HW #2 Turbulent Flows

5.19 Starting from the Reynolds equation (Eq. (4.12)) show that the mean-kinetic-energy equation (for $\bar{E} \equiv \frac{1}{2}\langle \boldsymbol{U}\rangle \cdot \langle \boldsymbol{U}\rangle$) is

$$\frac{\bar{D}\bar{E}}{\bar{D}t} + \nabla \cdot \bar{\boldsymbol{T}} = -\mathcal{P} - \bar{\varepsilon}, \tag{5.134}$$

where

$$\mathcal{P} \equiv -\langle u_i u_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j}, \tag{5.135}$$

$$\bar{T}_i \equiv \langle U_j \rangle \langle u_i u_j \rangle + \langle U_i \rangle \langle p \rangle / \rho - 2\nu \langle U_j \rangle \bar{S}_{ij}. \tag{5.136}$$

5.20 By subtracting the Reynolds equations (Eq. (4.12)) from the Navier–Stokes equation (Eq. (2.35)), show that the fluctuating velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ evolves by

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_i}(U_i U_j - \langle U_i U_j \rangle) = \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}, \tag{5.137}$$

or

$$\frac{D u_j}{D t} = -u_i \frac{\partial \langle U_j \rangle}{\partial x_i} + \frac{\partial}{\partial x_i}\langle u_i u_j \rangle + \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\partial p'}{\partial x_j}, \tag{5.138}$$

where p' is the fluctuating pressure field ($p' = p - \langle p \rangle$). Hence show that the turbulent kinetic energy evolves by

$$\frac{\bar{D}k}{\bar{D}t} + \nabla \cdot \boldsymbol{T}' = \mathcal{P} - \varepsilon, \tag{5.139}$$

where

$$T_i' \equiv \tfrac{1}{2}\langle u_i u_j u_j \rangle + \langle u_i p' \rangle / \rho - 2\nu \langle u_j s_{ij} \rangle. \tag{5.140}$$

For the turbulent kinetic energy, this identity will be useful:

$$\overline{\upsilon u_J \frac{\partial^2 (u_J)}{\partial x_\iota \partial x_\iota}} = 2\upsilon \frac{\partial \overline{(u_J s_{\iota J})}}{\partial x_i} - \varepsilon$$

5.25 Obtain the following relationship between the dissipation ε and the pseudo-dissipation $\tilde{\varepsilon}$:

$$\begin{aligned}\varepsilon \equiv 2\nu\langle s_{ij}s_{ij}\rangle &= \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\rangle \\ &= \tilde{\varepsilon} + \nu\frac{\partial^2\langle u_i u_j\rangle}{\partial x_i\,\partial x_j}. \qquad (5.161)\end{aligned}$$

5.28 In homogeneous isotropic turbulence, the fourth-order tensor

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_\ell} \right\rangle$$

is isotropic, and hence can be written

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_\ell} \right\rangle = \alpha \delta_{ij} \delta_{k\ell} + \beta \delta_{ik} \delta_{j\ell} + \gamma \delta_{i\ell} \delta_{jk}, \tag{5.165}$$

where α, β, and γ are scalars. In view of the continuity equation $\partial u_i / \partial x_i = 0$, show that a relation between the scalars is

$$3\alpha + \beta + \gamma = 0. \tag{5.166}$$

By considering $(\partial / \partial x_j) \langle u_i \, \partial u_j / \partial x_\ell \rangle$ (which is zero on account of homogeneity) show that

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_\ell} \right\rangle$$

is zero, and hence

$$\alpha + \beta + 3\gamma = 0. \tag{5.167}$$

Show that Eq. (5.165) then becomes

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_\ell} \right\rangle = \beta \left(\delta_{ik} \delta_{j\ell} - \tfrac{1}{4} \delta_{ij} \delta_{k\ell} - \tfrac{1}{4} \delta_{i\ell} \delta_{jk} \right). \tag{5.168}$$

Show that

$$\left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle = \tfrac{1}{2} \beta, \quad \left\langle \left(\frac{\partial u_1}{\partial x_2} \right)^2 \right\rangle = 2 \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle, \tag{5.169}$$

$$\left\langle \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \right\rangle = \left\langle \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right\rangle = -\tfrac{1}{2} \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle, \tag{5.170}$$

$$\varepsilon = \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle = \tfrac{15}{2} \nu \beta = 15 \nu \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle. \tag{5.171}$$

11.1 Show that the Poisson equation for pressure (Eq. (2.42)) can alternatively be written

$$\frac{1}{\rho}\nabla^2 p = -\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} = -\frac{\partial^2 U_i U_j}{\partial x_i \partial x_j}. \tag{11.16}$$

Hence show that the mean pressure satisfies

$$\frac{1}{\rho}\nabla^2 \langle p \rangle = -\frac{\partial \langle U_i \rangle}{\partial x_j}\frac{\partial \langle U_j \rangle}{\partial x_i} - \frac{\partial^2 \langle u_i u_j \rangle}{\partial x_i \partial x_j}, \tag{11.17}$$

and that the fluctuation pressure satisfies Eq. (11.9).