HW #1 Turbulent Flows

2.2 Compute the form of λ_f in a general shear flow when $df/dr(0) \neq 0$. Are there any potential problems with this definition?

3.34 From the substitution $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$ and the definition of the two-point correlation (Eq. (3.160)), show that

$$R_{ij}(\boldsymbol{r}, \boldsymbol{x}, t) = R_{ji}(-\boldsymbol{r}, \boldsymbol{x}', t), \tag{3.168}$$

and hence, for a statistically homogeneous field,

$$R_{ij}(\boldsymbol{r}, t) = R_{ji}(-\boldsymbol{r}, t). \tag{3.169}$$

3.35 If $\boldsymbol{u}(\boldsymbol{x}, t)$ is divergence-free (i.e., $\nabla \cdot \boldsymbol{u} = 0$), show that the two-point

correlation (Eq. (3.160)) satisfies

$$\frac{\partial}{\partial r_j} R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = 0. \tag{3.170}$$

Show that, if, in addition, $\boldsymbol{u}(\boldsymbol{x}, t)$ is statistically homogeneous, then

$$\frac{\partial}{\partial r_j} R_{ij}(\boldsymbol{r}, t) = \frac{\partial}{\partial r_i} R_{ij}(\boldsymbol{r}, t) = 0. \tag{3.171}$$

Exercise 12.6. Let $R(\tau)$ and $S(\omega)$ be a Fourier transform pair. Show that $S(\omega)$ is real and symmetric if $R(\tau)$ is real and symmetric.

Exercise 12.7. Compute the power spectrum, integral time scale, and Taylor time scale when $R_{11}(\tau) = \overline{u_1^2}\exp(-\alpha\tau^2)\cos(\omega_o\tau)$, assuming that α and ω_o are real positive constants.

Hints:

1) use $\beta = \sqrt{\alpha}\left(\tau + \frac{i(\omega\mp\omega_0)}{2\alpha}\right)$ as integration variable for the power spectrum
2) Use Taylor expansion of $R_{11}(\tau)$ for $\tau \ll 1$ to obtain the Taylor time scale

Exercise 12.9. Derive the formula for the temporal Taylor microscale λ_t by expanding the definition of the temporal correlation function (12.17) into a two term Taylor series, and determining the time shift, $\tau = \lambda_t$, where this two term expansion equals zero.

$$R_{ij}(\tau) = \overline{u_i(t)u_j(t+\tau)} \text{ and } R_{11}(\tau) = \overline{u_1(t)u_1(t+\tau)}, \quad (12.17)$$

Additional hint for Exercise 12.7:

A second (and equivalent) means of describing the characteristics of turbulent fluctuations, which also complements the information provided by moments, is the *energy spectrum* $S_e(\omega)$ defined as the Fourier transform of the autocorrelation function $R_{11}(\tau)$:

$$S_e(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{11}(\tau)\exp\{-i\omega\tau\}\,d\tau. \quad (12.20)$$

Thus, $S_e(\omega)$ and $R_{11}(\tau)$ are a Fourier transform pair:

$$R_{11}(\tau) \equiv \int_{-\infty}^{+\infty} S_e(\omega)\exp\{+i\omega\tau\}\,d\omega. \quad (12.21)$$

The relationships (12.20) and (12.21) are not special for $S_e(\omega)$ and $R_{11}(\tau)$ alone, but hold for many function pairs for which a Fourier transform exists. Roughly speaking, a Fourier transform can be defined if the function decays to zero fast enough as its argument goes to infinity. Since $R_{11}(\tau)$ is real and symmetric, then $S_e(\omega)$ is real and symmetric (see Exercise 12.6). Substitution of $\tau = 0$ in (12.21) gives

$$\overline{u_1^2} \equiv \int_{-\infty}^{+\infty} S_e(\omega)d\omega. \quad (12.22)$$

This shows that the integrand increment $S_e(\omega)d\omega$ is the contribution to the variance (or fluctuation energy) of u_1 from the frequency band $d\omega$ centered at ω. Therefore, the function $S_e(\omega)$ represents the way fluctuation energy is distributed across frequency ω. From (12.20) it also follows that

$$S_e(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{11}(\tau)d\tau = \frac{\overline{u_1^2}}{\pi}\int_0^{\infty} r_{11}(\tau)d\tau = \frac{\overline{u_1^2}}{\pi}\Lambda_t,$$