Solution to Problem 4.1

We are given the energy spectrum:

$$E(k) = \frac{1}{\sqrt{2\pi}} \bar{u}^2 \lambda_g (k\lambda_g)^4 e^{-(k\lambda_g)^2/2}. \tag{1}$$

We need to find the characteristic wavenumbers k_e and k_d.

Finding k_e (Energy-containing Wavenumber)

The characteristic energy-containing wavenumber k_e corresponds to the peak of the energy spectrum. This is found by setting the derivative $dE(k)/dk = 0$:

$$\frac{d}{dk}\left[(k\lambda_g)^4 e^{-(k\lambda_g)^2/2}\right] = 0. \tag{2}$$

Using the product rule,

$$4(k\lambda_g)^3 e^{-(k\lambda_g)^2/2} + (k\lambda_g)^4 e^{-(k\lambda_g)^2/2}\left(-\frac{k\lambda_g}{1}\right) = 0. \tag{3}$$

Factoring out common terms,

$$(k\lambda_g)^3 e^{-(k\lambda_g)^2/2}\left(4 - (k\lambda_g)^2\right) = 0. \tag{4}$$

Setting the bracketed term to zero,

$$4 - (k\lambda_g)^2 = 0. \tag{5}$$

Solving for k,

$$k_e = \frac{2}{\lambda_g}. \tag{6}$$

Finding k_d (Dissipation Wavenumber)

The dissipation spectrum is given by:

$$D(k) = 2\nu k^2 E(k). \tag{7}$$

Substituting $E(k)$:

$$D(k) = 2\nu k^2 \cdot \frac{1}{\sqrt{2\pi}} \bar{u}^2 \lambda_g (k\lambda_g)^4 e^{-(k\lambda_g)^2/2}. \tag{8}$$

Simplifying,

$$D(k) = \frac{2\nu}{\sqrt{2\pi}} \bar{u}^2 \lambda_g (k\lambda_g)^6 e^{-(k\lambda_g)^2/2}. \tag{9}$$

To find k_d, we take the derivative of $D(k)$ and set it to zero:

$$\frac{d}{dk}\left[(k\lambda_g)^6 e^{-(k\lambda_g)^2/2}\right] = 0. \tag{10}$$

Using the product rule,

$$6(k\lambda_g)^5 e^{-(k\lambda_g)^2/2} + (k\lambda_g)^6 e^{-(k\lambda_g)^2/2}\left(-\frac{k\lambda_g}{1}\right) = 0. \tag{11}$$

Factoring out common terms,

$$(k\lambda_g)^5 e^{-(k\lambda_g)^2/2}\left(6 - (k\lambda_g)^2\right) = 0. \tag{12}$$

Setting the bracketed term to zero,

$$6 - (k\lambda_g)^2 = 0. \tag{13}$$

Solving for k,

$$k_d = \frac{\sqrt{6}}{\lambda_g}. \tag{14}$$

Distance Between k_e and k_d

Now we compute the distance:

$$k_d - k_e = \frac{\sqrt{6}}{\lambda_g} - \frac{2}{\lambda_g} \approx \frac{0.45}{\lambda_g} \tag{15}$$

Showing that the distance between k_d and k_e is inversely proportional to the Taylor microscale, i.e., for high Re flows the peaks are clearly separated.

Pope 6.4 eqn. 6.46 $\rightsquigarrow$ $g(r,t) = f(r,t) + \frac{1}{2} r \frac{\partial}{\partial r} f(r,t)$

If $\frac{\partial}{\partial r}[r\, f(r,t)] = f(r,t) + r \frac{\partial}{\partial r} f(r,t)$ then

$$\frac{1}{2} f(r,t) + \frac{1}{2}\frac{\partial}{\partial r}[r f(r,t)] = \frac{1}{2} f(r,t) + \frac{1}{2} f(r,t) + \frac{1}{2} r \frac{\partial}{\partial r} f(r,t)$$

$\Rightarrow$ $f(r,t) + \frac{1}{2} r \frac{\partial}{\partial r} f(r,t)$ recovers eqn. 6.46

$$\therefore \frac{1}{2}\left(f(r,t) + \frac{\partial}{\partial r}[r f(r,t)]\right) = f(r,t) + r \frac{\partial}{\partial r} f(r,t)$$

$$L_{22} = \int_0^\infty g(r,t)\, dr = \frac{1}{2}\int_0^\infty \left(f(r,t) + \frac{\partial}{\partial r}[r f(r,t)] \right) dr$$

$$L_{22} = \frac{1}{2}\int_0^\infty f(r,t)\, dr + \frac{1}{2}\int_0^\infty \frac{\partial}{\partial r}[r f(r,t)]\, dr = \frac{1}{2} L_{11} + \int_0^\infty \frac{\partial}{\partial r}[r f(r,t)]\, dr$$

$$\boxed{L_{22}\ \frac{1}{2} L_{11} + \frac{1}{2}[r f(r,t)]_0^\infty \Rightarrow \frac{1}{2} L_{11}}$$ Since $f(r,t) \to 0$ as $r \to \infty$ and $0 \cdot f(0,t) = 0$

Pope 6.6

$$g''(r,t) = \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left[f(r,t) + \frac{1}{2} r \frac{\partial}{\partial r} f(r,t)\right]\right] = \frac{\partial}{\partial r}\left[f'(r,t) + \frac{1}{2}f'(r,t) + \frac{1}{2} r f''(r,t)\right]$$

$$g''(r,t) = f''(r,t) + \frac{1}{2} f''(r,t) + \frac{1}{2} f''(r,t) + \frac{1}{2} r f'''(r,t)$$

$$\underline{g''(r,t) = 2f''(r,t) + \frac{1}{2} r f'''(r,t)}$$

$$g(r,t) \approx g(r,t) + \frac{1}{2} g''(r,t) r^2 \rightsquigarrow 0 \approx g(0,t) + \frac{1}{2}\lambda_g^2 (g''(0,t)) = 1 + \frac{1}{2}\lambda_g^2 g''(0,t)$$

$$\underline{\lambda_g = \left[-\frac{1}{\frac{1}{2} g''(0,t)}\right]^{1/2} = \left[-\frac{1}{2} g''(0,t)\right]^{-1/2}}$$

$$\Rightarrow \left[-\frac{1}{2}\left(2f''(r,t) + \frac{1}{2}(0)\cancel{f'''(0,t)}^{\,0}\right)\right]^{-1/2} = \left[-f''(0,t)\right]^{-1/2}$$

Substitute into equation

If $\lambda_f = \left[-\frac{1}{2} f''(0,t)\right]^{-1/2} \rightsquigarrow f''(0,t) = \lambda_f^{-2} \cdot -\frac{1}{2} = -\frac{1}{2\lambda_f^2}$

$$\lambda_g = \left[\frac{1}{2}\lambda_f^{-2}\right]^{-1/2} \Rightarrow \left(\frac{1}{2}\right)^{-1/2}\lambda_f = \underline{\lambda_f/\sqrt{2} = \lambda_g} \quad (\lambda_f = \sqrt{2}\,\lambda_g)$$

If $\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle = \frac{1}{2}\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle = \frac{2u'^2}{\lambda_f^2}$ by equation 6.56

$$\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle = \frac{4u'^2}{\lambda_f^2} = \frac{\cancel{4}^{\,2}u'^2}{\cancel{2}\lambda_g^2} \Rightarrow \boxed{\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle = \frac{2u'^2}{\lambda_g^2}}$$

6.31 $E_{22}(\kappa_1) = \frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 + \frac{\kappa_1^2}{\kappa^2}\right) d\kappa$, $E(\kappa) = -\kappa\left\{\frac{dE_{22}(\kappa)}{d\kappa} + \int_{\kappa}^{\infty} \frac{1}{\kappa_1}\frac{dE_{22}(\kappa_1)}{d\kappa_1} d\kappa_1\right\}$

Equation 6.218 relates E_{22} and E_{11} and Eqn. 6.216 relates E_{11} and $E(\kappa)$

(1) $E_{22}(\kappa_1) = \frac{1}{2}\left(E_{11}(\kappa_1) - \kappa_1 \frac{dE_{11}(\kappa_1)}{d\kappa_1}\right)$

(2) $E_{11}(\kappa_1) = \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \kappa_1^2/\kappa^2\right) d\kappa$

(3) $\int_{\kappa_1}^{\infty} -\frac{2E(\kappa)}{\kappa^3} d\kappa = \frac{1}{\kappa_1}\frac{dE_{11}}{d\kappa_1}$ *From class notes

Substitute (2) and (3) into (1)

$$E_{22}(\kappa_1) = \frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \kappa_1^2/\kappa^2\right) d\kappa - \frac{\kappa_1^2}{2}\int_{\kappa_1}^{\infty} \frac{-2E(\kappa)}{\kappa^3} d\kappa$$

$$E_{22}(\kappa_1) = \frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa} - \frac{\kappa_1^2 E(\kappa)}{\kappa^3} + \frac{2\kappa_1^2 E(\kappa)}{\kappa^3}\, d\kappa$$

Simplifying the integrand $\rightsquigarrow$ $\boxed{E_{22}(\kappa_1) = \frac{1}{2}\int_{\kappa_1}^{\infty} E(\kappa)/\kappa\left(1 + \kappa_1^2/\kappa^2\right) d\kappa}$

Now inverting the equation to solve for $E(\kappa)$

$$\frac{dE_{22}(\kappa_1)}{d\kappa_1} = \frac{1}{2}\left[\frac{d}{d\kappa_1}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 + \frac{\kappa_1^2}{\kappa^2}\right) d\kappa\right] \longrightarrow \text{Apply Leibniz theorem}$$

$$\frac{dE_{22}}{dk_1}\cdot 2 = \int_{k_1}^{\infty} \frac{2k_1 E(k)}{k^3}\, dk - \frac{2E(k_1)}{k_1} \quad \text{Assuming } k_1 = k \rightarrow \frac{dk_1}{dk} = 1$$

Use fundamental theorem of calculus to remove integral

$$\frac{2\,dE_{22}(k_1)}{dk_1}\cdot\frac{d}{dk} = \frac{d}{dk}\int_{k_1}^{\infty}\frac{2k_1 E(k)}{k^3}\,dk - \frac{d}{dk}\left(\frac{2E(k)}{k}\right) = 2\frac{d}{dk}\left(\frac{dE_{22}(k_1)}{dk_1}\right)$$

Apply product Rule

Cancels out

$$\Rightarrow 2\left(-\frac{E(k_1)}{k_1^2} - \frac{dE((k))k_1}{dk_1}\right) \Rightarrow -2\left(\frac{E(k_1)}{k_1^2} + \overbrace{\frac{d}{dk_1}\left[E(k_1)k_1^{-1}\right]}\right) = -2\left(\cancel{\frac{E(k_1)}{k_1^2} - \frac{E(k_1)}{k_1^2}} + \frac{1}{k_1}\frac{dE(k_1)}{dk_1}\right)$$

$$\frac{d}{dk}\left\{\frac{dE_{22}(k_1)}{dk_1}\right\}\cdot 2 = -\cancel{2}\left(\frac{1}{k_1}\frac{dE(k_1)}{dk_1}\right) \longrightarrow \frac{dE(k_1)}{dk_1} = -k_1\frac{d}{dk}\left(\frac{dE_{22}}{dk_1}\right) = -k_1\frac{d^2E_{22}}{dk_1^2}$$

assuming $k_1 = k$

$$\int\frac{dE(k_1)}{dk_1}\,dk_1 = \int dE(k_1) = E(k_1) = E(k) = \int -k_1\frac{d^2E_{22}}{dk_1^2}\,dk_1$$

Integrate by parts w/ $u = k_1$ and $dv = \frac{d^2E_{22}}{dk_1^2}\,dk_1$

$$\int u\,dv = uv - \int v\,du \rightarrow -\int k_1\frac{d^2E_{22}}{dk_1^2}\,dk_1 = -\left(k_1\frac{dE_{22}}{dk_1} - \int\frac{dE_{22}}{dk_1}\,dk_1\right)$$

$E(k_1) = -k_1\left(\frac{dE_{22}}{dk_1} - \frac{1}{k_1}E_{22}(k_1)\right)$ in which $-\frac{1}{k_1}E_{22}(k_1)$ becomes

$$-\frac{1}{k_1}E_{22}(k_1) = +\int_k^{\infty}\frac{1}{k_1}\frac{dE_{22}(k_1)}{dk_1}\,dk_1 = \int_k^{\infty}\frac{1}{k_1}\,dE_{22}(k_1) \Rightarrow \frac{1}{k_1}\left[E_{22}(k_1)\right]\Big|_k^{\infty}$$

$$= \frac{1}{\infty}\left[E_{22}(\infty)\right] - \frac{1}{k}\left[E_{22}(k)\right] = -\frac{1}{k}E_{22}(k) = -\frac{1}{k_1}E_{22}(k_1)$$

$*$ Since $E_{22}(\infty)/\infty$ is clearly $= 0$

$\therefore$ with $k_1 = k \Rightarrow$ $\boxed{E(k) = -k\left\{\frac{dE_{22}(k)}{dk} + \int_k^{\infty}\frac{1}{k_1}\frac{dE_{22}(k_1)}{dk_1}\,dk_1\right\}}$

6.32 $\int_0^{\infty} E(k)\,dk = K = \int_0^{\infty} C\varepsilon^{2/3}k^{-5/3}f_L(kL)f_\eta(k\eta)\,dk$

$K = C\varepsilon^{2/3}\int_0^{\infty} k^{-5/3}f_L(kL)f_\eta(k\eta)\,dk$ Switching the integration variable to $d(kL)$ . . .

$\frac{d}{dk}(kL) = L \rightarrow d(kL) = L\,dk$ and $dk = \frac{1}{L}d(kL)$

$K = C\varepsilon^{2/3}\int_0^{\infty} k^{-5/3}L^{-1}f_L(kL)f_\eta(k\eta)\,d(kL) \longrightarrow$ Treat $f_\eta(k\eta)$ as a constant $f_\eta(k\eta) = 1$ for very high Re over most of k-space

$$k = C\varepsilon^{2/3}\int_0^\infty \frac{\kappa^{-5/3}}{L} f_L(\kappa L)\, d(\kappa L) \rightarrow \text{Regroup variables}$$

$$k = C(\varepsilon L)^{2/3}\int_0^\infty (\kappa L)^{-5/3} f_L(\kappa L)\, d(\kappa L)$$ which is the same as before

$$\frac{1}{L} = L^{2/3} \cdot L^{-5/3} = L^{2/3-5/3} = \underline{L^{-3/3} = 1/L}$$

$$\therefore \boxed{k = C(\varepsilon L)^{2/3}\int_0^\infty (\kappa L)^{-5/3} f_L(\kappa L)\, d(\kappa L)}$$

w/ the assumption that $f_\eta(\kappa\eta) \rightarrow 1$ for most the wavenumber space at very large Re

Integral of 6.250 $= \int_0^\infty (\kappa L)^{-5/3}\left[\frac{\kappa L}{\sqrt{(\kappa L)^2 + C_L}}\right]^{5/3+2} d(\kappa L) \Rightarrow \int_0^\infty \frac{(\kappa L)^2}{((\kappa L)^2 + C_L)^{11/6}}\, d(\kappa L)$

letting $\kappa L = x \rightarrow d(\kappa L) = dx$

$$\int_0^\infty \frac{(\kappa L)^2}{\{(\kappa L)^2 + C_L\}^{11/6}}\, d(\kappa L) = \int_0^\infty \frac{x^2}{\{x^2 + C_L\}^{11/6}}\, dx$$

Assuming the denominator $(x^2 + C_L)^{11/6} = C_L^{1/3}(x^2+1)^{11/6}$

$$\therefore C_L^{-1/3}\int_0^\infty \frac{x^2}{(x^2+1)^{11/6}}\, dx = \int_0^\infty (\kappa L)^{-5/3} f_L(\kappa L)\, d(\kappa L) \approx 1.262\, C_L^{-1/3}$$

Therefore $k \approx C(\varepsilon L)^{2/3} \cdot 1.262\, C_L^{-1/3} = 1.5(\varepsilon L)^{2/3} \cdot 1.262\, C_L^{-1/3}$

$$k \approx 1.893(\varepsilon L)^{2/3} C_L^{-1/3} \Rightarrow C_L^{-1/3} \approx \frac{k}{1.893(\varepsilon L)^{2/3}} \rightarrow C_L \approx \left(\frac{k}{1.893(\varepsilon L)^{2/3}}\right)^{-3}$$

$C_L \approx \frac{(1.893)^3(\varepsilon L)^2}{k^3}$ using $Re_L \equiv \frac{k^{1/2}L}{\nu} = \frac{k^2}{\varepsilon\nu}$ to show

$\varepsilon L k^{1/2} = k^2 \rightarrow \varepsilon L = k^{3/2} \Rightarrow (\varepsilon L)^2 = k^3$ Substitute into equation

$$C_L \approx \frac{(1.893)^3(k)^3}{k^3} = 1.893^3 = 6.783 \quad \boxed{\therefore C_L \approx 6.783}$$

Since eqn. 6.250 and 6.251 assume a very large Re, C_L become a constant value after simplifying two equations to solve for C_L.

6.33 $\mathcal{E} = \int_0^\infty 2\nu\kappa^2 E(\kappa)\,d\kappa = \int_0^\infty 2\nu\kappa^2\left(C\varepsilon^{2/3}\kappa^{-5/3} f_L(\kappa L) f_\eta(\kappa\eta)\right) d\kappa$

$\frac{d(\kappa\eta)}{d\kappa} = \frac{\partial}{\partial\kappa}(\kappa\eta) = \eta \rightarrow d\kappa = \frac{1}{\eta} d(\kappa\eta)$

Assuming $f_L(\kappa L) \rightarrow 1$ for most of wavenumber range contributing to the dissipation ($\mathcal{E}$).

$\Rightarrow \mathcal{E} = 2C\nu\mathcal{E}^{2/3}\eta^{-1}\int_0^\infty \kappa^{1/3} f_\eta(\kappa\eta)\,d(\kappa\eta)$

$\kappa^{1/3}\eta^{-1} = (\kappa\eta)^{1/3}\cdot\eta^{-4/3}$ Substitute $\Rightarrow \boxed{\mathcal{E} = 2C\nu\mathcal{E}^{2/3}\eta^{-4/3}\int_0^\infty (\kappa\eta)^{1/3} f_\eta(\kappa\eta)\,d(\kappa\eta)}$

If $f_\eta(\kappa\eta) = \exp(-\beta_0\kappa\eta) \rightarrow \int_0^\infty (\kappa\eta)^{1/3}\exp(-\beta_0\kappa\eta)\,d(\kappa\eta)$

letting $x = \kappa\eta \rightarrow \int_0^\infty x^{1/3} e^{-\beta_0 x}\,dx = \beta_0^{-4/3}\Gamma(4/3) = \beta_0^{-4/3}(0.8930)$

Therefore:

$\mathcal{E} = 2C\nu\mathcal{E}^{2/3}\eta^{-4/3}(\beta_0^{-4/3}\cdot 0.8930) \rightarrow \beta_0^{-4/3} = \frac{\mathcal{E}^{1/3}\eta^{4/3}}{2\nu(1.5)(\mathcal{E}^{2/3})\cdot 0.8930}$

Using $\eta = (\nu^3/\mathcal{E})^{1/4} \rightarrow \eta^{4/3} = \frac{\nu}{\mathcal{E}^{1/3}}$ Substitute into equation

$\beta_0^{-4/3} = \frac{\mathcal{E}^{1/3}\left(\frac{\cancel{\nu}}{\mathcal{E}^{1/3}}\right)}{2(1.5)\cancel{\nu}\cdot 0.8930} = \frac{1}{2.679} \rightarrow \beta_0 = \left(\frac{1}{2.679}\right)^{-3/4} \approx 2.094$

Constant

$\boxed{\beta_0 \text{ converges to a constant } 2.094 \text{ at high Re}}$

Pao Spectrum $\rightarrow f_\eta = \exp\left[-\tfrac{3}{2}C(\kappa\eta)^{4/3}\right]$

$\mathcal{E} = 2C\nu\mathcal{E}^{2/3}\eta\int_0^\infty(\kappa\eta)^{1/3} f_\eta(\kappa\eta)\,d(\kappa\eta) \Rightarrow 2C\cancel{\nu}\mathcal{E}^{2/3}\left(\frac{\mathcal{E}^{1/3}}{\cancel{\nu}}\right)\int_0^\infty x^{1/3} f_\eta(x)\,dx$

$\mathcal{E} = 2C\mathcal{E}\int_0^\infty x^{1/3} f_\eta(x)\,dx$. Integral must be $\frac{1}{2C}$ for equation to be consistent

$\frac{1}{2C} = \int_0^\infty x^{1/3} e^{-\frac{3}{2}(C)x^{4/3}}\,dx \xrightarrow{\text{manipulate}} \int_0^\infty \frac{-1}{2C}\frac{d}{dx}\left(e^{-\frac{3}{2}C x^{4/3}}\right)dx = \frac{1}{2C}$

$\frac{-1}{2C}\frac{d}{dx}\left(e^{-\frac{3}{2}Cx^{4/3}}\right) = \frac{-1}{2C}\left(-\tfrac{3}{2}C\cdot\tfrac{4}{3}x^{1/3}\right)e^{-\frac{3}{2}Cx^{4/3}} = \frac{-1}{\cancel{2C}}\left(-\cancel{2C}x^{1/3}\right)e^{-\frac{3}{2}Cx^{4/3}} = x^{1/3}e^{-\frac{3}{2}Cx^{4/3}}$

recovers the original integrand $\rightarrow$

$$\therefore \int_0^\infty -\frac{1}{2c}\,\frac{d}{dx}\left(e^{-\frac{3}{2}cx^{4/3}}\right)dx = -\frac{1}{2c}\int_0^\infty d\left(e^{-\frac{3}{2}cx^{4/3}}\right)$$

$$= -\frac{1}{2c}\left[e^{-\frac{3}{2}cx^{4/3}}\right]_0^\infty = 0 - \left(-\frac{1}{2c}\right) = \frac{1}{2c}$$ Hence,

$$\varepsilon = 2c\varepsilon\int_0^\infty x^{1/3} f_\eta(x)\,dx = \cancel{2c}\,\frac{1}{\cancel{2c}}\,\varepsilon = \varepsilon$$ Pao's Spectrum is Consistent w/ 6.256