\infty, \|\mathbf{l}_{|v|}\|_\infty, \|\mathbf{n}_{|w|}\|_\infty \le m \\ j_1,\ldots,j_{|u|}, l_1,\ldots,l_{|v|}, n_1,\ldots,n_{|w|} \neq 0}} \right.$$

$$\left. \times C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}, \mathbf{s}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u; \mathbf{d}) \phi_{v\mathbf{l}_{|v|}}(\mathbf{d}_v; \boldsymbol{\mu}_{\mathbf{D}}) \varphi_{w\mathbf{n}_{|w|}}(\mathbf{s}_w; \boldsymbol{\mu}_{\mathbf{S}}) + y_{\varnothing}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{D}}, \boldsymbol{\mu}_{\mathbf{S}}) \right] \psi_{u\mathbf{j}_{|u|}}\left(\mathbf{x}_u; \mathbf{d}'\right)$$
$$\times \phi_{v\mathbf{l}_{|v|}}\left(\mathbf{d}_v; \boldsymbol{\mu}_{\mathbf{D}}'\right) \varphi_{w\mathbf{n}_{|w|}}\left(\mathbf{s}_w; \boldsymbol{\mu}_{\mathbf{S}}'\right) f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}') f_{\mathbf{D}}\left(\mathbf{d}; \boldsymbol{\mu}_{\mathbf{D}}'\right) f_{\mathbf{S}}\left(\mathbf{s}; \boldsymbol{\mu}_{\mathbf{S}}'\right) d\mathbf{x}\, d\mathbf{d}\, d\mathbf{s}, \tag{83}$$

which are applicable for $u \subseteq \{1, \ldots, N\}$, $v \subseteq \{1, \ldots, M_d\}$, $w \subseteq \{1, \ldots, M_s\}$ and $1 \le |u| + |v| + |w| \le S$, become approximate, although convergent. Simply replacing $\boldsymbol{\mu}_{\mathbf{D}}'$ and $\boldsymbol{\mu}_{\mathbf{S}}'$ with $\mathbf{d}'$ and $\mathbf{s}'$, respectively, in equations (82) and (83) leads to the PDD coefficients for the new design. Furthermore, the integrals in equations (82) and (83) consist of finite-order polynomial functions of at most S variables and can be evaluated inexpensively without having to compute the original function y for the new design. Therefore, new stochastic analyses, all using S-variate, mth-order augmented PDD approximation of y, are conducted with little additional cost during all design iterations, drastically curbing the computational effort of solving an RDO/RBDO subproblem.

5.3 Proposed multipoint single-step design process

When the multipoint approximation is combined with the single-step procedure, the result is an accurate and efficient design process to solve the RDO and RBDO problems defined by equations (1) and (2). Using the single-step procedure, the design solution of an individual

RDO/RBDO subproblem becomes the initial design for the next RDO/RBDO subproblem. Then, the move limits are updated, and the optimization is repeated iteratively until an optimal solution is attained. The method is schematically depicted in Figure 1. Given an initial design $(\mathbf{d}_0, \mathbf{s}_0)$, a sequence of design solutions, obtained successively for each subregion $\mathcal{D}^{(q)}$ and using the S-variate, mth-order augmented PDD approximation, leads to an approximate optimal solution $(\tilde{\mathbf{d}}^*, \tilde{\mathbf{s}}^*)$ of the RDO/RBDO problem. In contrast, an augmented PDD approximation constructed for the entire design space $\mathcal{D}$, if it commits large approximation errors, may possibly lead to a premature or an erroneous design solution. The multipoint approximation in the proposed methods overcomes this quandary by adopting smaller subregions and local augmented PDD approximations, whereas the single-step procedure diminishes the computational requirement as much as possible by recycling the PDD expansion coefficients.

When $S \to N + M$, $m \to \infty$, $m' \to \infty$, and $q \to \infty$, the moments, reliability, and their design sensitivities by the augmented PDD approximations converge to their exactness, yielding coincident solutions of the original RDO/RBDO problems [equations (1) and (2)] and RDO/RBDO subproblems [equations (77) and (78)]. However, if the subregions are sufficiently small, then for finite and possibly low values of S and m, equations (77) or (78) is expected to generate an accurate solution of equations (1) or (2), the principal motivation for developing the augmented PDD methods.

The augmented PDD methods in conjunction with the combined multi-point, single-step design process are outlined by the following steps. The flow chart of this method is shown in Figure 2.

Step 1. Select an initial design vector $(\mathbf{d}_0, \mathbf{s}_0)$. Define tolerances $\epsilon_1 > 0$, $\epsilon_2 > 0$, and $\epsilon_3 > 0$. Set the iteration $q = 1$, $(\mathbf{d}, \mathbf{s})_0^{(q)} = (\mathbf{d}_0, \mathbf{s}_0)$. Define the subregion size parameters $0 < \beta_{d,k}^{(q)} \leq 1$, $k = 1, \ldots, M_d$, and $0 < \beta_{s,p}^{(q)} \leq 1$, $p = 1, \ldots, M_s$, describing the qth subregion defined in equation (76). Denote the subregion's increasing history by a set $H^{(0)}$ and set it to empty. Set two designs $(\mathbf{d}, \mathbf{s})_f = (\mathbf{d}_0, \mathbf{s}_0)$ and $(\mathbf{d}_0, \mathbf{s}_0)_{f,\ last} \neq (\mathbf{d}_0, \mathbf{s}_0)$ such that $||(\mathbf{d}_0, \mathbf{s}_0)_f - (\mathbf{d}_0, \mathbf{s}_0)_{f,last}||_2 > \epsilon_1$. Set $(\mathbf{d}, \mathbf{s})_*^{(0)} = (\mathbf{d}_0, \mathbf{s}_0)$, $q_{f,\ last} = 1$ and $q_f = 1$. Usually, a feasible design should be selected to be the initial design $(\mathbf{d}_0, \mathbf{s}_0)$. However, when an infeasible initial design is chosen, a new feasible design can be obtained during the iteration if the initial subregion size parameters are large enough.

Step 2. Select (q = 1) or use (q > 1) the PDD truncation parameters S and m. At $(\mathbf{d}, \mathbf{s}) = (\mathbf{d}, \mathbf{s})_0^{(q)}$, generate the augmented PDD expansion coefficients, $y_\emptyset(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{S})$ and $C_{uvw\mathbf{j}_{|u|}\mathbf{l}_{|v|}\mathbf{n}_{|w|}}(\mathbf{d}, \boldsymbol{\mu}_\mathbf{D}, \boldsymbol{\mu}_\mathbf{S})$, where $\emptyset \neq u \subseteq \{1, \ldots, N\}$, $\emptyset \neq v \subseteq \{1, \ldots, M_d\}$,

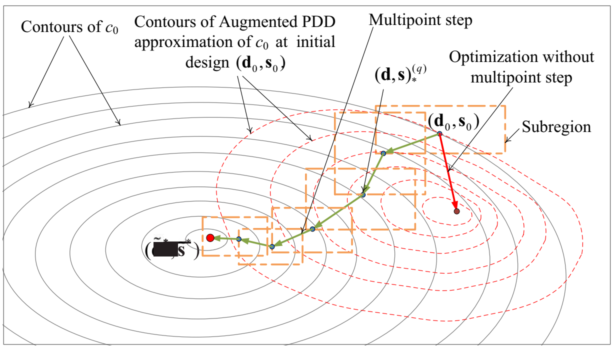


Figure 1.
A schematic description of the multi-point, single-step design process

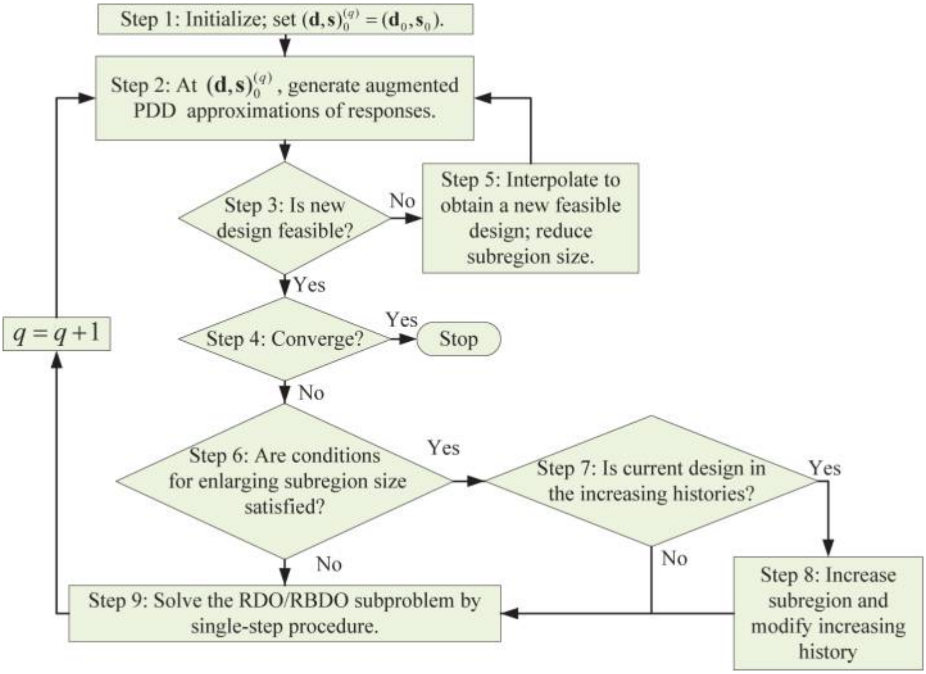


Figure 2.
A flow chart of the proposed multi-point, single-step design process

$\varnothing \neq w \subseteq \{1,\ldots,M_s\}$, $1 \leq |u| + |v| + |w| \leq S$, $\|\mathbf{j}_{|u|}\|_\infty, \|\mathbf{l}_{|v|}\|_\infty, \|\mathbf{n}_{|w|}\|_\infty \leq m$, using dimension-reduction integration with $R = S$, $n = m + 1$, leading to S-variate, mth-order augmented PDD approximations $\tilde{y}_{l,S,m}^{(q)}(\mathbf{X};\mathbf{d},\mathbf{s})$ of $y_l(\mathbf{X};\mathbf{d},\mathbf{s})$ and $\tilde{c}_{l,S,m}^{(q)}(\mathbf{d},\mathbf{s})$ of $c_l(\mathbf{d},\mathbf{s})$, $l = 0,1,\ldots,K$, in Equations. (77) or (78). For RDO, calculate the expansion coefficients of score functions, $s_{k,\varnothing}(\mathbf{d})$ and $D_{i_k,j}(\mathbf{d})$, where $k = 1,\ldots,M$ and $j = 1,\ldots,m'$, analytically, if possible, or numerically, resulting in m' th-order Fourier-polynomial approximations of $s_k(X_{i_k};\mathbf{d})$, $k = 1,\ldots,M$.

Step 3. If q = 1 and $\tilde{c}_{l,S,m}^{(q)}\left((\mathbf{d},\mathbf{s})_0^{(q)}\right) < 0$ for l = 1, …, K, then go to Step 4. If q > 1 and $\tilde{c}_{l,S,m}^{(q)}\left((\mathbf{d},\mathbf{s})_0^{(q)}\right) < 0$ for $l = 1,\ldots,K$, then set $(\mathbf{d},\mathbf{s})_{f,last} = (\mathbf{d},\mathbf{s})_f$, $(\mathbf{d},\mathbf{s})_f = (\mathbf{d},\mathbf{s})_0^{(q)}$, $q_{f,last} = q_f$, $q_f = q$ and go to Step 4. Otherwise, go to Step 5.

Step 4. If $\|(\mathbf{d},\mathbf{s})_f - (\mathbf{d},\mathbf{s})_{f,last}\|_2 < \epsilon_1$ or $\left|\left[\tilde{c}_{0,S,m}^{(q)}\left((\mathbf{d},\mathbf{s})_f\right) - \tilde{c}_{0,S,m}^{(q_{f,last})}\left((\mathbf{d},\mathbf{s})_{f,last}\right)\right] / \tilde{c}_{0,S,m}\ (q)\ \left((\mathbf{d},\mathbf{s})_f\right)\right| < \epsilon_3$, then stop and denote the final optimal solution as $\left(\tilde{\mathbf{d}}^*,\tilde{\mathbf{s}}^*\right) = (\mathbf{d},\mathbf{s})_f$. Otherwise, go to Step 6.

Step 5. Compare the infeasible design $(\mathbf{d},\mathbf{s})_0^{(q)}$ with the feasible design (d,s)f and interpolate between $(\mathbf{d},\mathbf{s})_0^{(q)}$ and (d,s)f to obtain a new feasible design and set it as $(\mathbf{d},\mathbf{s})_0^{(q+1)}$. For dimensions with large differences between $(\mathbf{d},\mathbf{s})_0^{(q)}$ and (d,s)f, interpolate aggressively. Reduce the size of the subregion $\mathcal{D}^{(q)}$ to obtain new subregion $\mathcal{D}^{(q+1)}$. For dimensions with large differences between $(\mathbf{d},\mathbf{s})_0^{(q)}$ and (d,s)f, reduce aggressively. Also, for dimensions with large differences between the sensitivities of $\tilde{c}_{l,Sm}^{(q)}\left((\mathbf{d},\mathbf{s})_0^{(q)}\right)$ and $\tilde{c}_{l,Sm}^{(q-1)}\left((\mathbf{d},\mathbf{s})_0^{(q)}\right)$, reduce aggressively. Update q = q + 1 and go to Step 2.

Step 6. If the subregion size is small, that is, $\beta_{d,k}^{(q)}(d_{k,U}-d_{k,L}) < \epsilon_2$, or $\beta_{s,p}^{(q)}(s_{p,U}-s_{p,L}) < \epsilon_2$, and $(\mathbf{d},\mathbf{s})_*^{(q-1)}$ is located on the boundary of the subregion, then go to Step 7. Otherwise, go to Step 9.

Step 7. If the subregion centered at $(\mathbf{d},\mathbf{s})_0^{(q)}$ has been enlarged before, that is, $(\mathbf{d},\mathbf{s})_0^{(q)} \in H^{(q-1)}$, then set $H^{(q)} = H^{(q-1)}$ and go to Step 9. Otherwise, set $H^{(q)} = H^{(q-1)} \cup \{(\mathbf{d},\mathbf{s})_0^{(q)}\}$ and go to Step 8.

Step 8. For coordinates of $(\mathbf{d},\mathbf{s})_0^{(q)}$ located on the boundary of the subregion and $\beta_{d,k}^{(q)}(d_{k,U}-d_{k,L}) < \epsilon_2$, or $\beta_{s,p}^{(q)}(s_{p,U}-s_{p,L}) < \epsilon_2$, increase the sizes of corresponding components of $\mathcal{D}^{(q)}$; for other coordinates, keep them as they are. Set the new subregion as $\mathcal{D}^{(q+1)}$.

Step 9. Solve the design problem in equations (77) or (78) using the single-step PDD procedure. In so doing, recycle the PDD expansion coefficients obtained from Step 2 in equations (82) and (83), producing approximations of the objective and constraint functions that stem from single calculation of these coefficients. To evaluate the gradients, recalculate the Fourier expansion coefficients of score functions as needed. Denote the optimal solution by $(\mathbf{d},\mathbf{s})_*^{(q)}$ and set $(\mathbf{d},\mathbf{s})_0^{(q+1)} = (\mathbf{d},\mathbf{s})_*^{(q)}$. Update q = q + 1 and go to Step 2.

6. Numerical examples

Four examples are presented to illustrate the proposed methods developed in estimating design sensitivities and solving various RDO/RBDO problems involving mixed design variables. The objective and constraint functions are either elementary mathematical functions or relate to engineering problems, ranging from simple structures to complex FEA-aided mechanical designs. Both size and shape design problems are included. The PDD expansion coefficients were estimated by dimension-reduction integration with the mean input as the reference point, $R = S$, and the number of integration points $n = m + 1$, where S and m vary depending on the problem. More specifically, the truncation parameters S and m depend on the dimensional structure and nonlinearity of a stochastic response. The higher the values of S and m, the higher the accuracy, and also the computational cost that is endowed with an Sth order polynomial computational complexity. In the case that the dimensional hierarchy or nonlinearity is not known *a priori*, an adaptive sparse PDD method is recommended, which performs global sensitivity analysis (Ren *et al.*, 2016; Rahman, 2011; Song *et al.*, 2016) based on the Sobol indices and determine these truncation parameters automatically by progressively drawing in higher-variate or higher-order contributions as appropriate. Interested readers are referred to authors' previous work (Ren *et al.*, 2016; Rahman, 2011).

As the distributional design variables describe both means and standard deviations of Gaussian random variables, the order m' used for Fourier expansion coefficients of score functions in Example 1 is two. However, in Example 4, where the distributional design variables are the means of truncated Gaussian random variables, m' is one. In Examples 1 through 4, orthonormal polynomials, consistent with the probability distributions of input random variables, were used as bases. For the Gaussian distribution, the Hermite polynomials were used. For random variables following non-Gaussian probability distributions, such as the Lognormal distribution in Example 3 and truncated Gaussian distribution in Example 4, the orthonormal polynomials were obtained either analytically when possible or numerically, exploiting the Stieltjes procedure DOT (2011). The sample size for the embedded MCS is 10^6 in all examples. The multi-point, single-step design procedure was used in Examples 3 and 4 for solving RDO and RBDO problems. The tolerances, initial subregion size and threshold parameters for the multi-point, single-step procedure are as follows: (1) $\epsilon_1 = 0.01$, $\epsilon_2 = 2$, $\epsilon_3 = 0.005$ (Example 3); $\epsilon_1 = 0.01$, $\epsilon_2 = 2$, $\epsilon_3 = 0.05$ (Example 4); (2) $\beta_{d,1}^{(1)} = \ldots = \beta_{d,M_d}^{(1)} = \beta_{s,1}^{(1)} = \ldots = \beta_{s,M_s}^{(1)} = 0.5$. The optimization algorithm selected is sequential quadratic programming (DOT, 2001) in Examples 3 and 4.

6.1 Example 1: sensitivities of moments

The first example involves calculating sensitivities of the first two moments of a polynomial function:

$$\begin{aligned} y(\mathbf{X};\mathbf{d},\mathbf{S}) = {} & 13.2(X_1 + X_2 + \mu + \sigma + s_1 + s_2) + 0.18(X_1 + X_2)3 + 0.31X_1^2X_2s_1 \\ & + 0.25X_2^2s_1\mu + 0.11X_1s_2\sigma + 0.4s_1^2s_2\mu^2, \end{aligned} \tag{84}$$

where X_1 and X_2 are two independent and identically distributed Gaussian random variables, each with the same mean μ and standard deviation σ. The distributional and structural design vectors are $\mathbf{d} = (\mu,\ \sigma)^T$ and $\mathbf{s} = (s_1,s_2)^T$, respectively. The affiliated random vector $\mathbf{D} = (D_1,D_2)^T$ is selected to be Gaussian, where the components D_1, D_2 are independent with the same standard deviation of 1 but different mean values $\mathbb{E}_1[D_1] = d_1 = \mu$ and $\mathbb{E}_1[D_2] = d_2 = \sigma$. The affiliated fictitious random vector $\mathbf{S} = (S_1,S_2)^T$ is also normally distributed with the independent components S_1, S_2, which have the same fictitious standard deviation of 1 but different mean values $\mathbb{E}_2[S_1] = s_1$ and $\mathbb{E}_2[S_2] = s_2$.

Table I presents the approximate sensitivities of the first two moments $\mathbb{E}_\mathbf{d}[y(\mathbf{X};\mathbf{d},\mathbf{S})]$ and $\mathbb{E}_\mathbf{d}\left[y^2(\mathbf{X};\mathbf{d},\mathbf{S})\right]$ at $\mathbf{d} = \mathbf{d}_0 = (0.4,1)^T$ and $\mathbf{s} = \mathbf{s}_0 = (0.55,\ 0.48)^T$, obtained by the proposed augmented PDD methods (Equations. (46),(47),(62), and (63)). Three sets of estimates stemming from univariate ($S = 1$), bivariate ($S = 2$) and trivariate ($S = 3$) third-order PDD approximations of y are included. The exact solution, which exists for this problem, is also included in Table II. The univariate PDD, listed in the second column, provides satisfactory estimates for all sensitivities, requiring only 26 function evaluations. Although the bivariate approximation is more expensive than the univariate approximation, the former generates highly accurate solutions, as expected. The function y, being both trivariate and a cubic polynomial, is exactly reproduced by the trivariate ($S = 3$), third-order ($m = 3$) augmented PDD approximation when orthonormal polynomials consistent with Gaussian probability measures are used. Therefore, the trivariate, third-order augmented PDD approximation, along with the proposed sensitivity analysis method, reproduces the exact solution. Although the third-order, bivariate augmented PDD approximation is unable to reproduce the original function exactly, it provides highly accurate sensitivity results for almost all cases, which are the same as the exact or trivariate results up to at least six significant digits. The only exception is for the sensitivity of the second moment with respect to s_1, which has about one percent error. Comparing the computational efforts, 1,546 function evaluations were required by trivariate PDD to produce the exact results, whereas 26 and 266 function evaluations were incurred by the univariate and bivariate approximations, respectively. Therefore, the univariate augmented PDD method furnishes very accurate and highly efficient estimates of the first two moment sensitivities.

6.2 Example 2: sensitivities of failure probability

For the second example, consider two performance functions:

$$y_1(\mathbf{X}; s) = -s + 1 + \frac{X_1^2 X_2^2}{5s^2} \tag{85}$$

and

$$y_2(\mathbf{X}; s) = -1 + \frac{5s^4}{X_1^2 + 8X_2 + 5}, \tag{86}$$

where the random vector $\mathbf{X}$ comprises two independent Gaussian random variables, X_1 and X_2, with the same standard deviation of 0.3 but different mean values

	Univariate (m = 3)	Augmented PDD Bivariate (m = 3)	Trivariate (m = 3)	Exact
$\partial m^{(1)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial \mu$	41.6183	43.0063	43.0063	43.0063
$\partial m^{(1)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial \sigma$	15.1955	15.1955	15.1955	15.1955
$\partial m^{(1)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial s_1$	13.2696	13.4936	13.4936	13.4936
$\partial m^{(1)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial s_2$	13.2634	13.2634	13.2634	13.2634
$\partial m^{(2)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial \mu$	3700.9977	3895.1957	3895.1957	3895.1957
$\partial m^{(2)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial \sigma$	2201.2607	2365.6375	2365.6375	2365.6375
$\partial m^{(2)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial s_1$	1161.5037	1188.3302	1198.9252	1198.9252
$\partial m^{(2)}(\mathbf{d}_0, \boldsymbol{S}_0)/\partial s_2$	1160.9547	1164.1960	1164.1960	1164.1960
No. of Func. Eval.	26	266	1546	–

Table I. Sensitivities of the first two moments at d_0 = (0.4,1)T and s_0 = (0.55,0.48)T

Methods	m	S	$P_{F,1}(s_0)$	$P_{F,2}(s_0)$	$\frac{\partial P_{F,1}(s_0)}{\partial s}$	$\frac{\partial P_{F,2}(s_0)}{\partial s}$	No. of Func. Eval. of y_1 and y_2
Augmented PDD	1	1	0.1122	0.0300	0.4180	−1.7590	14
		2	0.1130	0.0306	0.4290	−1.8100	38
		3	0.1117	0.0264	0.2320	−1.4630	54
	2	1	0.0921	0.0243	0.6155	−1.6224	20
		2	0.0932	0.0249	0.6570	−1.6830	74
		3	0.0908	0.0203	0.3268	−1.4601	128
	3	1	0.0921	0.0244	0.5998	−1.6942	26
		2	0.0932	0.0250	0.6496	−1.7117	122
		3	0.0908	0.0204	0.3167	−1.4677	250
FORM	N/A	N/A	0.0549	0.0192	0.3700	−1.2600	106
SORM	N/A	N/A	0.0552	0.0230	0.3800	−1.4700	134
Crude-MCS	N/A	N/A	0.0905	0.0204	0.3228	−1.4139	4×10^8

Table II. Sensitivities of probability of failure at s_0 = 2

$\mathbb{E}_{\mathbf{d}}[X_1] = 7.5$ and $\mathbb{E}_{\mathbf{d}}[X_2] = 1$. The sole structural design variable is s. The corresponding affiliated fictitious random variable S is selected to be Gaussian with fictitious mean $\mathbb{E}_2[S] = s$ and fictitious standard deviation σ_s. The objective of this example is to evaluate the accuracy of the proposed augmented PDD methods [equation (74)] in calculating sensitivities of the failure probabilities $P_{F,1}(s)$: = $P[y_1(\mathbf{X};s)<0]$ and $P_{F,2}(s)$: = $P[y_2(\mathbf{X};s) < 0]$. The perturbation size for finite-difference approximation is taken as $\Delta s = 0.001$.

Table II exhibits the sensitivities of the failure probabilities $P_{F,1}(s)$ and $P_{F,2}(s)$ with respect to the structural design variable s calculated at $s = s_0 = 2$. It contains the estimates of the sensitivities by the univariate ($S = 1$), bivariate ($S = 2$) and trivariate ($S = 3$) third-order augmented PDD approximations of y_1 and y_2, with $\sigma_s = 0.0005$. Combined with the different values of m, which are $m = 1$, $m = 2$ and $m = 3$, a total of nine cases were examined to study the convergence with respect to m and the truncation S. The results by crude MCS, is also listed in the last row to verify the approximate solutions. Reasonably accurate results are obtained by the third-order, bivariate and trivariate augmented PDD approximations, incurring 128 and 250 function evaluations, respectively. In addition, the first-order, trivariate augmented PDD provides less accurate, but still effective estimates of sensitivities with only 26 function evaluations. It is important to note that the orders of σ_s and Δs have to be similar to achieve satisfactory estimates of sensitivities, as found, at least, in this particular example.

FORM and the second-order reliability method (SORM) have been used extensively by engineers for nearly two decades due to accuracy and efficiency. Therefore, the results by FORM and SORM are also listed in Table II for a comparison. The estimations by both FORM and SORM for the failure probabilities $P_{F,1}(s)$ exhibit significant errors of around 40 per cent, requiring 108 and 134 function evaluations, respectively. All nine cases of augmented PDD approximation provide much more accurate estimates of $P_{F,1}(s)$ than FORM and SORM. It is worth to note that our first-order, univariate augmented PDD requires only 14 function evaluations to provide a better estimation on $P_{F,1}(s)$. When m and S increase, the table shows that the accuracy of the augmented PDD are simultaneously improved for most of cases. Specially, when $m = 2$ and $S = 2$, the proposed method provides much more accurate evaluations for all two failure probabilities and their sensitivities than ones by FORM and SORM, and with less function evaluation than SORM.

6.3 Example 3: size and configuration design of a six-Bay, twenty-one-bar truss

The third example demonstrates how RBDO problems with constraints limiting the system reliability can be efficiently solved by the proposed method. A linear-elastic, six-bay, twenty-one-bar truss structure, with geometric properties shown in Figure 3 is simply supported at nodes 1 and 12 and is subjected to a concentrated load of 56,000 lb (249,100 N) at node 7. The truss material is made of an aluminum alloy with the Young's modulus $E = 10^7$ psi (68.94 GPa). Considering the symmetry of the structure, the random input is selected as $\mathbf{X} = (X_1, \ldots, X_{11})^T \in \mathbb{R}^{11}$, where $X_i, i = 1, \ldots, 11$, is the cross-sectional area of the ith truss member. The random variables are independent and lognormally distributed with means μ_i in^2 and standard deviations $\sigma_i = 0.1$ in^2, $i = 1, \ldots, 11$. As depicted in Figure 3, the structural design vector $\mathbf{s} = (s_1, s_2)^T$ describes the node locations, where s_1 represents the horizontal location of nodes 2, 3, 10 and 11, and s_2 represents the horizontal location of nodes 4, 5, 8 and 9. Let $v_{\max}(\mathbf{X};\mathbf{s})$ and $\sigma_{\max}(\mathbf{X};\mathbf{s})$ denote the maximum vertical displacement of all nodes and maximum axial stress in all truss members, respectively, determined from linear-elastic FEA. The permissible displacement and stress are limited to $d_{\text{allow}} = 0.266$ in (6.76

mm) and σ_{allow} = 37,680 psi (259.8 MPa), respectively. The system-level failure set is defined as $\Omega_F := \{\mathbf{x}: \{y_1(\mathbf{x};\mathbf{s}) < 0\} \cup \{y_2(\mathbf{x};\mathbf{s}) < 0\}\}$, where the performance functions:

$$y_1(\mathbf{X};\mathbf{s}) = 1 - \frac{|v_{\max}(\mathbf{X};\mathbf{s})|}{d_{\text{allow}}},\ y_2(\mathbf{X}) = 1 - \frac{|\sigma_{\max}(\mathbf{X};\mathbf{s})|}{\sigma_{\text{allow}}}. \tag{87}$$

Due to the symmetry of the structure and loads, the distributional design vector is $\mathbf{d} = (\mu_1, \ldots, \mu_{11})^T \in \mathcal{D} \subset \mathbb{R}^{11}$. The objective is to minimize the volume of the truss structure subject to a system reliability constraint, limiting the maximum vertical displacement and the maximum axial stress. Therefore, the RBDO problem is formulated to:

$$\begin{aligned}
&\min_{(\mathbf{d},\mathbf{s})\in\mathcal{D}}\ c_0(\mathbf{d},\mathbf{s}) = V(\mathbf{d},\mathbf{s}),\\
&\text{subject to}\ \ c_1(\mathbf{d},\mathbf{s}) = P_{\mathbf{d}}\left[\{y_1(\mathbf{X};\mathbf{s}) < 0\} \cup \{y_2(\mathbf{X};\mathbf{s}) < 0\}\right] - \Phi(-3) \le 0,\\
&1 \le d_k \le 30,\ k = 1, \ldots, 11,\\
&8 \le s_1 \le 12,\ 18 \le s_2 \le 22,
\end{aligned} \tag{88}$$

where $V(\mathbf{d}, \mathbf{s})$ is the total volume of the truss. The initial value of the distributional design vector is $\mathbf{d}_0$ = (15,15,15,15,15,15,15,15,15,15,15)^T in^2 ($\times 2.54^2$ cm^2), and the initial value of the structural design vector is $\mathbf{s}_0$ = (10,10)^T in ($\times$2.54 cm). The approximate optimal solution is denoted by $\left(\tilde{\mathbf{d}}^*; \tilde{\mathbf{s}}^*\right) = \left(\tilde{d}_1^*, \tilde{d}_2^*, \ldots, \tilde{d}_{11}^*; \tilde{s}_1^*, \tilde{s}_2^*\right)^T$. The affiliated fictitious random vectors $\mathbf{D}$ and $\mathbf{S}$ are selected to be Gaussian, and their components are independent with the same fictitious standard deviation of 0.0005 but different mean vectors $\mathbb{E}_1[\mathbf{D}] = \mathbf{d}$ and $\mathbb{E}_2[\mathbf{S}] = \mathbf{s}$. The perturbation sizes of d_k and s_p for finite-difference approximation of sensitivities of failure probabilities are taken as Δd_k = 0.001 and Δs_p = 0.001, respectively, for k = 1, . . ., 11 and p = 1, 2.

The proposed multi-point, single-step design procedure was applied to solve this problem, using bivariate, second-order augmented PDD approximations for the underlying stochastic and design sensitivity analysis. The second column of Table III summarizes the values of design variables, objective function and constraint function for the optimal design, all generated by the augmented PDD method. The objective function c_0 is reduced from 3044.47 in^3 at initial design to 1049.02 in^3 at optimal design – an almost 66 per cent change. At optimum, the constraint function c_1 is –0.21 $\times$ 10^{-3} and is, therefore, close to being active. Most of the design variables have undergone moderate to significant changes from their

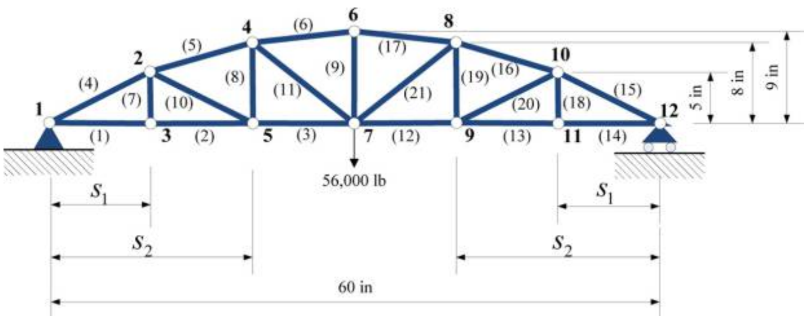


Figure 3.
A six-bay, twenty-one-bar truss structure (Example 3)

initial values, prompting substantial modifications of sizes and configurations of the truss structures. For further scrutinizing the optimum, the results by the crude MCS method, adopting the optimum solution by the proposed augmented PDD method as the initial design, are listed in the last column of Table III. The negligible difference between the results of the proposed PDD method and the results of the corresponding crude MCS method demonstrates the accuracy of the proposed method. Comparing the computational efforts, only 7420 FEA were required to produce the results of the proposed method in Table III, whereas 846 million FEA (samples) were incurred by crude MCS. Therefore, the proposed augmented PDD methods provide not only highly accurate but also vastly efficient, solutions of this mixed RBDO problem.

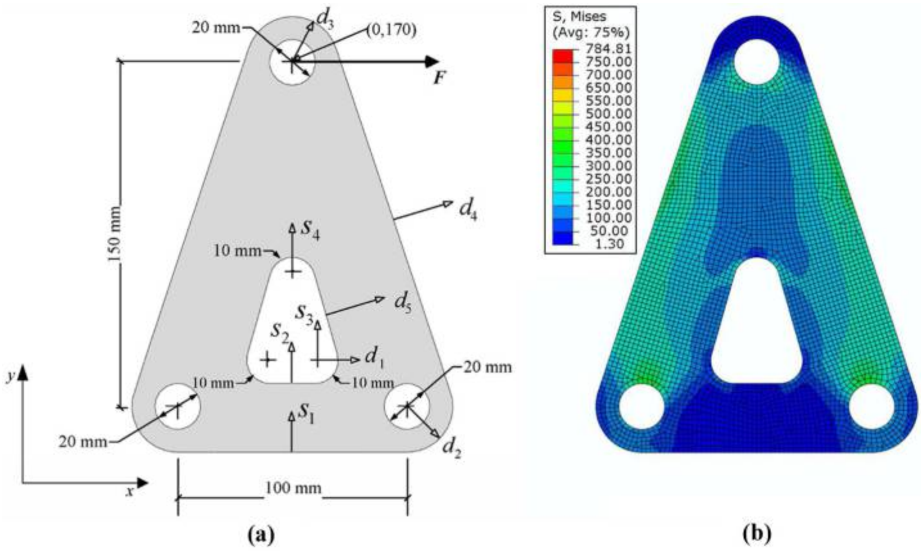
6.4 Example 4: shape design of a three-hole bracket

The final example involves shape design optimization of a two-dimensional, three-hole bracket, where five random shape parameters, X_i, $i = 1, \ldots, 5$, describe its inner and outer boundaries, while maintaining symmetry about the central vertical axis. The distributional design variables, $d_k = \mathbb{E}_{\mathbf{d}}[X_k]$, $i = 1, \ldots, 5$ are the means of these five independent random variables, with Figure 4(a) depicting the initial design of the bracket geometry at the mean values of the shape parameters. The structural design variables, s_p, $p = 1, \ldots, 4$ are four deterministic shape parameters shown in Figure 4(a), along with the random shape parameters defining the geometry of the three-hole bracket. The bottom two holes are fixed, and a deterministic horizontal force $F = 15,000$ N is applied at the center of the top hole. The bracket material has a deterministic mass density $\rho = 7810 \text{kg/m}^3$, deterministic

Table III. Optimization results for the six-Bay, twenty-one-bar truss problem

	Augmented PDD $S = 2, m = 2$	Crude MCS [(b)]
$\tilde{d}_1^*$, in^2	7.6858	7.6665
$\tilde{d}_2^*$, in^2	7.7138	7.7005
$\tilde{d}_3^*$, in^2	4.3102	4.3101
$\tilde{d}_4^*$, in^2	4.7163	4.7162
$\tilde{d}_5^*$, in^2	4.9026	4.9025
$\tilde{d}_6^*$, in^2	4.2936	4.2935
$\tilde{d}_7^*$, in^2	6.0545	6.0544
$\tilde{d}_8^*$, in^2	5.0385	5.0384
$\tilde{d}_9^*$, in^2	6.2239	6.2239
$\tilde{d}_{10}^*$, in^2	4.5967	4.5967
$\tilde{d}_{11}^*$, in^2	3.3725	3.3723
$\tilde{s}_1^*$, in	12.0000	12.0000
$\tilde{s}_2^*$, in	19.1703	19.1702
c_0, in^3	1049.02	1048.35
c_1 [(a)]	-0.2100×10^{-3}	-0.5300×10^{-4}
No. of FEA	7420	846,000,000

Sources: (a) The constraint values are calculated by MCS with 10^6 sample size; (b) Crude MCS: initial design is set to the optimal solution of augmented PDD, i.e., the optimal solution in the second column



Notes: (a) Design parametrization; (b) von Mises stress at initial design

Figure 4.
A three-hole bracket

elastic modulus E = 207.4 GPa, deterministic Poisson's ratio v = 0.3, and deterministic uniaxial yield strength S_y = 800 MPa. The objective is to minimize the second-moment properties of the mass of the bracket by changing the shape of the geometry such that the maximum von Mises stress $\boldsymbol{\sigma}_{e,\max}(\mathbf{X};\mathbf{s})$ does not exceed the yield strength S_y of the material with 99.875 per cent probability if y_1 is Gaussian. Mathematically, the RDO for this problem is defined to:

$$\begin{aligned}
&\min_{(\mathbf{d},\mathbf{s})\in\mathcal{D}}\ c_0(\mathbf{d},\mathbf{s}) = 0.5\frac{\mathbb{E}_{\mathbf{d}}[y_0(\mathbf{X};\mathbf{s})]}{\mathbb{E}_{\mathbf{d}_0}[y_0(\mathbf{X};\mathbf{s})]} + 0.5\frac{\sqrt{\mathrm{var}_{\mathbf{d}}[y_0(\mathbf{X};\mathbf{s})]}}{\sqrt{\mathrm{var}_{\mathbf{d}_0}[y_0(\mathbf{X};\mathbf{s})]}},\\
&\text{subject to } c_1(\mathbf{d},\mathbf{s}) = 3\sqrt{\mathrm{var}_{\mathbf{d}}[y_1(\mathbf{X};\mathbf{s})]} - \mathbb{E}_{\mathbf{d}}[y_1(\mathbf{X};\mathbf{s})] \le 0,\\
&10\,\mathrm{mm} \le d_1 \le 30\,\mathrm{mm},\ 12\,\mathrm{mm} \le d_2 \le 30\,\mathrm{mm},\\
&12\,\mathrm{mm} \le d_3 \le 30\,\mathrm{mm},\ -15\,\mathrm{mm} \le d_4 \le 10\,\mathrm{mm},\\
&-8\,\mathrm{mm} \le d_5 \le 15\,\mathrm{mm},\ 0\,\mathrm{mm} \le s_1 \le 14\,\mathrm{mm},\\
&17\,\mathrm{mm} \le s_2 \le 35\,\mathrm{mm},\ 30\,\mathrm{mm} \le s_3 \le 40\,\mathrm{mm},\\
&50\,\mathrm{mm} \le s_4 \le 140\,\mathrm{mm},
\end{aligned} \tag{89}$$

where

$$y_0(\mathbf{X};\mathbf{s}) = \rho\int_{\mathcal{D}'(\mathbf{X};\mathbf{s})} d\mathcal{D}' \tag{90}$$

and

$$y_1(\mathbf{X};\mathbf{s}) = S_y - \boldsymbol{\sigma}_{e,\max}(\mathbf{X};\mathbf{s}) \tag{91}$$

are two random response functions, and $\mathbb{E}_{\mathbf{d}_0}[y_0(\mathbf{X};\mathbf{s})]$ and $\mathrm{var}_{\mathbf{d}_0}[y_0(\mathbf{X};\mathbf{s})]$ are the mean and variance, respectively, of y_0 at the initial design $(\mathbf{d}_0, \mathbf{s}_0) = (0,30,10,40,20,20,75,0,0)^T$ mm of the design vector $(\mathbf{d},\mathbf{s}) = (d_1,\ldots,d_5,s_1,\ldots,s_4)^T \in \mathcal{D} \subset \mathbb{R}^9$. The corresponding mean and standard deviation of y_0 of the original design, calculated by the bivariate, first-order augmented PDD method, are 0.3415 and 0.00136 kg, respectively. Figure 4(b) portrays the contours of the von Mises stress calculated by FEA of the initial bracket design, which comprises 11,908 nodes and 3914 eight-noded quadrilateral elements. A plane stress condition was assumed. The approximate optimal solution is denoted by $\left(\tilde{\mathbf{d}}^*,\tilde{\mathbf{s}}\right) = \left(\tilde{d}_1^*,\ldots,\tilde{d}_5^*,\tilde{s}_1^*,\ldots,\tilde{s}_4^*\right)^T$. The corresponding affiliated fictitious random vectors $\mathbf{D}$ and $\mathbf{S}$ are selected to be Gaussian, and their components are independent with the same fictitious standard deviation of 0.2 but different mean vectors $\mathbb{E}_1[\mathbf{D}] = \mathbf{d}$ and $\mathbb{E}_2[\mathbf{S}] = \mathbf{s}$.

Due to their finite bounds, the random variables X_i, $i = 1, \ldots,5$, were assumed to follow truncated Gaussian distributions with densities:

$$f_{X_i}(x_i) = \frac{\phi\left(\frac{x_i - d_i}{\sigma_i}\right)}{\Phi\left(\frac{D_i}{\sigma_i}\right) - \Phi\left(-\frac{D_i}{\sigma_i}\right)} \tag{92}$$

when $a_i \le x_i \le b_i$ and *zero* otherwise, where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution and PDFs, respectively, of a standard Gaussian variable, and $\sigma_i = 0.2$; and $a_i = d_i - D_i$ and $b_i = d_i + D_i$ are the lower and upper bounds, respectively, of X_i. To avoid unrealistic designs, the bounds were chosen with $D_i = 2$, which is consistent with the bound constraints of design variables stated in equation (89).

The proposed multi-point, single-step PDD design procedure was applied to solve this problem, employing three univariate and one bivariate augmented PDD approximations for the underlying stochastic analysis: (1) $S = 1, m = 1$; (2) $S = 1, m = 2$; (3) $S = 1, m = 3$; and (4) $S = 2$, $m = 1$. Table IV summarizes the optimization results by all four choices of the truncation parameters. The optimal design solutions rapidly converge as S or m increases. The univariate, first-order ($S = 1$, $m = 1$) PDD method, which is the most economical method, produces an optimal solution reasonably close to those obtained from higher-order univariate or bivariate PDD methods. For instance, the largest deviation from the average values of the objective function at four optimum points is only 3.8 per cent. It is important to note that the coupling between the single-step procedure and multi-point approximation is essential to find optimal solutions of this practical problem using low-variate, low-order augmented PDD approximations.

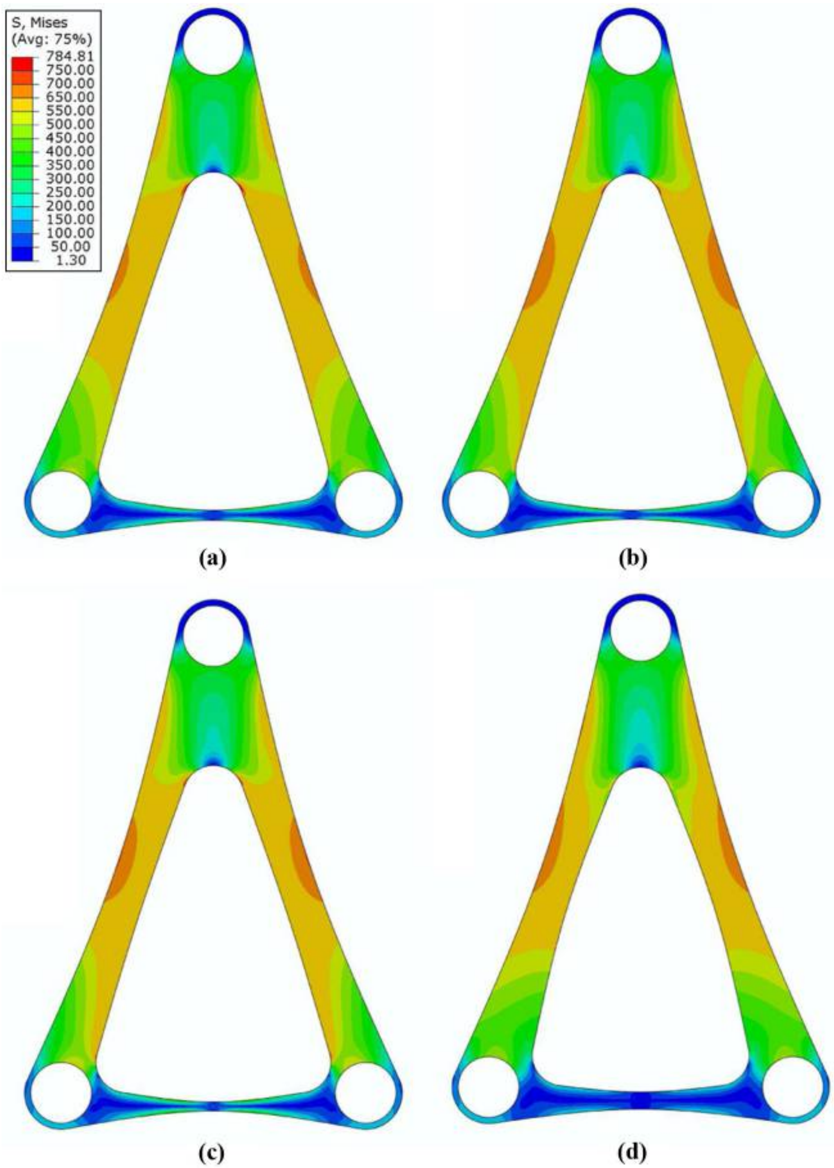
Figure 5(a) through 5(d) illustrates the contour plots of the von Mises stress for the four optimal designs at the mean values of random shape parameters. Regardless of S or m, the overall area of an optimal design has been substantially reduced, mainly due to significant alteration of the inner boundary and moderate alteration of the outer boundary of the bracket. All nine design variables have undergone moderate to significant changes from their initial values. The optimal masses of the bracket vary as 0.1204, 0.1184, 0.1178 and 0.1278 kg – about a 63 per cent reduction from the initial mass

Results	Augmented PDD method			
	Univariate $m = 1$	Univariate $m = 2$	Univariate $m = 3$	Bivariate $m = 1$
$\tilde{d}_1^*$, mm	27.7537	28.0521	28.5815	26.8853
$\tilde{d}_2^*$, mm	12.0030	12.0000	12.0000	12.0000
$\tilde{d}_3^*$, mm	12.0003	12.0000	12.0000	12.0000
$\tilde{d}_4^*$, mm	−13.7431	−13.9282	−13.9025	−14.3121
$\tilde{d}_5^*$, mm	14.7886	14.9982	15.0000	15.0000
$\tilde{s}_1^*$, mm	13.6741	13.9833	14.0000	13.6256
$\tilde{s}_2^*$, mm	17.0081	17.0096	17.0000	17.0000
$\tilde{s}_3^*$, mm	30.0606	30.0002	30.0000	30.0000
$\tilde{s}_4^*$, mm	118.1092	117.6801	117.5495	124.1864
$\tilde{c}_0\left(\tilde{\mathbf{d}}^*\right)^{(a)}$	0.6668	0.6638	0.6628	0.6895
$\tilde{c}_1\left(\tilde{\mathbf{d}}^*\right)^{(a)}$	−1.8819	−14.1435	−18.3799	−10.1967
$\mathbb{E}_{\tilde{c}^*}[y_0(\mathbf{X};\mathbf{s})]^{(a)}$, kg	0.1204	0.1184	0.1178	0.1278
$\sqrt{\mathrm{var}_{\tilde{d}^*}[y_0(\mathbf{X};\mathbf{s})]}^{(a)}$, kg	0.00138	0.00138	0.00138	0.00135
No. of iterations	35	21	37	19
No. of FEA	665	588	1.369	3.078

Source: (a) The objective and constraint functions, $\mathbb{E}_{\tilde{d}^*}[y_0(\mathbf{X};\mathbf{s})]$ and $\sqrt{\mathrm{var}_{\tilde{d}^*}[y_0(\mathbf{X};\mathbf{s})]}$ at respective optima, were evaluated by respective approximations

Table IV. Optimization results for the three-hole bracket

of 0.3415 kg. The second-moment statistics at optimal designs are averages of all PDD solutions described earlier. The largest reduction of the mean is 62.57 per cent, whereas the slight average drop, 0.99 per cent, in the standard deviations, is attributed to the objective function that combines both the mean and standard deviation of y_0. Compared with the conservative design in Figure 4(b), larger stresses – for example, 800 MPa – are safely tolerated by the final designs in Figure 5(a) through 5(d).



Notes: (a) Univariate approximation (S = 1, m = 1); (b) univariate approximation (S = 1, m = 2); (c) univariate approximation (S = 1, m = 3); (d) bivariate approximation (S = 2, m = 1)

Figure 5.
von Mises stress contours at mean values of optimal bracket designs by the multi-point, single-step PDD method

7. Conclusion

A novel computational method, referred to as the augmented PDD method, is proposed for RDO and RBDO of complex engineering systems subject to mixed design variables comprising both distributional and structural design variables. The method involves a new augmented PDD of a high-dimensional stochastic response for statistical moment and reliability analyses; an integration of the augmented PDD, score functions and finite-difference approximation for calculating the sensitivities of the first two moments and the failure probability with respect to distributional and structural design variables; and standard gradient-based optimization algorithms, encompassing a multi-point, single-step design process. For RDO sensitivity analysis, the method capitalizes on a novel integration of the augmented PDD and score functions, providing analytical expressions of mean-square convergent approximations of the design sensitivities of the first two moments. For RBDO sensitivity analysis, the method uses the embedded MCS of the augmented PDD approximation and a finite-difference approximation to estimate the design sensitivities of the failure probability. In each variant of design optimization, both the stochastic responses, whether the first two moments or the failure probability, and their design sensitivities are determined concurrently from a single stochastic analysis or simulation. Moreover, the multi-point, single-step design process embedded in the proposed method facilitates a solution of an RDO/RBDO problem entailing mixed design variables with a large design space. Numerical results, including a shape design optimization of a three-hole bracket, indicate that the proposed methods provide accurate and computationally efficient sensitivity estimates and optimal solutions for general RDO and RBDO problems.

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