HW #1 Turbulent Flows

2.2 Compute the form of λ_f in a general shear flow when $df/dr(0) \neq 0$. Are there any potential problems with this definition?

3.34 From the substitution $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$ and the definition of the two-point correlation (Eq. (3.160)), show that

$$R_{ij}(\boldsymbol{r}, \boldsymbol{x}, t) = R_{ji}(-\boldsymbol{r}, \boldsymbol{x}', t), \tag{3.168}$$

and hence, for a statistically homogeneous field,

$$R_{ij}(\boldsymbol{r}, t) = R_{ji}(-\boldsymbol{r}, t). \tag{3.169}$$

3.35 If $\boldsymbol{u}(\boldsymbol{x}, t)$ is divergence-free (i.e., $\nabla \cdot \boldsymbol{u} = 0$), show that the two-point correlation (Eq. (3.160)) satisfies

$$\frac{\partial}{\partial r_j} R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = 0. \tag{3.170}$$

Show that, if, in addition, $\boldsymbol{u}(\boldsymbol{x}, t)$ is statistically homogeneous, then

$$\frac{\partial}{\partial r_j} R_{ij}(\boldsymbol{r}, t) = \frac{\partial}{\partial r_i} R_{ij}(\boldsymbol{r}, t) = 0. \tag{3.171}$$

Exercise 12.6. Let $R(\tau)$ and $S(\omega)$ be a Fourier transform pair. Show that $S(\omega)$ is real and symmetric if $R(\tau)$ is real and symmetric.

Exercise 12.7. Compute the power spectrum, integral time scale, and Taylor time scale when $R_{11}(\tau) = \overline{u_1^2}\exp\left(-\alpha\tau^2\right)\cos(\omega_o\tau)$, assuming that α and ω_o are real positive constants.

Hints:

1) use $\beta = \sqrt{\alpha}\left(\tau + \frac{i(\omega \mp \omega_0)}{2\alpha}\right)$ as integration variable for the power spectrum
2) Use Taylor expansion of $R_{11}(\tau)$ for $\tau \ll 1$ to obtain the Taylor time scale

Exercise 12.9. Derive the formula for the temporal Taylor microscale λ_t by expanding the definition of the temporal correlation function (12.17) into a two term Taylor series, and determining the time shift, $\tau = \lambda_t$, where this two term expansion equals zero.