HW # 1

$\boxed{\text{PROB 2.2}}$

$$f(r) = 1 + r\frac{df}{dr}(0) + \frac{r^2}{2!}\frac{d^2f}{dr^2}(0) + \dots$$

$r = \lambda_f$ Evaluate intersection with $y=0$ axis:

$$0 = 1 + \lambda_f \frac{df}{dr}(0) + \frac{\lambda_f^2}{2!}\frac{d^2f}{dr^2}(0)$$

Solve for λ_f:

$$\lambda_f = \frac{-\frac{df}{dr}(0) \pm \sqrt{\left(\frac{df}{dr}(0)\right)^2 - 4\frac{d^2f}{dr^2}(0)}}{2\frac{d^2f}{dr^2}(0)}$$

Depending on the value of the discriminant, the solution could become a complex number $\Rightarrow$ cannot represent Taylor microscale physically.

$\boxed{\text{PROB 3.34}}$

$$\underline{x}' = \underline{x} + \underline{r}$$

By definition $\boxed{R_{ij}(\underline{r}, \underline{x}, t) = \langle u_i(\underline{x}, t)\, u_j(\underline{x}+\underline{r}, t)\rangle}$

$$R_{ij}(\underline{r}, \underline{x}, t) = \langle u_i(\underline{x}, t)\, u_j(\underline{x}', t)\rangle$$

$$R_{ij}(\underline{r}, \underline{x}', t) = \langle u_i(\underline{x}'-\underline{r}, t)\, u_j(\underline{x}', t)\rangle$$

$$R_{ji}(\underline{r}, \underline{x}', t) = \langle u_j(\underline{x}', t)\, u_i(\underline{x}'-\underline{r}, t)\rangle$$

$$\boxed{\Rightarrow R_{ji}(-\underline{r}, \underline{x}', t) = \langle u_j(\underline{x}', t)\, u_i(\underline{x}'+\underline{r}, t)\rangle}$$

$\Rightarrow R_{ij}(\underline{r}, \underline{x}, t) = R_{ji}(-\underline{r}, \underline{x}', t)$

A field is statistically homogeneous if all statistics are invariant under a shift in position $\Rightarrow$ There is no dependence from $\underline{x}$ or $\underline{x}'$.

$\Rightarrow R_{ij}(\underline{r}, t) = R_{ji}(-\underline{r}, t)$ as demonstrated before.

$\boxed{\text{PROB } 3.35}$ $\underline{U}$ s.t. $\nabla \cdot \underline{U} = 0 \Rightarrow$ Show that $\frac{\partial R_{ij}}{\partial r_j}(\underline{x}, \underline{r}, t) = 0$

If $\underline{U}$ statistically homogeneous:

$$\frac{\partial R_{ij}}{\partial r_j}(\underline{r}, t) = \frac{\partial}{\partial r_i} R_{ij}(\underline{r}, t) = 0$$

$$R_{ij}(\underline{r}, \underline{x}, t) = \langle U_i(\underline{x}, t)\, U_j(\underline{x} + \underline{r}, t) \rangle$$

$$\frac{\partial R_{ij}}{\partial r_j} = \langle U_i(\underline{x}, t) \frac{\partial U_j(\underline{x} + \underline{r}, t)}{\partial (x+r)_k} \frac{\partial (x+r)_k}{\partial r_j} \rangle$$

$$\frac{\partial R_{ij}}{\partial r_j} = \langle U_i(\underline{x}, t) \frac{\partial U_j(x + r, t)}{\partial (x+r)_k} \delta_{jk} \rangle$$

$$\frac{\partial R_{ij}}{\partial r_j} = \langle U_i(\underline{x}, t) \frac{\partial U_j(x + r, t)}{\partial (x+r)_j} \rangle = \langle U_i(\underline{x}, t)\, \nabla \cdot \underline{U}(\underline{x} + \underline{r}, t) \rangle$$

$$\nabla \cdot \bar{U} = 0 \Rightarrow \frac{\partial R_{ij}}{\partial r_j} = 0$$

If $\underline{U}$ statistically homogeneous:

$$R_{ij}(\underline{r}, t) = \langle U_i(t)\, U_j(\underline{r}, t) \rangle$$

$$\frac{\partial R_{ij}}{\partial r_i} = \langle \underbrace{\frac{\partial u_i}{\partial r_i}(t)}_{\substack{\emptyset \text{ because} \\ u_i \text{ does not depend} \\ \text{on } r_i}} u_j(r,t) \rangle = 0$$

$$\frac{\partial R_{ij}}{\partial r_j}(\underline{r},t) = \langle u_i(t) \frac{\partial u_j(\underline{r},t)}{\partial r_j} \rangle =$$

$$= \langle u_i(t) \underbrace{\nabla \cdot \underline{u}(\underline{r},t)}_{\emptyset} \rangle = 0$$

$$\Rightarrow \frac{\partial R_{ij}}{\partial r_i} = \frac{\partial R_{ij}}{\partial r_j}$$

$$\boxed{\text{PROBLEM } 12.6}$$

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau)\, d\tau$$

$$e^{-i\omega\tau} = \cos(\omega\tau) - i \sin(\omega\tau)$$

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cos(\omega\tau) - i\sin(\omega\tau))\, R(\tau)\, d\tau$$

$\cos(-\omega\tau) = \cos(\omega\tau) \Rightarrow$ even function

$\sin(-\omega\tau) = -\sin(\omega\tau) \Rightarrow$ odd "

$R(\tau) \Rightarrow$ even function

$$\Rightarrow S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(\omega\tau)\, R(\tau)\, d\tau$$

$$S(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(-\omega\tau)\, R(\tau)\, d\tau = S(\omega)$$

$S(\omega)$ has symmetry with respect to $\omega = 0$.

If $\cos(\omega\tau)$ and $R(\tau)$ are real $\Rightarrow$ $S(\omega)$ is the integral $\Rightarrow$ it is a real function.

PROB 12.7

$$R_{11}(\tau) = \bar{U}_1^2 \exp(-\alpha\tau^2) \cos(\omega_0 \tau) \qquad \alpha, \omega_0 > 0$$

Power spectrum:

$$S_e(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{U}_1^2 \exp(-\alpha\tau^2) \cos(\omega_0\tau) \exp(-i\omega\tau)\, d\tau$$

$$\cos(\omega_0\tau) = \frac{1}{2}\left(\exp(i\omega_0\tau) + \exp(-i\omega_0\tau)\right)$$

$$S_e(\omega) = \frac{\bar{U}_1^2}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-\alpha\tau^2)}{2} \Big[\underbrace{\exp(i(\omega_0-\omega)\tau)}_{\text{first term}} + \underbrace{\exp(i(-\omega_0-\omega)\tau)}_{\text{second term}}\Big] d\tau$$

Let's focus on the exponents in the integral:

$$-\alpha\tau^2 - i(\omega \mp \omega_0)\tau =$$

$$= -\alpha\left(\tau^2 + \frac{i(\omega\mp\omega_0)}{\alpha}\tau + \frac{(\omega\mp\omega_0)^2}{4\alpha^2} - \frac{(\omega\mp\omega_0)^2}{4\alpha^2}\right)$$

$$= -\alpha\left(\tau + \frac{i(\omega\mp\omega_0)}{2\alpha}\right)^2 - \frac{(\omega\mp\omega_0)^2}{4\alpha}$$

$-$ first term
$+$ second term

Let $\beta = \sqrt{\alpha}\left(\tau + \frac{i(\omega\mp\omega_0)}{2\alpha}\right)$

The resulting exponent becomes:

$$-\beta^2 - \frac{(\omega \mp \omega_0)^2}{4\alpha}$$

$$\Rightarrow S_e(\omega) = \frac{\overline{u_1^2}}{4\pi\sqrt{a}} \exp\left[-\frac{(\omega-\omega_0)^2}{4\alpha}\right] \int_{-\infty}^{\infty} e^{-\beta^2} d\beta +$$

$$+ \frac{\overline{u_1^2}}{4\pi\sqrt{a}} \exp\left[-\frac{(\omega+\omega_0)^2}{4\alpha}\right] \underbrace{\int_{-\infty}^{\infty} e^{-\beta^2} d\beta}_{\sqrt{\pi}}$$

$$S_e(\omega) = \frac{\overline{u_1^2}}{4\sqrt{\pi a}} \left[\exp\left\{-\frac{(\omega-\omega_0)^2}{4\alpha}\right\} + \exp\left\{-\frac{(\omega+\omega_0)^2}{4\alpha}\right\}\right]$$

Integral Time scale:

$$\Lambda_t = \int_0^{\infty} \frac{R_{11}(\tau)}{R_{11}(0)} d\tau = \frac{1}{\overline{u_1^2}} \int_0^{\infty} R_{11}(\tau) d\tau$$

We can also relate it to the power spectrum:

$$S_e(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{11}(\tau) e^{-i\omega\tau} d\tau$$ $\Rightarrow$ use even function property

$$S_e(\omega) = \frac{2}{2\pi} \int_0^{\infty} R_{11}(\tau) e^{-i\omega\tau} d\tau$$

$$S_e(0) = \frac{2}{2\pi} \int_0^{\infty} R_{11}(\tau) d\tau$$

$$\Rightarrow \boxed{\Lambda_t = \frac{2\pi}{2\overline{u_1^2}} S_e(0)}$$

$$\Lambda_\tau = \frac{\pi}{\overline{u_1^2}} \frac{\overline{u_1^2}}{4\sqrt{\pi a}} \, 2\exp\left\{-\frac{w_0^2}{4\alpha}\right\} = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left\{-\frac{w_0^2}{4\alpha}\right\}$$

Taylor Time scale:

Use Taylor expansion:

$$R_{11}(\tau) = \overline{u_1^2} \exp(-\alpha\tau^2)\cos(w_0\tau)$$

$$\exp(-\alpha\tau^2) \simeq 1 - \alpha\tau^2$$

$$\cos(w_0\tau) \simeq 1 - \frac{1}{2}(w_0\tau)^2$$

$$\Rightarrow R_{11}(\tau) \simeq \overline{u_1^2}\,(1-\alpha\tau^2)\left(1 - \frac{1}{2}(w_0\tau)^2\right) =$$

$$= \overline{u_1^2}\left(1 - \frac{1}{2}w_0^2\tau^2 - \alpha\tau^2 + \cancel{\frac{1}{2}\alpha w_0^2\tau^4}\right)$$

higher order

$$R_{11}(\tau) = \overline{u_1^2}\left(1 - \frac{1}{2}w_0^2\tau^2 - \alpha\tau^2\right)$$

$$\frac{dR_{11}(\tau)}{d\tau} = \overline{u_1^2}\left(-w_0^2\tau - 2\alpha\tau\right)$$

$$\frac{d^2R_{11}(\tau)}{d\tau^2} = \overline{u_1^2}\left(-w_0^2 - 2\alpha\right)$$

$$R_{11}(0) = \overline{u_1^2}$$

$$\lambda_\tau = \sqrt{\frac{-2R_{11}(0)}{\left.\frac{d^2R_{11}}{d\tau^2}\right|_{\tau=0}}} = \sqrt{\frac{-2\overline{u_1^2}}{-\overline{u_1^2}(w_0^2+2\alpha)}} = \sqrt{\alpha + \frac{w_0^2}{2}}^{\,-\frac{1}{2}}$$

PROB 12.9

$$R_{11}(\tau) = R_{11}(0) + \tau \frac{dR_{11}(\tau)}{d\tau}\Bigg|_{\tau=0} + \frac{\tau^2}{2!}\frac{d^2R_{11}(\tau)}{d\tau^2}\Bigg|_{\tau=0}$$

Autocorrelation $\Rightarrow$ even function $\Rightarrow \frac{dR_{11}}{d\tau} = 0$

Set $\tau = \lambda_\tau$ and find intersection at $y=0$

$$\Rightarrow 0 = R_{11}(0) + \frac{\lambda_\tau^2}{2}\frac{d^2R_{11}(\tau)}{d\tau^2}\Bigg|_0$$

$$\Rightarrow \lambda_\tau^2 = -\frac{2R_{11}(0)}{\frac{d^2R_{11}(\tau)}{d\tau^2}\Big|_0}$$