B6.1 $\bar{U}(y)$ Shear Flow → Predicted normal R.S. from Eqn 6.11

$$\sigma_t = -\tfrac{2}{3}\rho K I + \mu_t(\nabla\bar{U} + \nabla\bar{U}^t)$$
$$= -\tfrac{2}{3}\rho K I + \mu_t\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right) \rightarrow$$ $\bar{U}$ is only function of y

$$\rho\overline{u_i u_i} = -\tfrac{2}{3}\rho K(1)\ \mu_t\left(2\frac{\partial \bar{U}_i}{\partial x_i}\right)$$ when $i=j$ which gives the normal Reynolds stress

$\frac{\partial \bar{U}_i}{\partial x_i} = 0$ from continuity. Therefore the equation reduces to the single term, $\boxed{\rho\overline{u_i u_i} = -\tfrac{2}{3}\rho K}$ for normal Reynolds stresses

B6.3

Eqn 6.52 $$K_\infty = \frac{\sqrt{C_\mu}\,(C_{\varepsilon_2} - C_{\varepsilon_1})^2}{C_{\varepsilon_3}^{\,2}}\nu S$$

Eqn 6.53 $$\varepsilon_\infty = \frac{C_\mu (C_{\varepsilon_2} - C_{\varepsilon_1})^2}{C_{\varepsilon_3}^{\,2}}\nu S^2$$

$$d\varepsilon/dt = C_{\varepsilon_1} C_\mu K S^2 + C_{\varepsilon_3} R_T^{1/2}\,\varepsilon^2/K - C_{\varepsilon_2}\,\varepsilon^2/K$$

$$dK/dt = C_\mu\, K^2/\varepsilon\ S^2 - \varepsilon$$

Setting $d\varepsilon/dt$ and dK/dt both to zero for $t \to \infty$ will give the asymptotic limit

$$0 = C_{\varepsilon_1} C_\mu K_\infty S^2 + C_{\varepsilon_3} R_{T_\infty}^{1/2}\frac{\varepsilon_\infty^2}{K_\infty} - C_{\varepsilon_2}\frac{\varepsilon_\infty^2}{K_\infty}$$

$$0 = \frac{K_\infty^2}{\varepsilon_\infty} C_\mu S^2 - \varepsilon_\infty$$

$$R_{T_\infty} = K_\infty^2/\nu\varepsilon_\infty$$

$$0 = C_{\varepsilon_1} C_\mu K_\infty S^2 + C_{\varepsilon_3}\frac{K_\infty}{\nu^{1/2}\varepsilon_\infty^{1/2}}\frac{\varepsilon_\infty^2}{K_\infty} - C_{\varepsilon_2}\,\varepsilon_\infty^2/K_\infty$$

$$0 = C_\mu C_{\varepsilon_1} K_\infty S^2 + C_{\varepsilon_3}\varepsilon_\infty^{3/2}\nu^{-1/2} - C_{\varepsilon_2}\,\varepsilon_\infty^2/K_\infty$$ and $\varepsilon_\infty^2 = K_\infty^2 C_\mu S^2$ $(K_\infty^2 = \varepsilon_\infty^2/C_\mu S^2)$

$$0 = \frac{\varepsilon_\infty}{\cancel{C_\mu^{1/2}}\cancel{S}} C_\mu^{\cancel{2}} C_{\varepsilon_1} S^{\cancel{2}} + C_{\varepsilon_3}\varepsilon_\infty^{3/2}\nu^{-1/2} - C_{\varepsilon_2}\,\varepsilon_\infty^{\cancel{2}}/\cancel{\varepsilon}/C_\mu^{1/2}S$$

$$0 = C_\mu^{1/2} S\,\varepsilon_\infty\left[C_{\varepsilon_1} - C_{\varepsilon_2}\right] + C_{\varepsilon_3}\varepsilon_\infty^{3/2}\nu^{-1/2}$$

$$\varepsilon_\infty^{\cancel{3/2}\,1/2}\nu^{-1/2} C_{\varepsilon_3} = C_\mu^{1/2} S\cancel{\varepsilon_\infty}(C_{\varepsilon_2} - C_{\varepsilon_1}) \Rightarrow \varepsilon_\infty^{1/2} = \frac{\sqrt{C_\mu \nu}\, S(C_{\varepsilon_2} - C_{\varepsilon_1})}{C_{\varepsilon_3}}$$

$$\boxed{\therefore\ \varepsilon_\infty = \frac{C_\mu (C_{\varepsilon_2} - C_{\varepsilon_1})^2 \nu S^2}{C_{\varepsilon_3}^{\,2}},\quad K_\infty = \frac{\varepsilon_\infty}{C_\mu^{1/2} S} = \frac{\sqrt{C_\mu}\,(C_{\varepsilon_2} - C_{\varepsilon_1})^2 \nu S}{C_{\varepsilon_3}^{\,2}}}$$

6.4 Plot solutions of 6.50 and 6.51

* attached at end of PDF *

$$\frac{S k_\infty}{\varepsilon_\infty} \ (6.44) \quad \frac{P_\infty}{\varepsilon_\infty} = \frac{-\overline{uv}S}{\varepsilon_\infty} \ (6.45) = \frac{-\tau_{22}\nu^2 S^2}{\varepsilon_\infty}$$

$$\frac{dk}{dt} = C_\mu \frac{k^2}{\varepsilon} S^2 - \varepsilon$$

$$\frac{d\varepsilon}{dt} = C_1 C_\mu k S^2 + C_3 R_T^{1/2} \varepsilon^2/k - C_2 \varepsilon^2/k$$

Let $k^* = k/k_0$, $\varepsilon^* = \varepsilon/\varepsilon_0$

$S \equiv d\bar{U}/dy > 0$ and is constant

$t^* = St$

$$\frac{dk}{dt} \cdot \frac{dk^*}{dk} \cdot \frac{dt}{dt^*} = \frac{dk^*}{dt^*} = \frac{1}{k_0} \cdot \frac{1}{S}\left[\frac{dk}{dt}\right] \quad \left(\frac{d\varepsilon}{dt}\left[\frac{1}{\varepsilon_0 S}\right] = \frac{d\varepsilon^*}{dt^*}\right)$$

$$\frac{dk^*}{dt^*} = C_\mu \frac{k}{k_0}\left(\frac{k}{\varepsilon} S\right) - \frac{\varepsilon}{k_0 S}$$

$$\frac{d\varepsilon^*}{dt^*} = C_1 C_\mu \frac{k}{\varepsilon_0} S - \frac{\varepsilon}{\varepsilon_0}\frac{\varepsilon}{kS}\left(C_3 R_T^{1/2} - C_2\right)$$

New nondimensional Eqns. for $k^*(t^*)$ and $\varepsilon^*(t^*)$

k and ε have been shown to grow at same rate

$\therefore$ $(k/\varepsilon) \sim$ constant (and Sk/ε will also be constant)

$$\frac{dk^*}{dt^*} = C_\mu k^* C - k^*/C$$

$$\frac{d\varepsilon^*}{dt^*} = C_1 C_\mu C \varepsilon^* - \frac{\varepsilon^*}{C}\left(C_3 R_T^{1/2} - C_2\right)$$

$$C \approx k^*/\varepsilon^*$$

Let $R_T = \frac{k^2}{\nu\varepsilon} = \frac{k_0^2 k^{*2}}{\nu \varepsilon_0 \varepsilon^*} = R_{T0}\frac{k^{*2}}{\varepsilon^*}$

$$\frac{k^*}{\varepsilon^*} \approx \frac{k/k_0}{\varepsilon/\varepsilon_0} = \frac{C/S}{C/S_0} = S/S_0 = 1$$

$$R_T = R_{T_0}(1)k^*$$

Results of ODE solution are attached at the end.

6.5 If $\tau_{22} \sim (k/\varepsilon)$ as the eddy turnover time.

Substitute into scaled equations for 6.5

$$\frac{dk^*}{dt^*} = C_\mu k^* \tau_{22} S - \frac{k^*}{\tau_{22} S}$$

$$\frac{d\varepsilon^*}{dt^*} = C_1 C_\mu \tau_{22}(S)\varepsilon^* - \frac{\varepsilon^*}{\tau_{22} S}\left(C_3 R_T^{1/2} - C_2\right)$$

Since S and τ_{22} are constant this recovers the same form as derived in 6.4.

Thus a constant eddy turnover time achieves same behavior as before

P5.40

$$\frac{\bar{D}\langle U_j\rangle}{\bar{D}t} = \nu\nabla^2\langle U_j\rangle - \frac{\partial\langle u_i u_j\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial\langle P\rangle}{\partial x_j} \quad (1)$$

$$\frac{DU_j}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu\nabla^2\frac{\partial^2 U_i}{\partial x_j \partial x_j} \quad (2)$$

Subtract (1) from (2) $\rightarrow \frac{DU_j}{Dt} - \frac{\bar{D}\langle U_j\rangle}{\bar{D}t} = -\frac{1}{\rho}\frac{\partial P'}{\partial x_j} + \nu\nabla^2 u_j + \frac{\partial}{\partial x_i}\langle u_j u_i\rangle$

$$\Rightarrow \underbrace{\left(\frac{Du_j}{Dt} - \frac{D\langle U_j\rangle}{Dt}\right)}_{\frac{Du_j}{Dt}} - u_i\frac{\partial\langle U_j\rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial P'}{\partial x_j} + \nu\nabla^2 u_j + \frac{\partial}{\partial x_i}\langle u_j u_i\rangle$$

$\frac{Du_j}{Dt} \rightarrow$ Put everything else on RHS (3)

$$u_j\frac{Du_j}{Dt} = \left\langle\frac{1}{2}\frac{\partial(u_j u_j)}{\partial t} + \frac{1}{2}\frac{\partial(\langle U_i\rangle u_j u_j)}{\partial x_i}\right\rangle = \frac{dk}{dt}$$

$\langle u_j\frac{\partial P'}{\partial x_j}\rangle = \frac{\partial}{\partial x_j}\langle P' u_j\rangle = 0$ and $\underline{\nu\langle u_j\nabla^2 u_j\rangle = -\varepsilon}$

Thus, multiplying (3) by u_j obtains dk/dt equation

$$\Rightarrow dk/dt = P - \varepsilon \quad \text{where } P = -\langle u_i u_j\rangle\frac{\partial\langle U_j\rangle}{\partial x_i}$$

P6.36 $T_k(k) = E(k)\kappa(k) = E(k)\alpha^{-1}\varepsilon^{1/3}k^{5/3}$

$$0 = -d/dk\, T(k) - 2\nu k^2 E(k)$$

$$d/dk\, T(k) = -2\nu k^2 E(k) = d/dk\left(E(k)\alpha^{-1}\varepsilon^{1/3}k^{5/3}\right)$$

$$\frac{1}{k^{5/3}E(k)}\frac{d}{dk}\left(E(k)k^{5/3}\right) = -2\nu\alpha\varepsilon^{-1/3}k^{1/3}$$

let $x = E(k)k^{5/3} \rightarrow \int\frac{dx}{x} = \ln(x)$

$$\ln(x) = \ln(E(k)k^{5/3}) = \int -2\nu\varepsilon^{-1/3}k^{1/3}\alpha\, dk = -2\cdot\tfrac{3}{4}\alpha\nu k^{4/3}\varepsilon^{-1/3} + C$$

$$E(k) = \beta k^{-5/3}\cdot\exp\left(-\tfrac{3}{2}\alpha\nu k^{4/3}\varepsilon^{-1/3}\right)$$

$\eta = \left(\nu^3/\varepsilon\right)^{1/4} \rightarrow (k\eta)^{4/3} = k^{4/3}\frac{\nu^{3/4\cdot 4/3}}{\varepsilon^{1/4\cdot 4/3}} = \nu k^{4/3}\varepsilon^{-1/3}$

$$\boxed{\Rightarrow \beta k^{-5/3}\cdot\exp\left[-\tfrac{3}{2}\alpha(k\eta)^{4/3}\right]}$$

If $E(k) = C\varepsilon^{2/3}k^{-5/3}$ for Small $k\eta$ $\;C\varepsilon^{2/3}k^{-5/3} \approx \beta k^{-5/3}e^{0} \rightarrow 1$

$$\boxed{\therefore \beta = C\varepsilon^{2/3} \text{ as } k\eta \rightarrow 0}$$

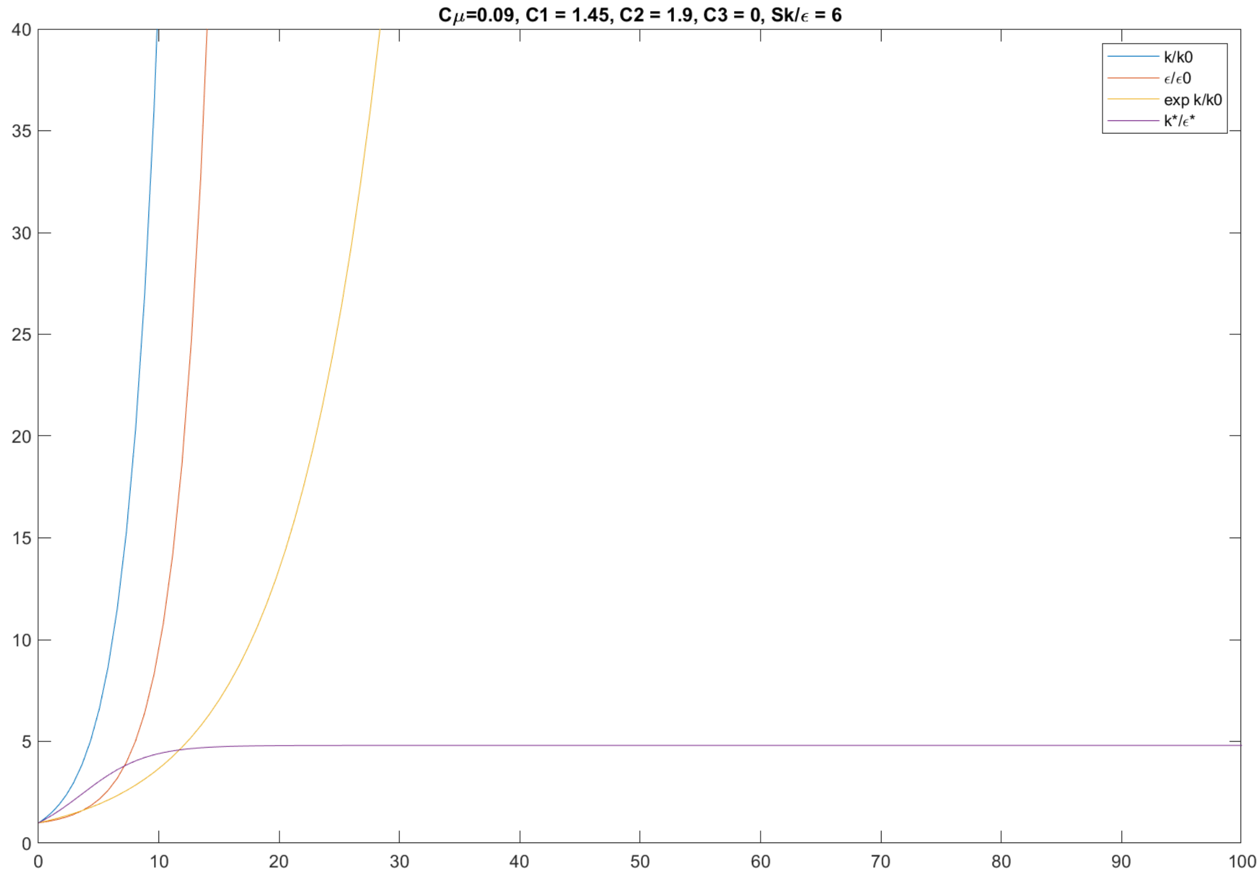
$$\int_0^\infty 2\nu k^2 E(k)\, dk = \varepsilon = \int_0^\infty 2\nu k^2 \left(\underbrace{C\varepsilon^{2/3}}_{\beta} k^{-5/3} \exp\left[-\tfrac{3}{2}\alpha (k\eta)^{4/3}\right] \right) dk$$

$$\Rightarrow 2\nu\beta \int_0^\infty k^{1/3} e^{-\frac{3}{2}\alpha (k\eta)^{4/3}}\, dk \Rightarrow 2\nu\beta \left[\frac{-3}{4} \frac{1}{\frac{3}{2}\alpha \eta^{4/3}} e^{-\frac{3}{2}\alpha (k\eta)^{4/3}} \right]_0^\infty$$

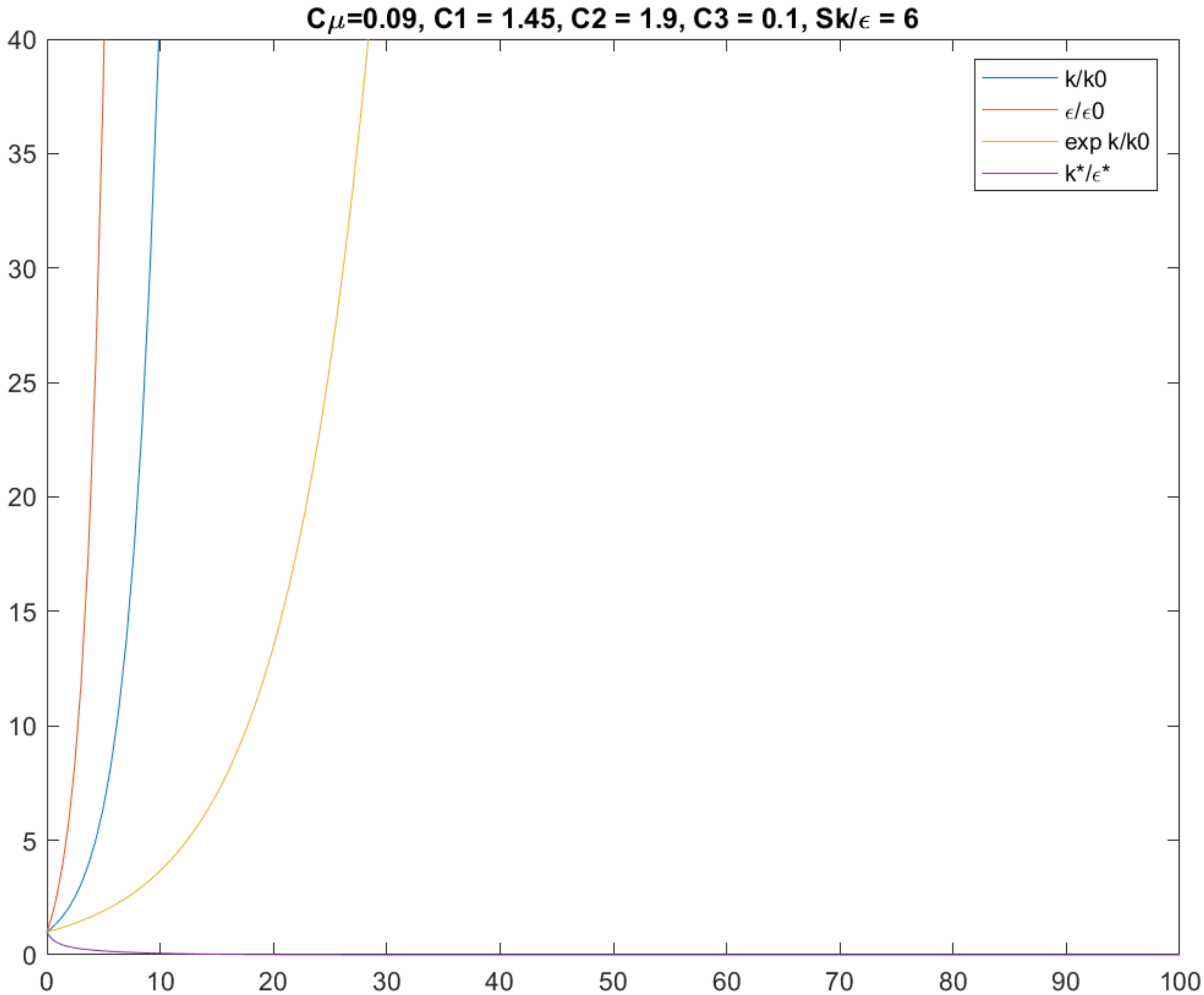
$$\varepsilon = 0 - \left[-\cancel{\tfrac{3}{2}}\nu\beta \frac{1}{\cancel{\tfrac{3}{2}}\alpha \eta^{4/3}} (1) \right] = \frac{\nu\beta}{\alpha \eta^{4/3}} = \frac{\nu C \varepsilon^{2/3}}{\alpha \left(\frac{\nu^3}{\varepsilon}\right)^{4/3 \cdot 1/4}}$$

$$\varepsilon = \frac{\cancel{\nu} C \varepsilon^{2/3}}{\alpha \cancel{\nu}/\varepsilon^{1/3}} = \frac{C}{\alpha}\varepsilon \rightarrow \text{Thus } \alpha = C$$

$$\boxed{E(k) = C\varepsilon^{2/3} k^{-5/3} \exp\left[-\tfrac{3}{2} C (k\eta)^{4/3}\right]}$$



No Vortex Stretching



With Vortex Stretching

The scaled ordinary differential equations for K-ε are sensitive to the initial conditions and constant values inserted into the ODE solver. It proved to be difficult to replicate the effects of the vortex stretching that introduces production of dissipation. Thus, both K and ε continued to grow exponentially at an even faster rate than the suggest exponential function $\frac{k}{k_0} = e^{0.13t^*}$. The results for asymptotic relationships $\frac{Sk}{\varepsilon} \approx 4.8$ and $\frac{P}{\varepsilon} \approx 2.0736$