B7.2

Eqn. 3.81 $\rightarrow$ $\rho = \varepsilon/2 - \overline{\nabla \underline{u} : (\nabla \underline{u})^T}$

$$\overline{\nabla \underline{u} : (\nabla \underline{u})^T} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{bmatrix} : \begin{bmatrix} \partial u/\partial x & \partial v/\partial x & \partial w/\partial x \\ \partial u/\partial y & \partial v/\partial y & \partial w/\partial y \\ \partial u/\partial z & \partial v/\partial z & \partial w/\partial z \end{bmatrix}$$

$$\Rightarrow \overline{\left(\frac{\partial u}{\cancel{\partial x}}\right)^2 \; 2\frac{\partial u}{\partial y}\frac{\partial v}{\cancel{\partial x}} + 2\frac{\partial u}{\cancel{\partial z}}\frac{\partial w}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2 + 2\cancel{\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}} + \left(\frac{\partial w}{\cancel{\partial z}}\right)^2}$$

For Channel flow $\rightarrow$ mean flow is only a function of y

$$\overline{\nabla \underline{u} : (\nabla \underline{u})^T} = \overline{\left(\frac{\partial v}{\partial y}\right)^2} \Rightarrow \frac{\partial^2 \overline{v^2}}{\partial y^2} \text{ for incompressible flow}$$

Thus $\rho = \varepsilon/2 - \dfrac{\partial^2 \overline{v^2}}{\partial y^2} \xrightarrow{\text{rearrange}} \boxed{\varepsilon/2 = \rho + \dfrac{\partial^2 \overline{v^2}}{\partial y^2}}$ Eqn. 9.45

B7.3 Find U_m/U_{cl} assuming eqn. 7.71 holds $\bar{U}/U_{cl} = \left(y/R_0\right)^{1/n}$ for $y=0$ wall, $y=R_0$ C.L.

U_m is the average velocity $\rightarrow$ $U_m = 2\pi\int^{R_0} r\,\bar{U}(r)\,dr \Big/ \pi R_0^2$

$$U_m = U_{CL}\frac{2\pi}{\pi R_0^2}\int_0^{R_0} r\left(1-\frac{r}{R_0}\right)^{1/n} dr = U_{CL}\left(\frac{2n^2}{(n+1)(2n+1)}\right) \text{ integrated from wolfram Alpha}$$

$$\boxed{\bar{U}_m/U_{cl} = \frac{2n^2}{(n+1)(2n+1)} \text{ using } n = 7 \Rightarrow U_m/U_{cl} \approx \frac{2(49)}{120} \approx 49/60}$$

B7.4 Eqn 7.62: $f = 0.266 R_e^{-1/4}$ and $f = 8\, R_\tau^2 / R_e^2$

$R_{e_{cl}} = \dfrac{U_{cl} R_0}{\nu}$, $R_e = \dfrac{U_m D}{\nu}$, $R_\tau = \dfrac{U_\tau D}{\nu}$, $U_\tau = \sqrt{\tau_w/\rho}$, $\tau_w = \mu \dfrac{\partial \bar{u}}{\partial y}\Big|_{wall}$

$$R_e = \frac{49/60\, U_{CL} D}{\nu} \Rightarrow \frac{49}{60}\frac{R_0 U_{CL}}{\nu} = \frac{49}{60} R_{e_{CL}} \rightarrow R_e^{-1/4} = \left(\frac{49}{60} R_{e_{CL}}\right)^{-1/4}$$

$$\boxed{f = 0.266\left(49/60\, R_{e_{CL}}\right)^{-1/4} \approx 0.28\, R_{e_{CL}}^{-1/4}}$$

B8.3 Using $\bar{U} = U_\infty (y/\delta)^{1/n}$ → Plug into integral

$\delta_1 = \int_0^\infty 1 - \frac{U}{U_\infty} dy$ and at $y=\delta \rightarrow U = U_\infty$

$\delta_1 = \int_0^\infty 1-(y/\delta)\,dy \Rightarrow \overset{\text{Assume } 0}{\cancel{c}} + \int_0^\delta 1-(y/\delta)^{1/n} dy$ let $u = (y/\delta)$, $du = 1/\delta\, dy$

$\delta_1 = \delta\int_0^1 1 - u^{1/n} du = \delta\left(u - \frac{n}{n+1} u^{n+1/n}\right)\Big|_0^1$

$\delta_1 = \delta\left(1-\frac{n}{n+1}\right) = \frac{\delta(n+1)-\delta n}{n+1} = \boxed{\frac{\delta}{n+1} = \delta_1}$

$\theta = \int_0^\infty \frac{\bar{U}}{U_\infty}\left(1-\frac{\bar{U}}{U_\infty}\right) dy \Rightarrow \int_0^\infty \left(\frac{y}{\delta}\right)^{1/n}\left(1-(y/\delta)^{\frac{1}{n}}\right) dy$

$\Rightarrow \delta\int_0^1 u^{1/n}(1-u^{1/n})\,du + c = \delta\int_0^1 u^{1/n} - u^{2/n}\, du + \cancel{c}^{\,0}$

$= \delta\left[\frac{n}{n+1}u^{n+1/n} - \frac{n}{n+2}u^{\frac{n+2}{n}}\right]_0^1 = \delta\left(\frac{n}{n+1} - \frac{n}{n+2}\right) = \delta\left(\frac{n^2+2n-n^2-n}{(n+1)(n+2)}\right)$

$\Rightarrow \boxed{\frac{\delta n}{(n+1)(n+2)} = \theta}$

B11.1 $\eta_{1/2}$ is η in wake flow where $f(\eta_{1/2}) = 0.5$ * Velocity defect = 1/2 of max

using $f(\eta) = e^{-R_d\eta^2/4} = 0.5 \rightarrow \ln(0.5) = -R_d\eta^2/4$

rearranging: $\frac{4\ln(0.5)}{-R_d} = \eta^2 \therefore \eta = \left[\frac{4\ln(0.5)}{-R_d}\right]^{1/2} = \left[\frac{2.773}{R_d}\right]^{1/2}$

using empirical value of 61.04 for R_d

$\eta = \sqrt{\frac{2.773}{61.04}} \approx \boxed{0.213 \approx \eta}$

B11.6

(Eqn 11.88) $\bar{U} = U_m\left(1 + \frac{\Delta U}{2U_m}\operatorname{erf}(\eta)\right) = U_m F'(\eta)$

$d\eta = \frac{1}{\ell}dy$

$d\eta/dy = 1/\ell$

$\overline{uv} = -U_m^2 g(\eta) = -\nu_t \frac{\partial \bar{U}}{\partial y} = -\nu_t \frac{U_m}{\ell} F''$

$F''(\eta) = \frac{1}{\ell}\frac{d}{d\eta}\left(1 + \frac{\Delta U}{2U_m}\operatorname{erf}(\eta)\right) = \frac{1}{\ell}\frac{\Delta U}{2U_m}\frac{\partial}{\partial\eta}(\operatorname{erf}(\eta)) = dF'(\eta)/\partial y$

$\frac{d(\operatorname{erf}(\eta))}{d\eta} = \frac{2}{\sqrt{\pi}}e^{-\eta^2}(1) - \cancel{(f(0)}^{0} + \cancel{\int_0^\eta \frac{\partial}{\partial\eta}\left(\frac{2}{\sqrt{\pi}}e^{-\xi^2}\right)d\xi} = \frac{2}{\sqrt{\pi}}e^{-\eta^2}$

$F''(\eta) = \frac{\Delta U}{2\ell U_m}\frac{2}{\sqrt{\pi}}e^{-\eta^2} = \frac{\Delta U}{\ell\sqrt{\pi}\,U_m}e^{-\eta^2}$ Now plug in to find $\overline{uv}$

$\overline{uv} = -\nu_t \cancel{U_m}\left(\frac{\Delta U}{\ell\sqrt{\pi}\cancel{U_m}}e^{-\eta^2}\right) = \boxed{-\nu_t \frac{\Delta U}{\ell\sqrt{\pi}}e^{-\eta^2} = \overline{uv}}$

where $\Delta U = (U_h - U_\ell)$ and $\eta = y/\ell$