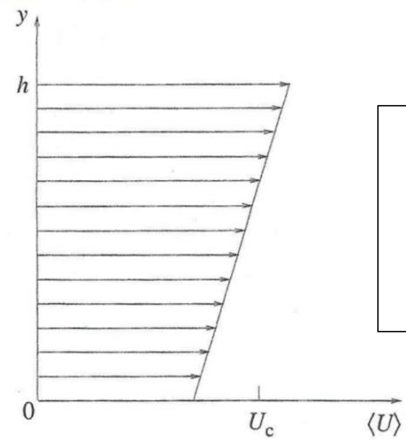
Chapter 6: Turbulent Transport and its Modeling

Part 3: Homogeneous Shear Flow

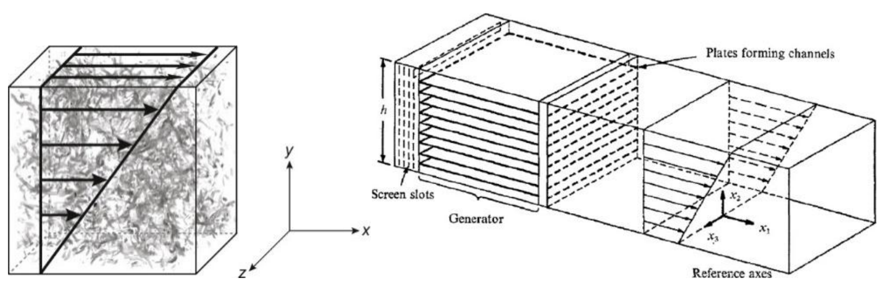
(1) Pope 5.4.5; also recall discussion Chapter 4 Part 7, shear-stress spectrum)

In homogeneous turbulence[1] $\underline{u}(\underline{x},t)$ and $p'(\underline{x},t)$ are statistically homogeneous and $\overline{U_{i,j}}$ must be uniform, although it may be $f(t)$ (Pope Ex. 5.41). In homogeneous shear flow $S = \overline{U_{i,j}}$ = constant, which can be realized in wind tunnel experiments by using screens controlling the inflow velocity profile.



$$\overline{U} = f(y)$$
$$\overline{V} = \overline{W} = 0$$
$$S = \frac{\partial \overline{U}}{\partial y} = \text{constant}$$

Fig. 5.30. A sketch of the mean velocity profile in homogeneous shear flow.



Moin and Chan: homogeneous shear flow DNS (left) and EFD setup (right).

Appendix A.2

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[1] Homogeneous turbulence: the time-averaged properties of the flow are uniform and independent of position → invariant under translation, i.e., shift in the origin of the coordinate system. For example, whereas $\overline{u^2}, \overline{v^2}, \overline{w^2}$ may differ from each other, each must be constant throughout the system, i.e., the time-averaged gradients of the fluctuating components, i.e.,

$$\frac{\partial}{\partial x_j}\overline{fluctuating\ terms} = 0$$

.

At x/h = 0, the RS are nearly uniform normal to the flow direction, which persists downstream. However, they show increasing values in the axial direction, which can be removed using a refence frame moving with the mean velocity $\overline{U}$ such that the turbulence is approximately homogeneous, as per Fig. 5.31.

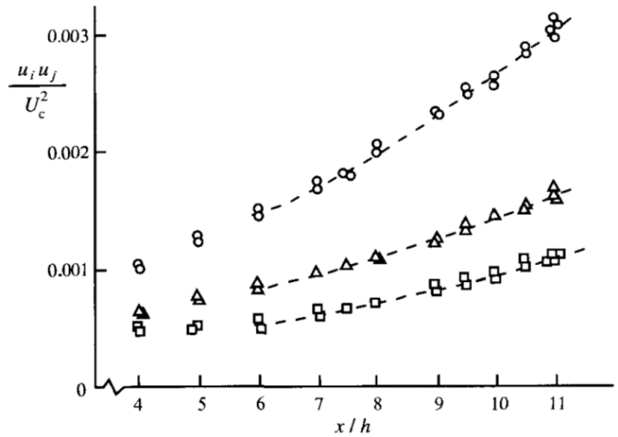


Fig. 5.31. Reynolds stresses against axial distance in the homogeneous-shear-flow experiment of Tavoularis and Corrsin (1981): ○, $\langle u^2\rangle$; □, $\langle v^2\rangle$; △, $\langle w^2\rangle$.

Despite axial variation, in frame of reference moving at U_c,

$$\overline{u_i u_j} \approx \begin{cases} \text{const} & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

An important conclusion is that after the initial development time the flow becomes self-similar when statistics are normalized by S and k, as shown in Table below.

Table 5.4. *Statistics in homogeneous turbulent shear flow from the experiments of Tavoularis and Corrsin (1981) and the DNS of Rogers and Moin (1987)*

	Tavoularis and Corrsin		Rogers and Moin
	$x/h = 7.5$	$x/h = 11.0$	$\mathcal{S}t = 8.0$
$\langle u^2\rangle/k$	1.04	1.07	1.06
$\langle v^2\rangle/k$	0.37	0.37	0.32
$\langle w^2\rangle/k$	0.58	0.56	0.62
$-\langle uv\rangle/k$	0.28	0.28	0.33
$-\rho_{uv}$	0.45	0.45	0.57
$\mathcal{S}k/\varepsilon$	6.5	6.1	4.3
$\mathcal{P}/\varepsilon$	1.8	1.7	1.4
$L_{11}\mathcal{S}/k^{1/2}$	4.0	4.0	3.7
$L_{11}/(k^{3/2}/\varepsilon)$	0.62	0.66	0.86

Correlation coefficient

$$\rho_{uv} = \frac{< uv >}{[< u >< v >^{1/2}}$$

Between $x/h = 7.5$ and 11, $k(t)$ increases by 65%, yet normalized Reynolds stresses nearly constant.

$\tau = k/\varepsilon =$ turbulent time scale nearly constant such that $\frac{Sk}{\varepsilon} \sim$ constant.

L_{11} increases by 30%, but when normalized nearly constant.

The TKE equation for homogeneous shear flow is (Pope Ex. 5.40):

$$\frac{dk}{dt} = P - \varepsilon$$

Such that ($\tau = k/\varepsilon$)

$$\frac{\tau}{k}\frac{dk}{dt} = \frac{P}{\varepsilon} - 1$$

Which has solution:

$$k(t) = k(0)\exp\left[\frac{t}{\tau}\left(\frac{P}{\varepsilon} - 1\right)\right]$$

Since $P/\varepsilon \approx 1.7$ and $S\tau \approx$ constant, $k(t)$ grows exponentially in time and both ε and $L = k^{3/2}/\varepsilon = k^{1/2}/\tau$ also grow exponentially.

Recall for grid turbulence (Chapter 5 Part 2) P = 0, and $k(t)$ and $\varepsilon(t)$ decay with time with decay exponents $-n$ and $-(n+1)$, respectively, with n between 1.15 and 1.45; and $L = k^{3/2}/\varepsilon$ grows in time with exponent (1-n/2) due faster decay smaller/faster motions.

(2) Bernard 6.6

Idealized flow such that $S = d\overline{U}/dy > 0$ =constant superimposed on homogeneous/isotropic turbulence.

k equation in homogeneous shear flow:

$$\frac{dk}{dt} = P - \varepsilon$$

Where the production term

$$P = -\overline{uv}\frac{d\overline{U}}{dy}$$

is positive, since $\overline{uv} < 0$ associated with $S > 0$.

ε equation in homogeneous shear flow:

$$\frac{d\varepsilon}{dt} = P_\varepsilon^1 + P_\varepsilon^2 + \underbrace{P_\varepsilon^4 - \Upsilon_\varepsilon}$$

Same as isotropic decay due homogeneous/isotropic turbulence assumption.

See Appendix A.4

For homogeneous isotropic turbulence

$$P_\varepsilon^1 + P_\varepsilon^2 = -\left(\varepsilon_{ij}^c + \varepsilon_{ij}\right)S = -2\varepsilon S = 2\varepsilon\frac{P}{\overline{uv}}$$

For homogeneous shear flow

$$-\frac{\overline{uv}}{k} \approx constant \approx 0.3$$

Such that:

$$P_\varepsilon^1 + P_\varepsilon^2 = C_{\varepsilon_1} P \frac{\varepsilon}{k}$$

For P_ε^4 and Υ_ε same expressions as isotropic turbulence:

$$P_\varepsilon^4 - \Upsilon_\varepsilon = S_k^* R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k} - G^* \frac{\varepsilon^2}{k}$$

And for the palenstrophy, assuming vortex stretching not preempted by dissipation, it is assumed that:

$$G^* = \left(S_k^* - C_{\varepsilon_3}\right)\sqrt{R_T} + C_{\varepsilon_2}$$

Where $C_{\varepsilon_3} = 0$ produces standard model for RANS, as per Chapter 5 Part 2 pg. 19. Therefore, the ε equation becomes:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon_1} P \frac{\varepsilon}{k} + C_{\varepsilon_3} R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k} - C_{\varepsilon_2} \frac{\varepsilon^2}{k}$$

And it needs to be solved in conjunction with

$$\frac{dk}{dt} = P - \varepsilon$$

Once a model is introduced for

$$P = -\overline{uv} \frac{d\overline{U}}{dy}$$

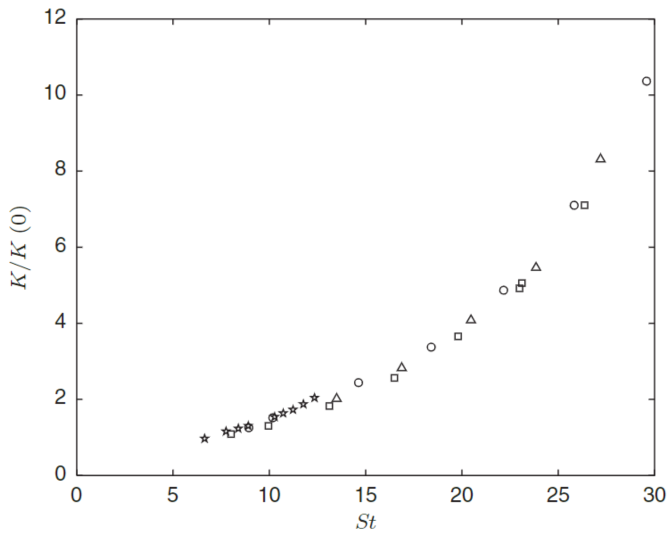


Figure 6.14 Measured $K/K(0)$ in homogeneous shear flow for $St < 30$ from [15]. Reprinted with permission of Cambridge University Press.

EFD and DNS show exponential growth K and ε, but $St < 30$; and following asymptotic relationships:

$$\frac{Sk}{\varepsilon} \approx 6 \quad (1)$$

$$\frac{P}{\varepsilon} \approx 1.8 \quad (2)$$

although in some cases LES statistics still not converged at $St = 30$.

Using $t^* = St$ and $k^*(St) = k(t)/k(0)$, the equation for k becomes

$$\frac{dk^*}{dt^*} = \frac{\varepsilon}{Sk}\left(\frac{P}{\varepsilon} - 1\right)k^* \quad (3)$$

And for $St < 30$, substituting Eqs. (1) and (2), the solution to Eq. (3) becomes

$$k^*(t^*) = e^{0.13t^*}$$

With similar analysis ε equation, with $C_{\varepsilon_3} = 0$, (i.e., neglecting vortex stretching term) results in an exponential growth for $\varepsilon^*(t^*)$.

Appendix A.1, Prob. 6.2 Bernard

Long time EFD and simulations not achievable. Two hypotheses put forward:

1) $P = \varepsilon$ such that k and ε asymptote to constant values (Townsend 1956)

2) k and ε continue exponential growth, which is not physical as unlimited growth k unrealistic → only pertains to ideal case.

To solve k and ε equations for long time growth $\overline{uv}$ model needed.

$$\overline{uv} = -\mathcal{T}_{22}\overline{v^2}\mathrm{S}$$

Where for isotropic turbulence $\overline{v^2} = 2k/3$.

Assume $\mathcal{T}_{22} \propto$ eddy turnover time k/ε, same as $k - \varepsilon$ model approach: $\mathcal{T}_{22} = \frac{3}{2}C_\mu\frac{k}{\varepsilon}$

Such that the set of equations becomes:

$$R_T = \frac{k^2}{\nu\varepsilon} = \frac{\sqrt{k}k^{3/2}/\varepsilon}{\nu}$$

$$\frac{dk}{dt} = C_\mu\frac{k^2}{\varepsilon}S^2 - \varepsilon$$

$$\frac{d\varepsilon}{dt} = C_{\varepsilon_1}C_\mu kS^2 + C_{\varepsilon_3}R_T^{\frac{1}{2}}\frac{\varepsilon^2}{k} - C_{\varepsilon_2}\frac{\varepsilon^2}{k}$$

And solution to these equations can be used to illustrate physics of homogeneous shear flow at long times.

$C_{\varepsilon_1} = 1.44$ homogeneous shear flow (value used k-ε turbulence model).

$C_{\varepsilon_2} = 1.92$, n = 1.09 EFD grid turbulence decay rate (value used k-ε turbulence model) A.4.

$C_\mu = 0.09$ equilibrium free shear flows (value used k-ε turbulence model).

$C_{\varepsilon_3} = 0$ → no vortex stretching.

$C_{\varepsilon_3} = 0.1$ → used to investigate vortex stretching in homogeneous shear flow.

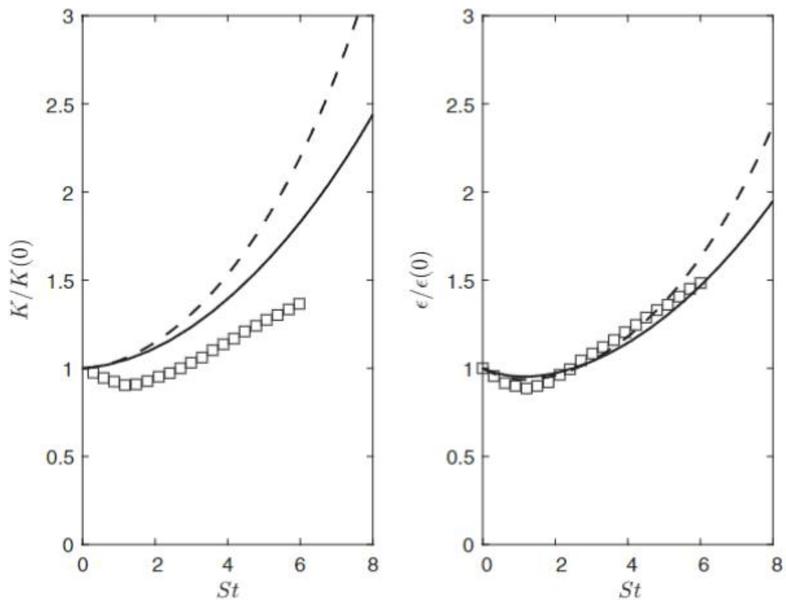


Figure 6.15 Computed solution for $K/K(0)$ (left) and $\epsilon/\epsilon(0)$ (right) in homogeneous shear flow: —, with vortex stretching; −−, without vortex stretching; o, LES calculation [17].

Fig. 6.15 shows EFD solutions for short time ($St < 10$), trends look similar.

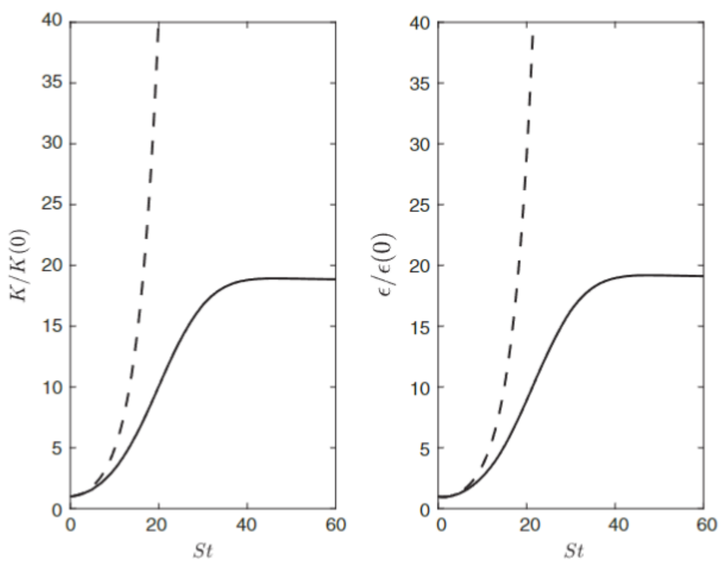


Figure 6.16 Computed solutions for $K/K(0)$ (left) and $\epsilon/\epsilon(0)$ (right) in homogeneous shear flow: —, with vortex stretching; −−, without vortex stretching.

Fig. 6.16 shows long time solutions ($St < 60$):

- $C_{\varepsilon_3} = 0 \rightarrow$ exponential growth
- $C_{\varepsilon_3} \neq 0 \rightarrow$ growth plateaus and $P = \varepsilon$ equilibrium achieved, as predicted by Townsend.

Asymptotic values for k and ε are found by setting $dk/dt = d\varepsilon/dt = 0$, resulting in:

$$k_\infty = \frac{\sqrt{C_\mu}\left(C_{\varepsilon_2} - C_{\varepsilon_1}\right)^2}{C_{\varepsilon_3}^2}\,\nu S$$

$$\varepsilon_\infty = \frac{C_\mu\left(C_{\varepsilon_2} - C_{\varepsilon_1}\right)^2}{C_{\varepsilon_3}^2}\,\nu S^2$$

Appendix A.3

The magnitude of the asymptotic values increases with the inverse square of C_{ε_3}, which highlights importance of vortex stretching as an additional source of dissipation.

$C_{\varepsilon_3} \neq 0$ likely most realistic physics, i.e., vortex stretching maintains independent physical process, as per issues discussed previously regarding the high Re equilibrium solution for self-similar flows.

(3) The Spectral view of the energy cascade (Pope 6.6)

For large Re, energy-containing and dissipative motions have clear separation of scales $L/\eta{\sim}Re^{3/4} \gg 1$ (L = $k^{3/2}/\varepsilon$ = $l_0 \approx 2L_{11}$) and bulk of TKE is contained in motions of length scale l, where $6L_{11} > l > \frac{1}{6}L_{11} = \frac{1}{12}l_0 = l_{EI}$, with characteristic velocity $k^{1/2}$.

Since $6L_{11}{\sim}\mathcal{L}$ large-scale motions anisotropic and f(geometry). Timescale $\tau = L/k^{1/2}$ is large compared to mean-flow time scale $\mathcal{L}/\overline{U}$ and f(flow history), i.e., smaller eddies turn over at a higher rate than the larger eddies.

∴ energy-containing motions do not have universal form arising from statistical equilibrium.

Anisotropy and production of turbulence confined to energy-containing motions and viscous dissipation is negligible.

Initial steps energy cascade, energy removed by inviscid processes (Production) and transferred to smaller scales $l < l_{EI}$ at rate $\mathcal{T}_{EI}{\sim}\frac{u_0^3}{l_0}$ which scales with $u_{rms}^3/L = k^{3/2}/L$. $\mathcal{T}_{EI}$ is not universal ∴ non-dimensional ratio $\mathcal{T}_{EI}/(k^{3/2}/L)$ is not universal.

Energy spectrum balance for homogeneous shear flow (Hinze Chapter 4):[2]

$$\frac{\partial}{\partial t}E(\kappa,t) = P_k(\kappa,t) - \frac{\partial}{\partial k}T_k(\kappa,t) - 2\nu\kappa^2 E(\kappa,t) \quad (1)$$

Rate of change energy spectrum	Production due to shear	Spectral transfer	Dissipation

$P_k =$ product of the mean velocity gradient $\partial\overline{U_i}/\partial x_j$ and anisotropic part of the spectrum tensor.

[2] Can be compared with Chapter 5 Part 3 $\mathcal{R}_{ij}$ equation for homogeneous turbulent flow, whereas Eq. (1) is for homogeneous shear flow.

$$P_k = \mathcal{E}(\kappa, t)\frac{d\overline{U}}{dy} = 2\pi\kappa^2\left[2\mathcal{E}_{12} - \kappa_1\frac{\partial \mathcal{E}_{i,i}}{\partial \kappa_2}\right]$$

Hinze 4-47, 4-46, 4-45, 4-37, and 4-30

$$P_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} P_k \, d\kappa$$

Represent contribution to the production from wave number range (κ_a, κ_b).

$$P = P_{(0,\infty)} \approx P_{(0,\kappa_{EI})}$$

i.e.,

$$\frac{P_{(\kappa_{EI},\infty)}}{P} \ll 1$$

Most of the anisotropy contained in energy-containing range.

$T_k(\kappa, t)$ represents the spectral energy transfer rate, i.e., net rate at which energy is transferred from modes of lower wave number than κ to those with wave numbers higher than κ.

Rate of gain of energy in (κ_a, κ_b) due to spectral transfer is:

$$\int_{\kappa_a}^{\kappa_b} -\frac{\partial}{\partial k} T_k(\kappa, t)\, d\kappa = T_k(\kappa_a) - T_k(\kappa_b)$$

Since for $T_k(\kappa_a = 0) = T_k(\kappa_b = \infty) = 0$, this term makes no contribution to the balance of TKE.

$$T_k = T_k^1 + T_k^2$$

T_k^1: Interaction triads of wave numbers

$$\underline{\kappa} + \underline{\kappa}' + \underline{\kappa}'' = 0$$

T_k^2: Kinematic effect of mean velocity gradients has on the spectrum.

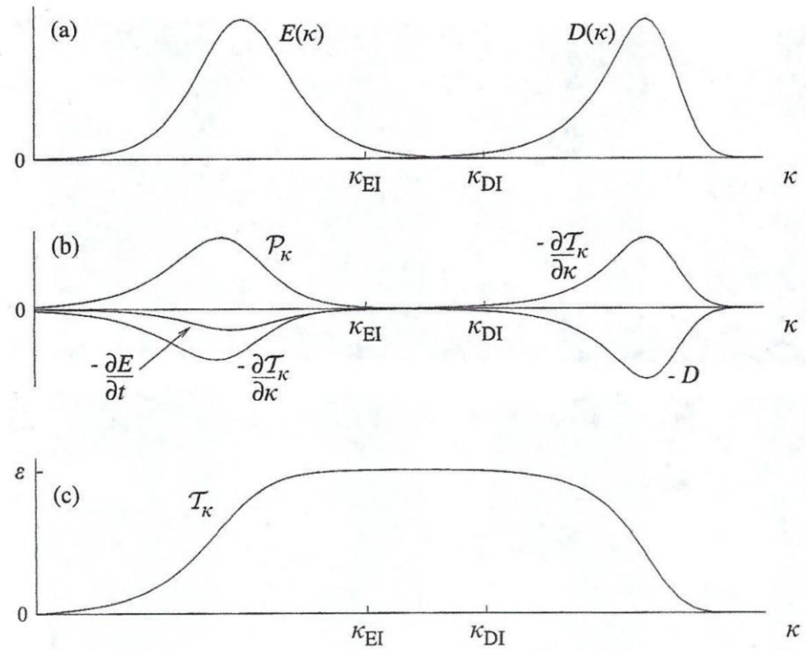


Fig. 6.28. For homogeneous turbulence at very high Reynolds number, sketches of (a) the energy and dissipation spectra, (b) the contributions to the balance equation for $E(\kappa, t)$ (Eq. (6.284)), and (c) the spectral energy-transfer rate.

1) For $\kappa < \kappa_{EI}$ in energy-containing range all terms significant except for dissipation. Assuming $k_{(0,\kappa_{EI})} \approx k$, $\varepsilon_{(0,\kappa_{EI})} \approx 0$ and $P_{(0,\kappa_{EI})} \approx P$, integration of Eq. (1) over the energy-containing range gives

$$\frac{dk}{dt} \approx P - T_{EI} \quad (2)$$

Where $T_{EI} = T_k(\kappa_{EI})$. Energy is generated by P and transferred to T_{EI}.

2) In the inertial subrange, $\kappa_{EI} < \kappa < \kappa_{DI}$, and spectral transfer only significant process so that integration of Eq. (1) from κ_{EI} to κ_{DI} gives:

$$0 \approx T_{EI} - T_{DI} \quad (3)$$

Where $T_{DI} = T_k(\kappa_{DI})$. Energy cascades without change in the inertial subrange.

3) In the dissipation range $\kappa > \kappa_{DI}$, integration of Eq. (1) from κ_{DI} to ∞ gives:

$$0 \approx T_{DI} - \varepsilon \quad (4)$$

Energy dissipates such that $T_{DI} = \varepsilon$.

Eqs. (2), (3) and (4) highlight the essential characteristics of the energy cascade and adding them together gives:

$$\frac{dk}{dt} = P - \varepsilon$$

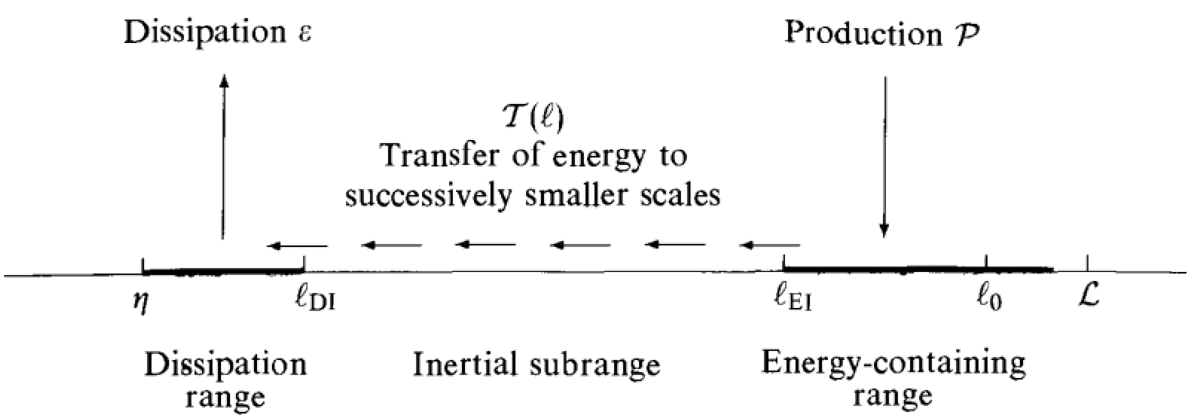


Fig. 6.2. A schematic diagram of the energy cascade at very high Reynolds number.

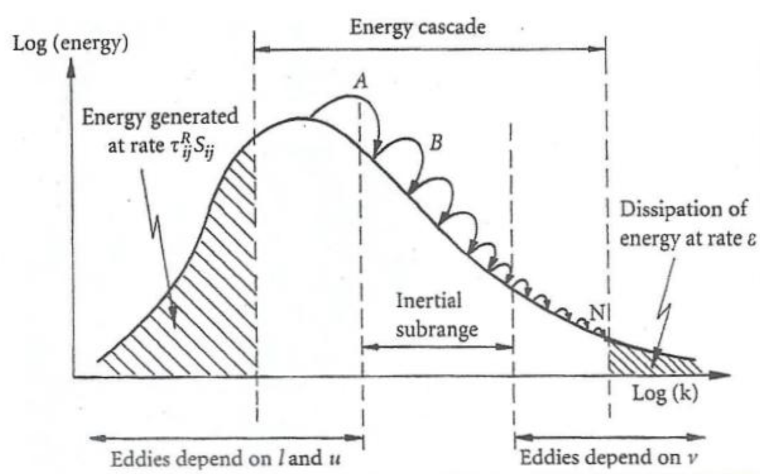


Figure 5.3 Schematic representation of the energy cascade.

The cascade timescale

Flow of energy in inertial subrange is analogous to incompressible continuity equation

$$A_1 V_1 = A_2 V_2$$

Through a variable area stream tube.

$T_{EI} = \frac{u_0^3}{l_0} = \left[\frac{m^2}{s^3}\right] = [\frac{m^2}{s^2}/s] =$ rate of change of energy, analogue to $Q = AV =$ flowrate = constant and A analogue to $E(\kappa) =$ energy per wave number = m^2/s^2/1/m = $\left[\frac{m^3}{s^2}\right]$. So, the speed (in units of wave number per time $[1/ms]$) at which energy travels through the cascade is:

$$\dot{\kappa}(\kappa) = \frac{T_{EI}}{E(\kappa)} = \frac{\varepsilon}{C\kappa^{-5/3}\varepsilon^{2/3}} = \frac{\kappa^{5/3}\varepsilon^{1/3}}{C}$$

And it can be noted that this speed increases rapidly with increasing κ.

It follows from the solution of (with speed in units of wave number per unit time):

$$\frac{d\kappa}{dt} = \dot{\kappa}$$

Integrated from κ_a to κ_b

$$\int_{\kappa_a}^{\kappa_b} \frac{C}{\kappa^{5/3}\varepsilon^{1/3}} d\kappa = t_{(\kappa_a,\kappa_b)}$$

Thus, the time needed for the energy to flow from κ_a to κ_b is:

$$t_{(\kappa_a,\kappa_b)} = \frac{3}{2} C\varepsilon^{-1/3}(\kappa_a^{-2/3} - \kappa_b^{-2/3})$$

And substituting for $\varepsilon^{-1/3}$:

$$\varepsilon^{-1/3} = \frac{1}{\left(\frac{u^3}{L}\right)^{1/3}} = \frac{L^{1/3}}{u} = \frac{L}{u}L^{-2/3} = \tau L^{-2/3}$$

$$t_{(\kappa_a,\kappa_b)} = \tau\frac{3}{2}C((\kappa_a L)^{-2/3} - (\kappa_b L)^{-2/3}) \qquad (5)$$

Using the relations

$$\kappa_{EI} = \frac{2\pi}{l_{EI}}, \qquad l_{EI} = \frac{1}{6}L_{11}, \qquad \frac{L_{11}}{L} \approx 0.4$$

Into Eq. (5) gives

$$\begin{aligned} t_{(\kappa_{EI},\infty)} &= \tau\frac{3}{2}C\left((\kappa_{EI}L)^{-\frac{2}{3}} - (\infty L)^{-\frac{2}{3}}\right) \\ &= \tau\frac{3}{2}C(\kappa_{EI}L)^{-2/3} \\ &\approx \tau\frac{3}{2}C\left(\frac{2\pi}{l_{EI}}\frac{L_{11}}{0.4}\right)^{-\frac{2}{3}} \\ &\approx \tau\frac{3}{2}C\left(\frac{12\pi}{L_{11}}\frac{L_{11}}{0.4}\right)^{-\frac{2}{3}} \\ &\approx \tau\frac{3}{2}C\left(\frac{12\pi}{0.4}\right)^{-\frac{2}{3}} \\ &\approx 0.0725\tau C \end{aligned}$$

Substituting C~1.5 = Kolmogorov universal constant

$$t_{(\kappa_{EI},\infty)} \approx 0.109\tau$$

$$t_{(\kappa_{EI},\infty)} \approx \frac{1}{10}\tau = \frac{1}{10}\frac{k}{\varepsilon}$$

i.e., lifetime of energy from $t_{EI} \rightarrow \infty = 1/10$ total lifetime, i.e., 90% of the energy lifetime $\frac{k}{\varepsilon}$ is in the energy containing range.

Spectral energy-transfer models

For $\kappa > \kappa_{EI}$,

$$0 = -\frac{d}{d\kappa}T_k(\kappa) - 2\nu\kappa^2 E(\kappa) \quad (6)$$

From 1940-1970 many models for $T_k(\kappa)$ to obtain $E(\kappa)$ using Eq. (6) → most of these models are non-local due to interaction of wave number triads in energy transfer.

Simplest local model Pao (1965):

$$\dot{k} \equiv \frac{T_k(\kappa)}{E(\kappa)} = f(\varepsilon, \kappa)$$

Using dimensional analysis

$$T_k(\kappa) = \dot{k}E(\kappa) = E(\kappa)\alpha^{-1}\varepsilon^{1/3}\kappa^{5/3}$$

Where $\alpha =$ constant.

Substituting this expression into Eq. (6) and integrating gives (Pope Ex. 6.36)

$$E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3}\exp[-\frac{3}{2}C(\kappa\eta)^{4/3}]$$

i.e., Pao energy spectrum for the dissipation range.

Appendix A

A.1

$$t^* = St$$

$$k^*(t^*) = k\left(\frac{t^*}{S}\right)$$

Similarly

$$\varepsilon^*(t^*) = \varepsilon\left(\frac{t^*}{S}\right)$$

Evolution equation for ε:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon_1} P \frac{\varepsilon}{k} + C_{\varepsilon_3} R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k} - C_{\varepsilon_2} \frac{\varepsilon^2}{k}$$

Written in non-dimensional form and assuming $C_{\varepsilon_3} = 0$:

$$S\frac{d\varepsilon^*}{dt^*} = C_{\varepsilon_1} P \frac{\varepsilon^*}{k^*} - C_{\varepsilon_2} \frac{(\varepsilon^*)^2}{k^*} \quad (1A)$$

Using

$$\frac{Sk}{\varepsilon} \approx 6$$

And

$$\frac{P}{\varepsilon} \approx 1.8$$

Eq. (1A) becomes:

$$\frac{d\varepsilon^*}{dt^*} = \left(C_{\varepsilon_1}\left(\frac{P}{\varepsilon^*}\frac{\varepsilon^*}{Sk^*}\right) - C_{\varepsilon_2}\left(\frac{\varepsilon^*}{Sk^*}\right)\right)\varepsilon^*$$

And substituting $C_{\varepsilon_1} = 1.45$ and $C_{\varepsilon_2} = 1.9$:

$$\frac{d\varepsilon^*}{dt^*} = \left(C_{\varepsilon_1}\frac{1.8}{6} - \frac{C_{\varepsilon_2}}{6}\right)\varepsilon^* = 0.1183\varepsilon^* \rightarrow \varepsilon^* = e^{0.1183t^*}$$

A.2

3.7 The Reynolds Stress Equations

The mean flow equations (2.25) contain the gradients of the Reynolds stresses, $\frac{\partial}{\partial x_j}\overline{u_i'u_j'}$, which require closure modeling. In addition, we saw earlier in this chapter that the evolution equations for the mean and turbulent kinetic energies (Equations (3.8) and (3.15), respectively) include the Reynolds stresses, which enter as part of the TKE production term. So far, we have used an eddy viscosity model to account for the effect of the Reynolds stresses on the mean flow. We now derive the actual governing equations for the Reynolds stresses to get further insights into how all the components of the Reynolds stresses are produced, transported, and dissipated.

The evolution equation for the fluctuating velocity, u_i', can be obtained by subtracting the mean momentum equations (3.1) from the Navier–Stokes equations for the total momentum:

$$\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_k}(u_i'u_k' + u_i'U_k + U_i u_k') = -\frac{\partial p'}{\partial x_i} + \cdots .$$

Summing the product of u_j' and the above evolution equation for u_i' with the product of u_i' and the evolution equation for u_j', and then taking the appropriate mean, gives us the evolution equation for the Reynolds stresses:

$$\frac{\partial\overline{u_i'u_j'}}{\partial t} + U_k\frac{\partial\overline{u_i'u_j'}}{\partial x_k} = -\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k} - \frac{\partial\overline{u_i'u_j'u_k'}}{\partial x_k}$$
$$- \frac{1}{\rho}\left[\overline{u_i'\frac{\partial p'}{\partial x_j}} + \overline{u_j'\frac{\partial p'}{\partial x_i}}\right] + \nu\left[\overline{u_i'\frac{\partial^2 u_j'}{\partial x_k\partial x_k}} + \overline{u_j'\frac{\partial^2 u_i'}{\partial x_k\partial x_k}}\right]. \qquad (3.32)$$

The velocity–pressure gradient term can be written as

$$\frac{\partial}{\partial x_j}\overline{p'u_i'} - \overline{p'\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_i}\overline{p'u_j'} - \overline{p'\frac{\partial u_j'}{\partial x_i}} = \frac{\partial}{\partial x_j}\overline{p'u_i'} + \frac{\partial}{\partial x_i}\overline{p'u_j'} - 2\overline{p's_{ij}'}.$$

The first two terms on the right-hand side are pressure–diffusion terms, while the last term is called the **pressure–strain correlation**, which plays an important role in the exchange of energy between turbulence intensities (see Sections 3.7.1 and 3.7.2). The viscous term can be written as

$$-2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \nu\frac{\partial^2\overline{u_i'u_j'}}{\partial x_k\partial x_k}.$$

Finally, using the continuity equation, the convection term can be written as $U_k\frac{\partial}{\partial x_k}\overline{u_i'u_j'} = \frac{\partial}{\partial x_k}\overline{u_i'u_j'}U_k$. These rearrangements of the velocity–pressure gradient, convection, and viscous terms render further simplification of the Reynolds stress equations in homogeneous flows, which will be taken up next.

3.7.1 Homogeneous Shear Flow and Pressure–Strain Correlations

In spatially homogeneous turbulent flows, the spatial derivatives of all turbulence statistics are zero, and the Reynolds stress equations become

$$\frac{\partial}{\partial t}\overline{u_i'u_j'} = -\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k} + \overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}. \quad (3.33)$$

For homogeneous shear flow (Section 2.2.3), $U_1 = Sx_2$, while $U_2 = U_3 = 0$. This yields the following Reynolds stress equations:

$$\frac{\partial}{\partial t}\overline{u_1'^2}/2 = -\overline{u_1'u_2'}S + \overline{\frac{p'}{\rho}\frac{\partial u_1'}{\partial x_1}} - \nu\overline{\frac{\partial u_1'}{\partial x_k}\frac{\partial u_1'}{\partial x_k}}, \quad (3.34)$$

$$\frac{\partial}{\partial t}\overline{u_2'^2}/2 = \overline{\frac{p'}{\rho}\frac{\partial u_2'}{\partial x_2}} - \nu\overline{\frac{\partial u_2'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}}, \quad (3.35)$$

$$\frac{\partial}{\partial t}\overline{u_3'^2}/2 = \overline{\frac{p'}{\rho}\frac{\partial u_3'}{\partial x_3}} - \nu\overline{\frac{\partial u_3'}{\partial x_k}\frac{\partial u_3'}{\partial x_k}}, \quad (3.36)$$

$$\frac{\partial}{\partial t}\overline{u_1'u_2'} = -\overline{u_2'^2}S + \overline{\frac{p'}{\rho}\left(\frac{\partial u_1'}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}\right)} - 2\nu\overline{\frac{\partial u_1'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}}. \quad (3.37)$$

Note that the equation for $\overline{u_1'^2}$ has a production term $-\overline{u_1'u_2'}S$, but the equations for $\overline{u_2'^2}$ and $\overline{u_3'^2}$ do not have production terms. Since the viscous terms are negative definite, $\overline{u_2'^2}$ and $\overline{u_3'^2}$ can only be maintained through the pressure–strain terms. In addition,

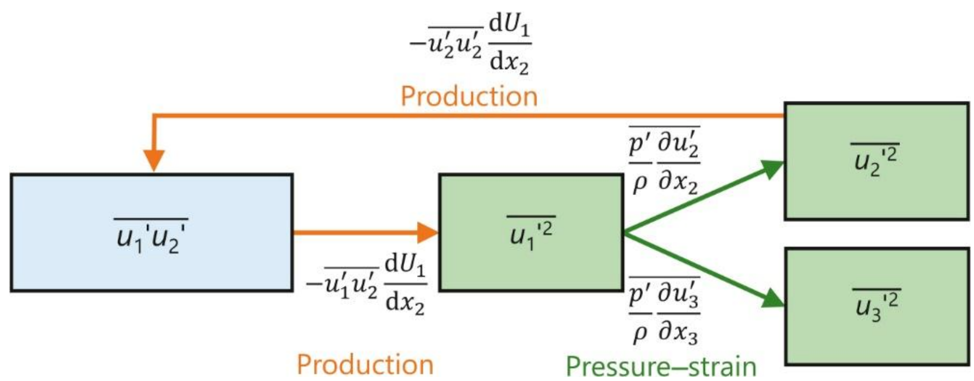
$$\overline{p'\frac{\partial u_1'}{\partial x_1}} + \overline{p'\frac{\partial u_2'}{\partial x_2}} + \overline{p'\frac{\partial u_3'}{\partial x_3}} = \overline{p'\frac{\partial u_i'}{\partial x_i}} = 0.$$

For $\overline{u_2'^2}$ and $\overline{u_3'^2}$ not to decay, $\overline{p'\frac{\partial u_2'}{\partial x_2}}$ and $\overline{p'\frac{\partial u_3'}{\partial x_3}}$ must be positive, implying that $\overline{p'\frac{\partial u_1'}{\partial x_1}} < 0$. Since only the $\overline{u_1'^2}$ equation has a production term, $\overline{p'\frac{\partial u_1'}{\partial x_1}}$ must be the conduit for energy transfer from $\overline{u_1'^2}$ to $\overline{u_2'^2}$ and $\overline{u_3'^2}$. Finally, note that $\overline{u_1'u_2'} < 0$ has a “production” term $(-\overline{u_2'^2}S)$ that is negative definite.

The pressure–strain terms are crucial in the intercomponent transfer of energy and the transport of Reynolds stresses, and will have to be carefully treated in turbulence closure models for the Reynolds stresses. They maintain an intricate balance that is critical to maintaining turbulence.

For example, if insufficient energy is transferred through the pressure–strain term to $\overline{u_2'^2}$ to overcome viscous dissipation in its governing equation, then $\overline{u_2'^2}$ would decay. This, in turn, reduces production in the $\overline{u_1' u_2'}$ equation, which diminishes production of $\overline{u_1'^2}$, eventually leading to the complete decay of turbulent fluctuations. This balance between the turbulence intensities is illustrated in Figure 3.6.

Fig. 3.6 Schematic illustrating the intercomponent transfer of TKE through the Reynolds shear stress in a homogeneous shear flow.



$$\frac{\partial^2 (u_i u_j)}{\partial x_k \partial x_k} = \frac{\partial}{\partial x_k}\left[\frac{\partial (u_i u_j)}{\partial x_k}\right]$$

$$= \frac{\partial}{\partial x_k}\left(u_i \frac{\partial u_j}{\partial x_k} + u_j \frac{\partial u_i}{\partial x_k}\right)$$

$$= u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} + \frac{\partial u_i \partial u_j}{\partial x_k \partial x_k} + \frac{\partial u_j}{\partial x_k}\frac{\partial u_i}{\partial x_k} + u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k}$$

$$= u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} + u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k} + 2 \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}$$

$$\Rightarrow u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} + u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{\partial^2 (u_i u_j)}{\partial x_k \partial x_k} - 2 \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}$$

A.3

$$\frac{dk}{dt} = C_\mu \frac{k^2}{\varepsilon} S^2 - \varepsilon = 0$$
$$\frac{d\varepsilon}{dt} = C_{\varepsilon_1} C_\mu k S^2 + C_{\varepsilon_3} R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k} - C_{\varepsilon_2} \frac{\varepsilon^2}{k} = 0$$

From the k equation:

$$\varepsilon \;=\; C_\mu \;\frac{k^2}{\varepsilon} S^2$$

$$\varepsilon^2 \;=\; C_\mu \;k^2 S^2$$

$$\varepsilon \;=\; \sqrt{C_\mu}\; kS$$

From the ε equation:

$$C_{\varepsilon_1} C_\mu k S^2 + C_{\varepsilon_3} R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k} - C_{\varepsilon_2} \frac{\varepsilon^2}{k} = 0$$

Where:

$$R_T = \frac{k^2}{\nu\varepsilon}$$

Such that:

$$C_{\varepsilon_1} C_\mu k S^2 + C_{\varepsilon_3} \frac{k}{\sqrt{\nu\varepsilon}} \frac{\varepsilon^2}{k} - C_{\varepsilon_2} \frac{\varepsilon^2}{k} = 0$$

$$C_{\varepsilon_1} C_\mu k S^2 + C_{\varepsilon_3} \frac{\varepsilon^{3/2}}{\sqrt{\nu}} - C_{\varepsilon_2} \frac{\varepsilon^2}{k} = 0$$

Substitute $\varepsilon \;=\; \sqrt{C_\mu}\; kS$:

$$C_{\varepsilon_1} C_\mu k S^2 + C_{\varepsilon_3} \frac{C_\mu^{3/4}\; k^{3/2}\, S^{3/2}}{\sqrt{\nu}} - C_{\varepsilon_2} \frac{C_\mu\; k^2 S^2}{k} = 0$$

$$\left(C_{\varepsilon_1} - C_{\varepsilon_2}\right) C_\mu k S^2 + C_{\varepsilon_3} \frac{C_\mu^{3/4} \ k^{3/2} \ S^{3/2}}{\sqrt{\nu}} = 0$$

$$C_{\varepsilon_3} \frac{C_\mu^{3/4} \ k^{3/2} \ S^{3/2}}{\sqrt{\nu}} = \left(-C_{\varepsilon_1} + C_{\varepsilon_2}\right) C_\mu k S^2$$

$$k^{1/2} = \frac{\left(-C_{\varepsilon_1} + C_{\varepsilon_2}\right) C_\mu S^2 \sqrt{\nu}}{C_\mu^{3/4} \ C_{\varepsilon_3} \ S^{3/2}} = \frac{\left(-C_{\varepsilon_1} + C_{\varepsilon_2}\right) C_\mu^{1/4} S^{1/2} \sqrt{\nu}}{C_{\varepsilon_3}}$$

$$k = \frac{\left(C_{\varepsilon_2} - C_{\varepsilon_1}\right)^2 \sqrt{C_\mu}}{C_{\varepsilon_3}^2} S \nu$$

And

$$\varepsilon = \sqrt{C_\mu} k S = \frac{\left(C_{\varepsilon_2} - C_{\varepsilon_1}\right)^2 C_\mu}{C_{\varepsilon_3}^2} S^2 \nu$$

A.4

Implications for Turbulence Modeling

Part 1 Eq. (1) and (2a,b):

$$\frac{dk}{dt} = -\varepsilon$$

$$\frac{d\varepsilon}{dt} = P_\varepsilon^4 - \Upsilon_\varepsilon = \nu P_\zeta^4 - \nu \Upsilon_\zeta = \left(S_k^* R_T^{\frac{1}{2}} - G^*\right)\frac{\varepsilon^2}{k}$$

For general applications, it is assumed that,

$$S_k^* R_T^{\frac{1}{2}} - G^* = -C_{\varepsilon_2} \quad (20)$$

Compare high Re equilibrium solution

$$\frac{d\varepsilon}{dt} = -2\frac{\varepsilon^2}{k} \quad (19)$$

$$R_{T\infty}^* = \left(\frac{G_0^* - 2}{S_{k_0}^*}\right)^2 \quad (7)$$

Where C_{ε_2} is constant. This leads to the model,

$$P_\varepsilon^4 - \Upsilon_\varepsilon = \left(S_k^* R_T^{\frac{1}{2}} - G^*\right)\frac{\varepsilon^2}{k} = -C_{\varepsilon_2}\frac{\varepsilon^2}{k}$$

Which yields an equation identical to Eq. (19), i.e., high R_E equilibrium, except that the constant on the RHS is $-C_{\varepsilon_2}$.

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon_2}\frac{\varepsilon^2}{k}$$

Further insight provided by fact that for high Re we have shown that:

$$G^* \sim \sqrt{R_T}$$

which is consistent with and justifies/explains Eq. (20), i.e., cancels the vortex stretching term. Recall again for isotropic turbulence,

$$\frac{d\varepsilon}{dt} = S_k^* R_T^{1/2}\frac{\varepsilon^2}{k} - G^*\frac{\varepsilon^2}{k}$$

$$= \left(S_k^* R_T^{1/2} - G^*\right)\frac{\varepsilon^2}{k}$$

$$= \left[S_k^* R_T^{1/2} - \left(S_k^* R_T^{1/2} + C_{\varepsilon 2}\right)\right]\frac{\varepsilon^2}{k} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

i.e., coefficient $-C_{\varepsilon_2}$ chosen to cancel vortex stretching term, which is the assumption in Eq. (20) is equivalent to imposing equilibrium structure on the turbulent decay process, which imposes a decay law of the form,

$$k \sim t^{-\frac{1}{C_{\varepsilon_2}-1}}$$

i.e., C_{ε_2} sets the decay rate. e.g., for $C_{\varepsilon_2} = \frac{11}{6}(1.83)$, Saffman $t^{-6/5}$ law recovered.

Other values can be achieved via specification of C_{ε_2}. In all cases, without vortex stretching. If vortex stretching is included, then eventually t^{-1} decay law will develop.

$C_{\varepsilon_2} = 1.40$, n=2.50: Final decay low Re similarity. No vortex stretching.

$C_{\varepsilon_2} = 1.50$, n=2.00: 0th law turbulence

$C_{\varepsilon_2} = 1.70$, n=1.43: Kolmogorov decay law

$C_{\varepsilon_2} = 1.83$, n=1.20: Saffman decay law

$C_{\varepsilon_2} = 1.92$, n = 1.09 EFD grid turbulence decay rate (value used k-ε turbulence model) A.4.

$C_{\varepsilon_2} = 2.00$, n=1.00: Final decay high Re similarity. Vortex stretching ~ dissipation with slight edge towards dissipation.

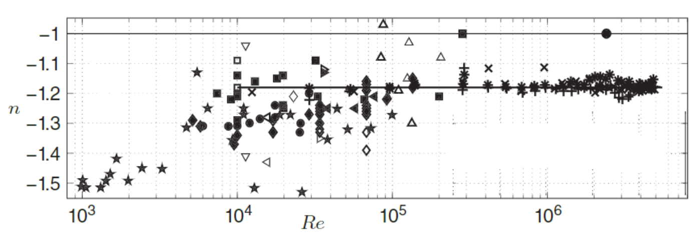


Figure 5.6 Measured power law exponents in decaying homogeneous turbulence from numerous experiments [9]. Filled symbols represent traditional turbulence behind a grid of bars. Open symbols are other turbulence sources. The symbols with ×,*, and + are from three different probes used in decaying turbulence behind a grid of bars for which the Reynolds number was changed only by altering the viscosity of the working fluid. Used by permission of AIP.

High Re Equilibrium: fixed point solution

For $G_0^* > 2$ and $R_{T_\infty}^* \neq 0$, R_{T_0} approaches $R_{T_\infty}^*$ from above or below depending on whether R_{T_0} is < or > $R_{T_\infty}^*$.

$$\frac{d\varepsilon}{dt} = S_{k_0}^* R_T^{1/2} \frac{\varepsilon^2}{k} - G_0^* \frac{\varepsilon^2}{k} \qquad (18) = (3b)$$

But

$$S_{k_0}^* = \frac{(G_0^* - 2)}{\sqrt{R_{T_\infty}^*}}$$

$$R_{T_\infty}^* = \left(\frac{G_0^* - 2}{S_{k_0}^*}\right)^2 \qquad (7)$$

Therefore, as R_T approaches $R_{T_\infty}^*$:

$$\frac{d\varepsilon}{dt} = \underbrace{(G_0^* - 2)\frac{\varepsilon^2}{k}}_{\text{Vortex stretching}} - \underbrace{G_0^* \frac{\varepsilon^2}{k}}_{\text{Dissipation of } \varepsilon}$$

And thus

$$\frac{d\varepsilon}{dt} = -2\frac{\varepsilon^2}{k} \qquad (19)$$

According to Eq. (7), very large $R_{T_\infty}^*$ requires large G_0^*, assuming EFD $S_{k_0}^*$ (≈ 0.3) values. $R_{T_\infty}^* = 10^4$ gives $G_0^* = 25$, which gives near balance between vortex stretching and dissipation of ε terms in $\frac{d\varepsilon}{dt}$ equation which is an important result → if $G_0^* \gg 2 \Rightarrow G_0^* - 2 \approx G_0^*$; thus, vortex stretching ~ dissipation with slight edge towards dissipation resulting in Eq. (19) with net dissipation coefficient -2, i.e., for high Re equilibrium (fixed-point) decay has nearly equal contributions vortex stretching and dissipation such that dissipation rate (per unit $\frac{\varepsilon^2}{k}$) assumes universal value, which is independent of G_0^*. Note that vortex stretching only implicitly included in $\frac{d\varepsilon}{dt}$ equation.