Chapter 6: Turbulent Transport and its Modeling

Part 1: Molecular Momentum Transport

$\rho\overline{u_i u_j}$ = turbulent momentum flux

$\rho\overline{uv} = x$ momentum ρu in y direction due to turbulent v, for $\underline{V} = (U + u, v, w)$

Classical ideas for modeling turbulent transport were based on molecular momentum transport for ideal (non-dense) gas: molecules far apart and inter-molecular forces are weak. Molecules in free flight with brief collisions at which time their direction and speed change.

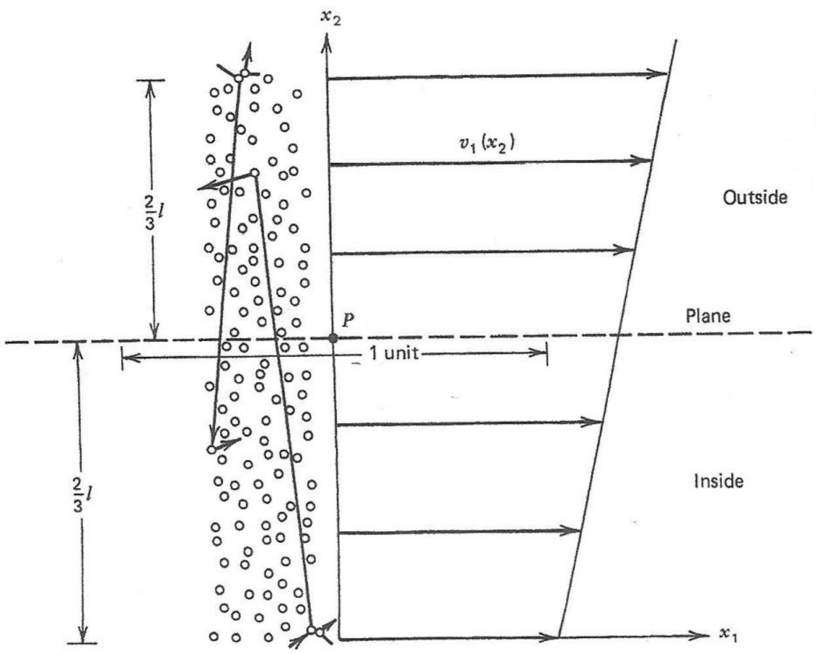


Figure 6.1 Molecular model of the viscosity of a gas.

Across plane separating the gas in two regions, the molecules do not attract or repel each other (contrasting to liquids); therefore, the primary source of shear stress is that due to microscopic transport of momentum due to random molecular motions.

Newtonian fluid stress rate of strain relationship

$$\tau_{21} = \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = \tau_{12}$$

$$= \mu\frac{\partial v_1}{\partial x_2} \qquad \text{For } \underline{V} = v_1(x_2)\hat{e}_1$$

Which can be derived for an ideal gas using four facts from kinetic theory:

1) Molecules that cross the plane $x_2 =$ constant begin their free flight on average at distance $\pm\frac{2}{3}l$ ($l =$ mean free path) from the plane.

2) Mean free path $f(d, n)$:

$$l = 1/(\sqrt{2}\pi d^2 n)$$

$d =$ molecular diameter $n =$ number density of molecules per unit volume

Since molecules have a distribution of speeds and they are moving relative to each other, $RHS = \frac{3}{4}RHS$ or if assume Maxwellian distribution of velocities RHS = 0.707·RHS.

3) Flux of velocities across x_2-plane per unit area $= \frac{1}{4}n\overline{v}$, where $\overline{v} =$ average molecular speed (without regard direction).

4) Average molecular speed

$$\overline{v} = \left[\frac{8kT}{\pi m}\right]^{1/2} = f(T)$$

$k =$ Boltzmann constant $m =$ molecular mass

Shear stress x_2-plane

$$\tau_{21} = \frac{x_1 \text{ force}}{\text{unit area}} = \text{net flux momentum across } x_2 \text{ plane}$$

x-momentum of one particle from above:

$$mv_1|_{x_2+\frac{2}{3}l} = m\left[v_1 + \frac{dv_1}{dx_2}\left(\frac{2}{3}l\right) + \cdots\right]_{x_2}$$

x-momentum of one particle from below:

$$mv_1|_{x_2-\frac{2}{3}l} = m\left[v_1 + \frac{dv_1}{dx_2}\left(-\frac{2}{3}l\right) + \cdots\right]_{x_2}$$

Net flux = difference between the momentum of a particle from above minus a particle from below times the rate that the particles cross a unit area $\frac{1}{4}n\bar{v}$:

$$\tau_{21} = \frac{1}{4}n\bar{v} \times \left[mv_1|_{x_2+\frac{2}{3}l} - mv_1|_{x_2-\frac{2}{3}l}\right]$$

$$= \frac{1}{4}n\bar{v} \times m\frac{4}{3}l\frac{dv_1}{dx_2}$$

i.e., comparing with $\tau_{ij} = \mu(v_{i,j} + v_{j,i})$

$$\mu = \frac{1}{3}n\bar{v}ml = \frac{2}{3d^2}\left[\frac{mkT}{\pi^3}\right]^{1/2} = \frac{1}{3}\rho\bar{v}l$$

$$= f(\text{molecular properties and T}) = f(\bar{v}, l)$$

$\mu \uparrow$ m $\uparrow$

$\mu \downarrow$ d $\uparrow$

$\mu \neq f(p)$

$\mu \uparrow \sqrt{T}\uparrow$

More complete theory includes inter molecular forces and better agreement T dependence.

Viscous liquids need more advanced models considering intermolecular forces but results in same $\tau_{ij} = \mu(v_{i,j} + v_{j,i})$ relationship.

Modelling Turbulent Transport by Analogy to Molecular Transport

Newtonian fluids (incompressible flow) NS

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij}$$
$$= -p\delta_{ij} + \tau_{ij}$$

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
μ = isotropic viscosity
= property of the fluid

vs. RANS

$$\overline{\sigma_{ij}} = -\overline{p}\delta_{ij} + \mu\left(\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial \overline{U_j}}{\partial x_i}\right) - \rho\overline{u_i u_j}$$

$$\overline{\sigma_{ij}} = -\overline{p}\delta_{ij} + \mu\left(\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial \overline{U_j}}{\partial x_i}\right) - \rho(\frac{2}{3}k\delta_{ij} + a_{ij})$$

$$\rho\frac{D\overline{U_i}}{Dt} = -\rho g\delta_{i3} - \frac{\partial}{\partial x_i}\left(\overline{p} + \frac{2}{3}\rho k\right) + \frac{\partial}{\partial x_j}[\mu\left(\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial \overline{U_j}}{\partial x_i}\right) - \rho a_{ij}]$$

In analogy the turbulent Reynolds stresses are modeled using the eddy viscosity concept

$$a_{ij} = \overline{u_i u_j} - \frac{2}{3}k\delta_{ij} = -\nu_t\underbrace{\left(\overline{U_{i,j}} + \overline{U_{j,i}}\right)} = -2\nu_t S_{ij}$$

Mean flow rate of strain

Anisotropic RS is modeled using isotropic eddy viscosity ν_t or $\mu_t = \rho\nu_t$, which may be contrasted with μ definition for ideal gas; however, no reason to believe turbulent motions are without directional biases that are not aligned with S_{ij}. Nonetheless eddy viscosity concept forms the basis of traditional RANS modeling, which focuses on modeling of ν_t.

EXAMPLE 4.8

Write out all the components of the stress tensor $\mathbf{T}$ in (x, y, z)-coordinates in terms of $\mathbf{u} = (u, v, w)$, and its derivatives.

Solution

Evaluate each component of (4.36) and abbreviate $S_{mm} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = \nabla\cdot\mathbf{u}$ to find:

$$\mathbf{T} = \begin{bmatrix} -p + 2\mu\dfrac{\partial u}{\partial x} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & -p + 2\mu\dfrac{\partial v}{\partial y} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & -p + 2\mu\dfrac{\partial w}{\partial z} + \left(\mu_v - \dfrac{2}{3}\mu\right)\nabla\cdot\mathbf{u} \end{bmatrix}$$

1D Shear flow: $\underline{V} = (\bar{U} + u, v, w) = \underline{V}(y)$

$$2S_{ij} = \begin{bmatrix} 0 & \bar{U}_y & 0 \\ \bar{U}_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S_{ij} = (\bar{U}_{i,j} + \bar{U}_{j,i})$$

$$\tilde{p} = \bar{p} + \tfrac{2}{3}k$$

$$a_{ij} = -2\nu_t S_{ij}$$

$$\bar{U}_t + \bar{U}\bar{U}_x + \bar{V}\bar{U}_y + \bar{W}\bar{U}_z = -\rho^{-1}\tilde{p}_x + \nu[\bar{U}_{xx} + \bar{U}_{yy} + \bar{U}_{zz}] - a_{ij}$$

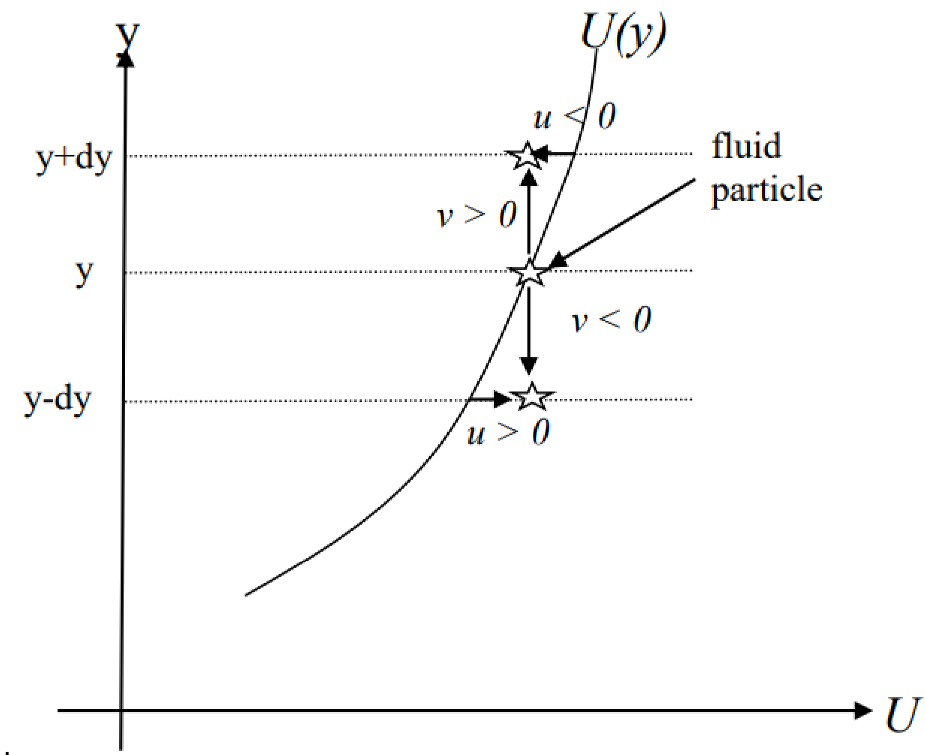
$$0 = -\rho^{-1}\tilde{p}_x + \nu\bar{U}_{yy} + \nu_t\bar{U}_{yy}$$

$$= -\rho^{-1}\tilde{p}_x + (\nu + \nu_t)\bar{U}_{yy}$$

For example, consider 1D shear flows (free shear flows, channel/pipe, and BL) for which most import RS is:

$$\frac{\tau_{12}}{\rho} = -\overline{uv} = \nu_t\frac{d\bar{U}}{dy}$$

Large scale turbulent eddies are most important in transporting momentum across the flow, which are mostly driven by inertia and pressure forces vs. viscosity. Assume $-\rho\overline{uv}$ due to turbulent eddies with transverse size l and intensity characterized by velocity scale u_0.



The fluid velocity is: $\underline{V}(y) = (U + u, v, w)$ with $\frac{dU}{dy} > 0$

If fluid particle retains its total velocity $\underline{V}$ from y to $y \pm dy$ gives, $U + u =$ constant → If U increases, u decreases and vice versa.

$v > 0 \quad \rightarrow \quad u < 0$
$v < 0 \quad \rightarrow \quad u > 0$ → $\overline{uv} < 0$

x-momentum tends towards decreasing y as turbulence diffuses gradients and decreases $\frac{dU}{dy}$

x-momentum transport in y direction, i.e., across y = constant AA per unit area

$M_{xy} = \int \rho \tilde{u} \underline{V} \cdot \underline{n}\, dA$, where $\tilde{u} = (U + u)$

$$\frac{d\overline{M_{xy}}}{dA} = \rho\overline{(U + u)v} = \rho U\overline{v} + \rho\overline{uv} = \rho\overline{uv}$$

i.e., $\rho\overline{u_i u_j}$ = average flux of i-momentum in j-direction = $\rho\overline{u_j u_i}$ = average flux of j-momentum in i-direction due to symmetry of the Reynolds stress tensor.

$$-\rho\overline{uv} = f\left(\rho, l, u_0, \frac{d\overline{U}}{dy}\right)$$

$$-\frac{\overline{uv}}{u_0^2} = f\left(\frac{l}{u_0}\frac{d\overline{U}}{dy}\right)$$

Dimensional analysis

Assume linear relationship and eddy viscosity:

$$-\overline{uv} = Cu_0^2 \frac{l}{u_0}\frac{d\overline{U}}{dy} = Clu_0 \frac{d\overline{U}}{dy}$$

i.e.,

$$\nu_t = Clu_0$$

Which is consistent with ideal gas theory: $\mu = \frac{1}{3}\rho\bar{u}l$

u_0 = turbulent velocity scale
l = turbulent length scale
ν_t has units m^2/s same as molecular viscosity.

The time scale for the large-scale turbulent eddy (turnover time):

$$l/u_0$$

And the time scale for the mean flow:

$$\left|\frac{d\overline{U}}{dy}\right|^{-1}$$

Since the turbulence produces the mean flow gradient, it can be assumed their times scales are proportional:

$$\frac{l}{u_0} = \left|\frac{d\overline{U}}{dy}\right|^{-1}, \text{ i.e., } u_0 = l\left|\frac{d\overline{U}}{dy}\right|$$

$$\nu_t = Clu_0 = Cl^2\left|\frac{d\overline{U}}{dy}\right| \quad \mu_t = C_\mu\rho\, l^2\left|\frac{d\overline{U}}{dy}\right|$$

$$-\overline{uv} = Cl^2\left|\frac{du}{dy}\right|\frac{d\overline{U}}{dy}$$

1) Prandtl mixing length theory. l depends on the type of flow.

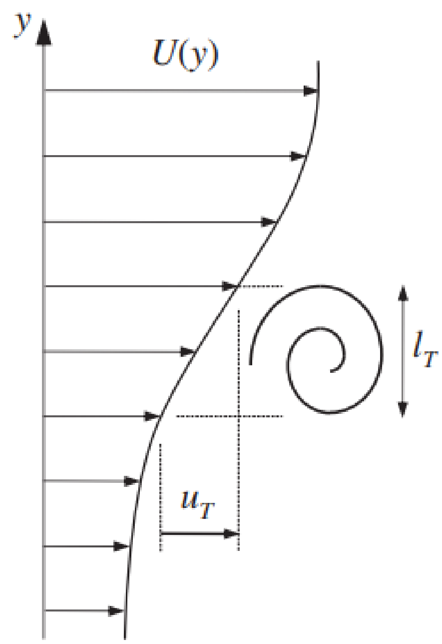


FIGURE 12.20 Schematic drawing of an eddy of size l_T in a shear flow with mean velocity profile $U(y)$. A velocity fluctuation, u or v, that might be produced by this eddy must be of order $l_T(dU/dy)$. Therefore, we expect that the Reynolds shear stress will scale like $\overline{uv} \sim l_T^2 (dU/dy)^2$.

$l \propto$ larger scale eddies

Free shear-flow: $l = c\delta$ where $c = f$(mixing layer, jet, wake) and δ = appropriate width viscous flow

Wall flows (channel/pipe and BL): $l = ky$, i.e., eddy size $\propto y$ near wall

$= c\delta$ away from the wall, δ = BL thickness

2) $k - \varepsilon$ model

$u = \sqrt{k}$ and $l = k^{3/2}/\varepsilon =$ length associated large eddy turnover time l/u

$\mu_t = C_\mu \rho \frac{k^2}{\varepsilon} \rightarrow$ additional equations needed to model k and ε.

Eddy viscosity concept is based on ideal gas molecular transport; thus, assumes:

1) Mixing occurs over well-defined mixing time.
2) Momentum preserved between collisions.
3) Linear velocity variation over the mixing length.

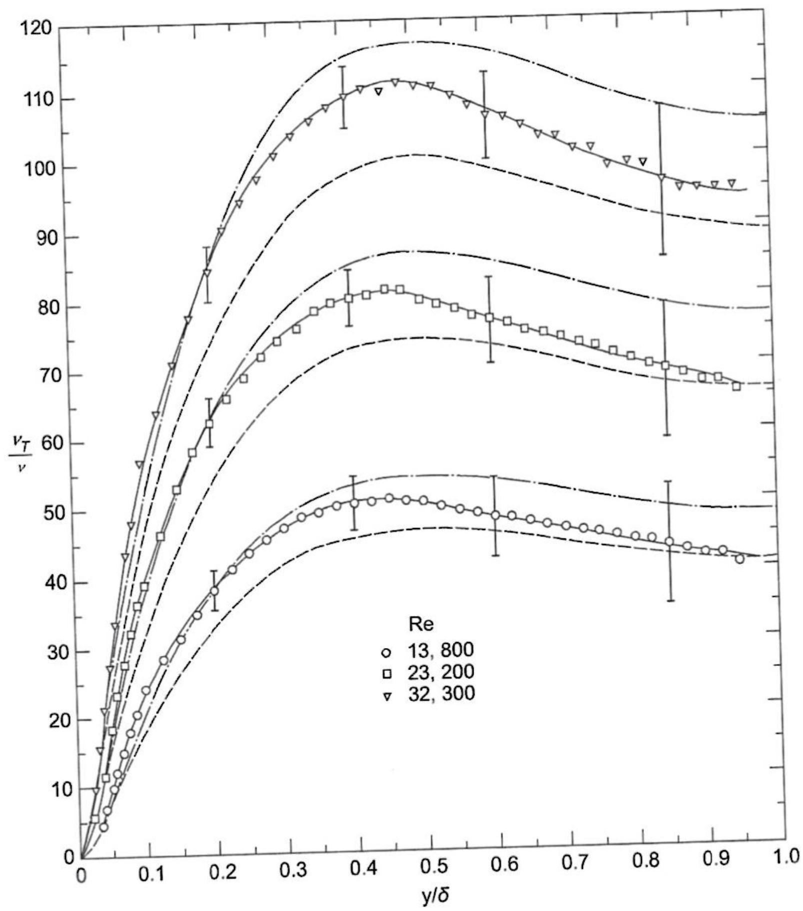


Fig. 2.22 Ratio of eddy viscosity to kinematic viscosity (turbulent Reynolds number, ν_T/ν) as a function of distance from the wall, normalized by the channel half-height, in a turbulent channel flow at several Reynolds numbers. The symbols represent data obtained from experimental measurements, while the different lines indicate estimates from tweaking a parameter in an eddy viscosity model (see (8.6)). (Data reproduced from Hussain and Reynolds (1975), adapted from figure 16(a))

Note that the ratio increases with Re.