# Chapter 6: Turbulent Transport and its Modeling

## Part 4: Vorticity Transport

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\frac{\partial\left(\frac{\overline{p}}{\rho} + k\right)}{\partial x_i} + \nu\nabla^2\overline{U_i} + \varepsilon_{ijk}\overline{u_j\omega_k}$$

Since

$$\frac{\partial \overline{u_i u_j}}{\partial x_j} = \frac{\partial k}{\partial x_i} - \varepsilon_{ijk}\overline{u_j\omega_k}$$

$$\boxed{\omega_i = \Omega_i - \overline{\Omega_i}}$$

Where $\varepsilon_{ijk} =$ alternating tensor equal to 1 when indices are even permutation of (123), -1 for odd permutation and 0 if any two of the indices are equal.

Note that $\delta_{ij}$ and $\varepsilon_{ijk}$ are the only isotropic 2nd and 3rd order tensors and there is no iotropic 1st order tensor.

$\varepsilon_{ijk} = \varepsilon_{jki}$ and $\varepsilon_{ijk} = \varepsilon_{kij}$, i.e., unchanged by moving indices two places right or left. Whereas movement one place changes sign: $\varepsilon_{ijk} = -\varepsilon_{ikj}$.

Also note relation between $\delta_{ij}$ and $\varepsilon_{ijk}$:

$$\sum_{k=1}^{3}\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

In the rotational form RANS equation, $\left(\overline{u_i u_j}\right)_j$ is replaced by the vorticity flux $\overline{u_j\omega_j} =$ rate at which $\omega_j$ is transported in the i$^{th}$ direction by $u_i$. $\overline{P} = \frac{\overline{p}}{\rho} + k$ can be solved similarly as $\overline{p}$ and since $k = 0$ on boundaries, forces and moments readily obtained.

Appendix A.2 provides scaling laws for the fluctuaing vorticity and its derivatives.

Assume unidirectional channel flow where $\underline{\overline{U}} = (\overline{U}, 0,0)$ and $\underline{\overline{\Omega}} = (0,0,\overline{\Omega_3})$ such that:

$$0 = -\frac{\partial \overline{P}}{\partial x} + \nu \frac{d^2 \overline{U}}{dy^2} + \overline{v\omega_3} - \overline{w\omega_2} \quad (1)$$

Taylor derived gradient transport law:

$$\overline{v\omega_3} = -\overline{vL_2} \frac{d\overline{\Omega_3}}{dy} \quad (2)$$

With $\overline{vL_2} = \mathcal{T}_{22}\overline{v^2}$, same as momentum gradient transport since $\nu_t$ independent from the quantity being transported.

Substituting Eq. (2) into (1) gives:

$$0 = -\frac{\partial \overline{P}}{\partial x} + \left(\nu + \mathcal{T}_{22}\overline{v^2}\right) \frac{d^2 \overline{U}}{dy^2}$$

$$\boxed{\overline{\Omega_3} = -\frac{d\overline{U}}{dy}}$$

Note that $\overline{w\omega_2} = 0$ for unidirectional channel flow and for gradient transport since $\overline{\Omega_2} = 0$.

Compare with similar equation using $\left(\overline{u_i u_j}\right)_j$ gradient transport model

$$0 = -\frac{\partial \overline{P}}{\partial x} + \frac{d}{dy}\left(\left(\nu + \mathcal{T}_{22}\overline{v^2}\right) \frac{d\overline{U}}{dy}\right)$$

This shows that in the vorticity transport $\nu_t$ is not differentiated; however, there are difficulties near boundaries since the vorticity flux and $\frac{d\overline{\Omega}}{dy}$ have opposite signs close to the wall, as shown by DNS and Fig. below, which results in unacceptable negative eddy viscosity.

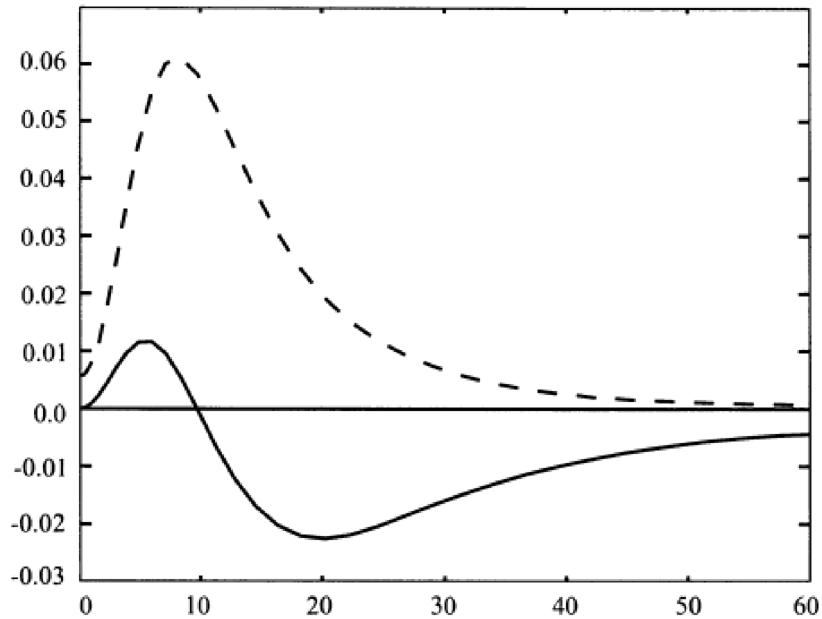


***Fig. 6.17*** *Wall-normal vorticity flux in channel flow.* —, $\overline{v\omega_3}$ ; – –, $d\overline{\Omega}/dy$.

As per turbulent momentum transport, but including possibility of 3D flow:

$$\omega_j^a = \Omega_j^a - \overline{\Omega_j^a}$$

$$\omega_j^b = \Omega_j^b - \overline{\Omega_j^b}$$

$$\omega_j^a - \omega_j^b = \left(\overline{\Omega_j^b} - \overline{\Omega_j^a}\right) + \left(\Omega_j^a - \Omega_j^b\right) \quad (3)$$

Starting from instantaneous vorticity equation

$$\frac{D\Omega_i}{Dt} = \frac{\partial \Omega_i}{\partial t} + U_j \frac{\partial \Omega_i}{\partial x_j} = \Omega_j \frac{\partial U_i}{\partial x_j} + \nu \nabla^2 \Omega_i$$

Integrating in time between $t - \tau$ and $t$:

$$\Omega_j^a - \Omega_j^b = \underbrace{\int_{t-\tau}^{t} \Omega_k(s) \frac{\partial U_j}{\partial x_k}(s) ds}_{\boxed{\text{Vortex stretching}}} + \underbrace{\int_{t-\tau}^{t} \nu \nabla^2 \Omega_j(s) ds}_{\boxed{\text{Viscous effects}}}$$

Define flux correlation as $\overline{u_i^a \omega_j^a}$ as was done for velocity ($\overline{u_a v_b}$) and using Eq. (3):

$$\overline{u_i^a \omega_j^a} = \cancel{\overline{u_i^a \omega_j^b}} + \underbrace{\overline{u_i^a \left(\overline{\Omega_j^b} - \overline{\Omega_j^a}\right)}}_{\boxed{1}} + \underbrace{\int_{t-\tau}^{t} \overline{u_i^a \Omega_k(s) \frac{\partial U_j}{\partial x_k}(s)} ds}_{\boxed{2}} + \underbrace{\int_{t-\tau}^{t} \nu \overline{u_i^a \nabla^2 \Omega_j(s)} ds}_{\boxed{3}} \quad (4)$$

Where it is assumed that $\tau$ is large enough that mixing condition $\overline{u_i^a \omega_j^b} = 0$ is satisfied.

Term 1 can be developed similarly as what was done for $\Phi_D$ in momentum transport.

Taylor series for $\overline{\Omega_j^b}$:

$$\overline{\Omega_j^b} = \overline{\Omega_j^a} - L_k \frac{d\overline{\Omega_j^a}}{dx_k} \quad (5)$$

Substituting Eq. (5) in Term 1 gives:

$$\overline{u_i^a \left(\overline{\Omega_j^b} - \overline{\Omega_j^a}\right)} = \overline{u_i^a \left(\cancel{\overline{\Omega_j^a}} - L_k \frac{d\overline{\Omega_j^a}}{dx_k} - \cancel{\overline{\Omega_j^a}}\right)} = -\overline{L_k u_i^a} \frac{d\overline{\Omega_j^a}}{dx_k} \quad (6)$$

Recall definition:

$$L_k = \int_{t-\tau}^{t} U_k\big(\underline{X}(s), s\big)\, ds = \int_{t-\tau}^{t} \Big(\overline{U_k}\big(\underline{X}(s), s\big) + u_k\big(\underline{X}(s), s\big)\Big)\, ds \quad (7)$$

And substitute Eq. (7) into (6) to obtain:

$$\overline{u_i^a \left(\overline{\Omega_j^b} - \overline{\Omega_j^a}\right)} = -\int_{t-\tau}^{t} \left(\cancel{\overline{\overline{U_k}\big(\underline{X}(s), s\big) u_i^a}} + \overline{u_k\big(\underline{X}(s), s\big) u_i^a}\right) ds \frac{d\overline{\Omega_j^a}}{dx_k}$$

Where the first term is 0 under the assumption that $\tau$ is large enough.

Therefore, term 1 becomes:

$$\overline{u_i^a\left(\overline{\Omega_j^b}-\overline{\Omega_j^a}\right)}=-\int_{t-\tau}^{t}\overline{u_k(s)u_i^a}\,ds\,\frac{d\overline{\Omega_j^a}}{dx_k}$$

For term 2, substituting mean and fluctuating quantities for $\Omega_k$ and $\frac{\partial U_j}{\partial x_k}$ gives:

$$\int_{t-\tau}^{t}\overline{u_i^a\left(\overline{\Omega_k}+\omega_k(s)\right)\left(\frac{\partial\overline{U_j}}{\partial x_k}+\frac{\partial u_j}{\partial x_k}(s)\right)}\,ds$$

$$=\int_{t-\tau}^{t}\overline{u_i^a\overline{\Omega_k}\frac{\partial\overline{U_j}}{\partial x_k}}+\overline{u_i^a\frac{\partial u_j}{\partial x_k}}(s)\overline{\Omega_k}+\overline{u_i^a\omega_k(s)\frac{\partial\overline{U_j}}{\partial x_k}}+\overline{u_i^a\omega_k(s)\frac{\partial u_j}{\partial x_k}}(s)\,ds$$

The first term is neglected as nonlinear in the mean flow; and the third and last term are neglected as considered higher order.

Therefore, term 2 becomes:

$$\int_{t-\tau}^{t}\overline{u_i^a\Omega_k(s)\frac{\partial U_j}{\partial x_k}}(s)ds=\int_{t-\tau}^{t}\overline{u_i^a\frac{\partial u_j}{\partial x_k}}(s)ds\,\overline{\Omega_k}$$

Term 3 is omitted for simplicity.

Equation 4 is equivalent to:

$$\overline{u_i^a \omega_j^a} = \underbrace{-\int_{t-\tau}^{t} \overline{u_k(s) u_i^a}\, ds \frac{d\overline{\Omega_j^a}}{dx_k}}_{\boxed{\text{gradient term}}} + \underbrace{\int_{t-\tau}^{t} \overline{u_i^a \frac{\partial u_j}{\partial x_k}(s) ds}\, \overline{\Omega_k}}_{\boxed{\text{vortex stretching term}}} \quad (8)$$

To obtain a more useful form of Eq. (8), it is necessary to introduce the Lagrangian integral scales $\mathcal{T}_{\alpha\beta}$ and $Q_{\alpha\beta\gamma}$ via

$$\overline{u_\alpha u_\beta}(\tau) = \int_{t-\tau}^{t} \overline{u_\alpha(\underline{x}, t) u_\beta(\underline{X}(s), s)}\, ds \quad \text{and}\ \mathcal{T}_{\alpha\beta} = \int_{-\infty}^{0} \overline{u_\alpha u_\beta}(\tau) d\tau \quad (9)$$

$$\overline{u_\alpha \frac{\partial u_\beta}{\partial x_\gamma}}(\tau) = \int_{t-\tau}^{t} \overline{u_\alpha \frac{\partial u_\beta}{\partial x_\gamma}(s)}\, ds \quad \text{and}\ Q_{\alpha\beta\gamma} = \int_{-\infty}^{0} \overline{u_\alpha \frac{\partial u_\beta}{\partial x_\gamma}}(\tau) d\tau \quad (10)$$

Such that Eq. (8) becomes:

$$\overline{u_\alpha \omega_\beta} = -\mathcal{T}_{\alpha k} \overline{u_\alpha u_k} \frac{\partial \overline{\Omega_\beta}}{\partial x_k} + Q_{\alpha\beta k} \overline{u_\alpha \frac{\partial u_\beta}{\partial x_k}}\, \overline{\Omega_k} \quad (11)$$

## Vorticity transport in Channel Flow

Consider now unidirectional shear flows with mean velocity $\overline{U}(y)$ and $\overline{\Omega_3} = -d\overline{U}/dy$ is the only non-zero mean vorticity component. With these assumptions five vorticity flux components are identically zero. The zero vorticity flux components are:

$$\overline{u\omega_1} = \overline{v\omega_2} = \overline{w\omega_3} = \overline{u\omega_2} = \overline{v\omega_1} = 0$$

Appendix A.1

The remaining correlations in Eq. (8) are non-zero and are given by:

$$\overline{w\omega_1} = Q_{313}\overline{w\frac{\partial u}{\partial z}}\overline{\Omega_3}$$

$$\overline{w\omega_2} = Q_{323}\overline{w\frac{\partial v}{\partial z}}\overline{\Omega_3}$$

$$\overline{v\omega_3} = -\mathcal{T}_{22}\overline{v^2}\frac{d\overline{\Omega_3}}{\partial y} + Q_{233}\overline{v\frac{\partial w}{\partial z}}\overline{\Omega_3}$$

$$\overline{u\omega_3} = -\mathcal{T}_{12}\overline{uv}\frac{d\overline{\Omega_3}}{dy} + Q_{133}\overline{u\frac{\partial w}{\partial z}}\overline{\Omega_3}$$

Where additional Lagrangian integral scales are defined according to Eqs. (9) and (10).

$\overline{w\omega_1}$ and $\overline{w\omega_2}$ do not originate in gradient physics and would be predicted to be zero if vortex stretching was neglected.

Using DNS best fit data as shown in figures uses $\mathcal{T}_{22}^{+} = 4.8$, $\mathcal{T}_{12}^{+} = 12.3$, $Q_{233}^{+} = 5.5$, $Q_{323}^{+} = 9.5$, $Q_{133}^{+} = 16.3$, and $Q_{313}^{+} = 0.95$.

Thus, including the first-order vortex stretching model, the essentials of turbulent vorticity flux can be accounted for. Gradient terms capture most of the transport away from the wall, while the stretching terms account for non-gradient transport for $\overline{v\omega_3}$ and $\overline{u\omega_3}$.

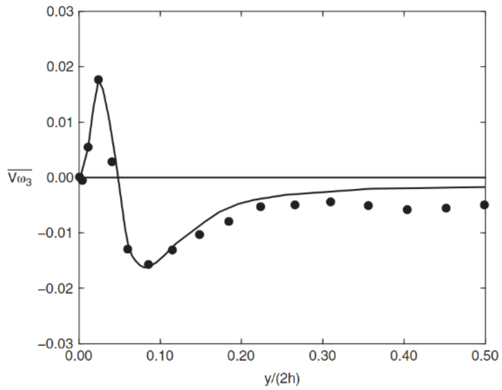


**Figure 6.17** $\overline{v\omega_3}$: •, DNS results; —, prediction from Eq. (6.62). From [21]. Copyright ©Springer

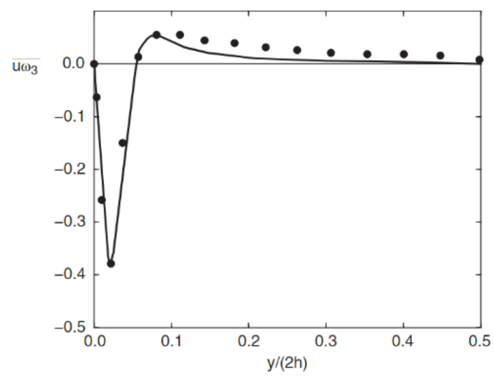


**Figure 6.18** $\overline{u\omega_3}$: •, DNS results; —, prediction from Eq. (6.63). From [21]. Copyright ©Springer-Verlag.

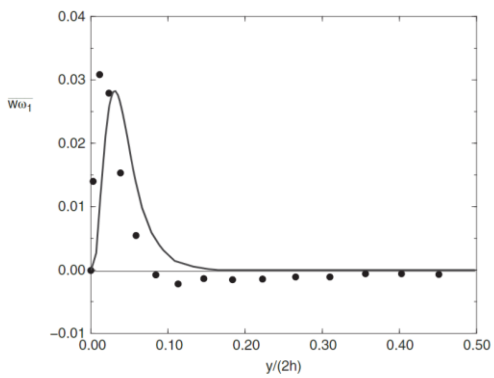


**Figure 6.19** $\overline{w\omega_1}$: •, DNS results; —, prediction from Eq. (6.60). From [21]. Copyright ©Springer-Verlag.

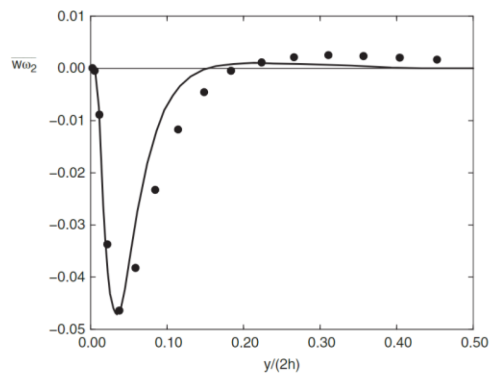


**Figure 6.20** $\overline{w\omega_2}$: •, DNS results; —, prediction from Eq. (6.61). From [21]. Copyright ©Springer-Verlag.

## Appendix A

## A.1

$$\overline{u\omega_1} = \overline{v\omega_2} = \overline{w\omega_3} = \overline{u\omega_2} = \overline{v\omega_1} = 0$$

Consider $\overline{u\omega_1}$:

$$\overline{u\omega_1} = -\mathcal{T}_{11}\overline{u_1u_1}\frac{\partial\overline{\Omega_1}}{\partial x_1} + Q_{111}\overline{u_1\frac{\partial u_1}{\partial x_1}}\,\overline{\Omega_1} - \mathcal{T}_{12}\overline{u_1u_2}\frac{\partial\overline{\Omega_1}}{\partial x_2} + Q_{112}\overline{u_1\frac{\partial u_1}{\partial x_2}}\,\overline{\Omega_2}$$
$$- \mathcal{T}_{13}\overline{u_1u_3}\frac{\partial\overline{\Omega_1}}{\partial x_3} + Q_{113}\overline{u_1\frac{\partial u_1}{\partial x_3}}\,\overline{\Omega_3}$$

It can be immediately recognized that all the terms, except for the last one, are identically zero, since $\overline{\Omega_1} = \overline{\Omega_2} = 0$ in a channel flow.

The last term

$$Q_{113}\overline{u_1\frac{\partial u_1}{\partial x_3}}\,\overline{\Omega_3}$$

Can be rewritten as

$$\overline{u\omega_1} = Q_{113}\overline{\frac{1}{2}\frac{\partial u_1^2}{\partial x_3}}\,\overline{\Omega_3} = 0 \quad (1A)$$

This expression shows that also the last term needs to be zero in order to satisfy the requirement of symmetry with respect to reflections in the $x - y$ plane.

Similarly, for $\overline{v\omega_2}$ and $\overline{w\omega_3}$, the only term containing $\overline{\Omega_3}$ is:

$$\overline{v\omega_2} = Q_{223}\overline{u_2\frac{\partial u_2}{\partial x_3}}\,\overline{\Omega_3} = Q_{223}\overline{\frac{\partial u_2^2}{\partial x_3}}\,\overline{\Omega_3} = 0 \quad (2A)$$

$$\overline{w\omega_3} = Q_{333}\overline{u_3\frac{\partial u_3}{\partial x_3}}\,\overline{\Omega_3} = Q_{333}\overline{\frac{\partial u_3^2}{\partial x_3}}\,\overline{\Omega_3} = 0 \quad (3A)$$

The remaining two vorticity fluxes are $\overline{u\omega_2}$ and $\overline{v\omega_1}$.

Recall vorticity components definition:

$$\omega_1 = w_y - v_z$$

$$\omega_2 = u_z - w_x$$

$$\omega_3 = v_x - u_y$$

Substituting them into Eqs. (1A), (2A), (3A):

$$\overline{u\omega_1} = \overline{uw_y} - \overline{uv_z} = 0 \rightarrow \overline{uw_y} = \overline{uv_z}$$

$$\overline{v\omega_2} = \overline{vu_z} - \overline{vw_x} = 0 \rightarrow \overline{vu_z} = \overline{vw_x}$$

$$\overline{w\omega_3} = \overline{wv_x} - \overline{wu_y} = 0 \rightarrow \overline{wv_x} = \overline{wu_y}$$

And using several identities appropriate to channel flow gives:

$$\overline{uw_y} = \overline{uv_z} = -\overline{vu_z} = -\overline{vw_x} = \overline{wv_x} = \overline{wu_y} = -\overline{uw_y} \quad (4A)$$

And the equality of the first and last terms in this relation implies that each of the correlations are zero.

For $\overline{u\omega_2}$ and $\overline{v\omega_1}$ the only term containing $\overline{\Omega_3}$ is:

$$\overline{u\omega_2} = Q_{123}\overline{u_1\frac{\partial u_2}{\partial x_3}}\overline{\Omega_3} = Q_{xyz}\overline{uv_z}\overline{\Omega_3} = 0$$

$$\overline{v\omega_1} = Q_{213}\overline{u_2\frac{\partial u_1}{\partial x_3}}\overline{\Omega_3} = Q_{213}\overline{vu_z}\overline{\Omega_3} = 0$$

And they are both zero according to the equalities shows in Eq. (4A).

**A.2**

Characteristic Scales of Vorticity

Estimation magnitude terms in turbulent enstrophy equation.

### 3.4 The Governing Equation for the Magnitude of Vorticity Fluctuations (or Turbulent Enstrophy)

Just like the equations for the kinetic energy, we can derive equations for $\Omega_i\Omega_i$ and $\overline{\omega_i'\omega_i'}$ by substituting $\omega_i = \Omega_i + \omega_i'$ and $u_i = U_i + u_i'$ into (2.66), and then averaging. For the mean squared vorticity fluctuations (or turbulent enstrophy) $\overline{\omega_i'\omega_i'}$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right) + U_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right) = \underbrace{-\overline{u_j'\omega_i'}\frac{\partial\Omega_i}{\partial x_j}}_{\text{Production}} - \underbrace{\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_j'\omega_i'\omega_i'}\right)}_{\text{Transport by turbulence}}$$
$$+ \underbrace{\overline{\omega_i'\omega_j's_{ij}'}}_{\substack{\text{Stretching by}\\ \text{strain fluctuations}}} + \underbrace{\overline{\omega_i'\omega_j'}S_{ij}}_{\substack{\text{Stretching by}\\ \text{mean strain}}} + \Omega_j\overline{\omega_i's_{ij}'} + \underbrace{\nu\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right)}_{\text{Viscous transport}} - \underbrace{\nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}}}_{\text{Viscous dissipation}}. \tag{3.24}$$

Note that the production term appears with the opposite sign in the mean vorticity equation, as was the case for kinetic energy. As mentioned in Section 2.6, differentiation amplifies small scales. Thus, we expect that spatial derivatives of fluctuating velocities in general, and $\omega_i'$ in particular, are dominated by the small scales of turbulence, whereas the derivatives of mean flow quantities are determined by large scales. Hence, different terms in the equation above have widely different orders of magnitude at high Reynolds numbers.

On the right-hand side of (3.24), the third term is dominant because it involves the product of three velocity derivatives taken at fine scales, whereas the other non-viscous terms contain at most two. Of the viscous terms, the second is a product of two second derivatives of velocity and is dominant. Retaining the dominant terms yields

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right) + U_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right) \approx \overline{\omega_i'\omega_j's_{ij}'} - \nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}}. \tag{3.25}$$

In homogeneous stationary turbulence, the left-hand side of (3.25) is zero, yielding

$$\boxed{\overline{\omega_i'\omega_j's_{ij}'} \approx \nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}}.} \tag{3.26}$$

The right-hand side of (3.26) is positive, suggesting that the stretching term is positive in the mean. That is, **stretching dominates compression on average**. (Refer to Figure 5.12 for an illustration of the spatial distribution of vorticity.) Note also that, unlike the equation for $\overline{u_i'u_i'}$ in homogeneous stationary turbulence, where the production term – which involves the mean flow – is dominant, the equation for $\overline{\omega_i'\omega_i'}$ in homogeneous stationary turbulence is not dominated by the mean.

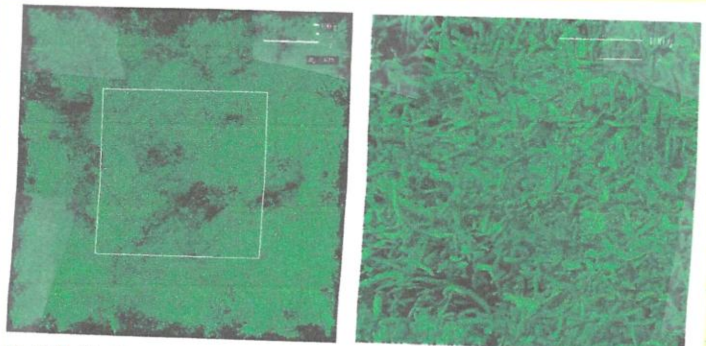

Fig. 5.12 Vorticity isosurfaces from direct numerical simulation of homogeneous isotropic turbulence ($4096^3$ grid with a Taylor microscale $Re_\lambda = 675$). The image on the right is a 16× zoomed-in view of the image on the left. In the figure legend, $L = l$ is the integral length scale. (Image credit: Ishihara, Gotoh and Kaneda (2007), adapted from figures 2(a) and 3(d))

velocity Scale: $u^2 = \frac{1}{3}q^2 = \overline{u_1^2} \sim \sqrt{k}$

length Scale: $\ell = u^3/\varepsilon = q^{3/2}/\varepsilon$

$\lambda$ = Taylor Micro Scale $\quad \varepsilon = \frac{30\nu\overline{u^2}}{\lambda^2}$

$s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \sim u/\lambda$

$S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \sim u/\ell$

$\Omega_i = \epsilon_{ijk} U_{k,j} \sim u/\ell$

$\overline{\omega_i \omega_i} = 2\overline{r_{ij} r_{ij}} \qquad r_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$

For homogeneous turbulence: $\frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} = \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}} = \overline{s_{ij}s_{ij}} - \overline{r_{ij}r_{ij}} = 0$

∴ $\overline{s_{ij}s_{ij}} = \overline{r_{ij}r_{ij}}$

ie $\overline{\omega_i \omega_i} = 2\overline{s_{ij}s_{ij}}$

Thus: $\omega_i \sim u/\lambda$ units are $s^{-1}$

$\Rightarrow \lambda/u$ time scale $\omega_i$

but does not provide proper length scale needed dissipation term turbulent enstrophy equation.

Weak scaling for $\omega_{i,j} = \frac{\partial \omega_i}{\partial x_j}$. If use $\lambda$ & $u$ in (3.26)

$$\overline{\omega_i \omega_j S_{ij}} = \nu\, \overline{\omega_{i,j}\, \omega_{i,j}}$$

$$u^3/\lambda^3 = \nu\, u^2/\lambda^4 \Rightarrow Re_\lambda \sim 1$$

which is not correct $\eta/\lambda \sim Re_\lambda^{-1/2} \sim \frac{u_\eta}{u_{rms}}$ (chapter 4 Part 3 pg 12) $Re_\eta = 1$ (Chapter 4 Part ① pg 4)

∴ define length scale associated $\omega_{i,j}$ as $\delta$

$$\frac{\partial \omega_i}{\partial x_j} = \frac{u}{\lambda \delta}$$

$$\Rightarrow \quad \nu \frac{u^2}{\lambda^2 \delta^2} = O\left(u^3/\lambda^3\right) \qquad \nu \frac{\lambda}{u \delta^2} = 1 \qquad \frac{\nu}{u\lambda} = \frac{\delta^2}{\lambda^2}$$

$$Re_\lambda^{-1/2} = \frac{\delta}{\lambda}$$

$$\delta/\lambda = O\left(Re_\lambda^{-1/2}\right) = \frac{\eta}{\lambda} \Rightarrow \delta \sim \eta$$

$$\frac{u_\eta}{u} = O\left(Re_\lambda^{-1/2}\right) \Rightarrow u \sim u_\eta$$

$$\omega_i \sim u_\eta/\eta \Rightarrow \frac{\partial \omega_i}{\partial x_j} \sim \frac{u_\eta}{\eta^2}$$

showing $\omega_i$ & $\omega_{i,j}$ scale with Kolmogorov variables.

Moin P157

$$\omega_k = \varepsilon_{kij} \frac{\partial u_j}{\partial x_i}$$

$$\omega_k \omega_k = \varepsilon_{kij} \frac{\partial u_j}{\partial x_i} \varepsilon_{kab} \frac{\partial u_b}{\partial x_a} = \varepsilon_{kij} \varepsilon_{kab} \frac{\partial u_j}{\partial x_i} \frac{\partial u_b}{\partial x_a}$$

$$= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}) \frac{\partial u_b}{\partial x_a} \frac{\partial u_j}{\partial x_i}$$

$$= \delta_{ia}\delta_{jb} \frac{\partial u_b}{\partial x_a} \frac{\partial u_j}{\partial x_i} - \delta_{ib}\delta_{ja} \frac{\partial u_b}{\partial x_a} \frac{\partial u_j}{\partial x_i}$$

$$= \delta_{ia} \frac{\partial u_j}{\partial x_a} \frac{\partial u_j}{\partial x_i} - \delta_{ib} \frac{\partial u_b}{\partial x_j} \frac{\partial u_j}{\partial x_i} = \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

$$\Rightarrow \boxed{\omega_k \omega_k = \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}}$$

$$r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$r_{ij} r_{ij} = \frac{1}{4}\left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} - 2 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i}\right)$$

Note: dummy indices, $i \leftrightarrow j$, then $\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i}$

$$r_{ij} r_{ij} = \frac{1}{4}\left(2 \frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} - 2 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}\right)$$

$$\Rightarrow r_{ij} r_{ij} = \frac{1}{2} \omega_k \omega_k \quad \text{or} \quad \boxed{\omega_i \omega_i = 2 r_{ij} r_{ij}}$$

drop primes

$$\frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left[\overline{\frac{\partial (u_i u_j)}{\partial x_j}}\right] = 0$$ homogeneous turbulence

$$\Rightarrow \frac{\partial}{\partial x_i}\left( u_i \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j}\right)$$ ($\frac{\partial u_j}{\partial x_j} \to 0$)

$$= \frac{\partial}{\partial x_i}\left( u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + u_j \frac{\partial^2 u_i}{\partial x_i \partial x_j}$$ ($\frac{\partial^2 u_i}{\partial x_i \partial x_j} \to 0$)

Therefore: $$\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} = \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} = 0$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$S_{ij}S_{ij} - r_{ij}r_{ij} = \frac{1}{4}\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_j}{\partial x_i}\right)^2 + 2\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right] - \frac{1}{4}\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_j}{\partial x_i}\right)^2 - 2\frac{u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right]$$

$$= \frac{1}{4}\left(4\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = 0$$

$$S_{ij}S_{ij} - r_{ij}r_{ij} = 0 \Rightarrow \boxed{S_{ij}S_{ij} = r_{ij}r_{ij}}$$