Chapter 5: Energy Decay in Isotropic Turbulence

Part 5: Energy Spectrum Equation via Fourier Analysis of the Velocity Field

Transfer physics in analyzed based on Fourier analysis in wave number space of the NS equations, which shows that the energy transfer occurs due to interactions between scales at specific combinations of wave numbers. Whereas previous approach used $\mathcal{R}_{ij}$ and K-H equation leading to $k(r,t)$ and $T(\kappa,t)$ analysis.

Fourier-series representation

The velocity field can be expressed as:

$$\underline{u}(\underline{x},t) = \sum_{\underline{\kappa}} e^{i\underline{\kappa}\cdot\underline{x}}\,\underline{\hat{u}}(\underline{\kappa},t) \quad (1) \qquad \boxed{\underline{\kappa} = 2\pi\underline{n}/L}$$

Where $\underline{n} = (n_1, n_2, n_3)$ and n_i are integers with $-\infty \le n_i \le \infty$.

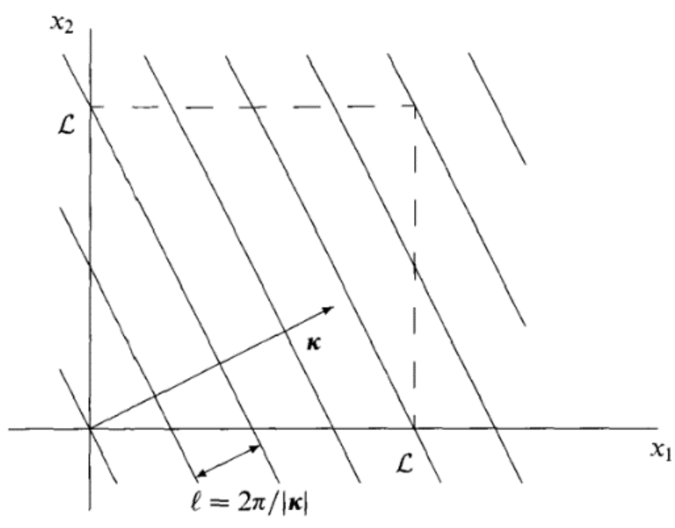


Fig. 6.8. A sketch of the Fourier mode corresponding to $\kappa = \kappa_0(4, 2, 0)$. The oblique lines show the crests, where $\Re(e^{i\kappa\cdot x}) = \cos\kappa\cdot x$ is unity.

The Fourier coefficients of the velocity are:

$$\widehat{u_j}(\underline{k}, t) = \mathcal{F}_k\{u_j(\underline{x}, t)\}$$

$$= \langle u_j(\underline{x}, t), e^{-i\underline{k}\cdot\underline{x}}\rangle_\forall$$

$$= \frac{1}{\forall}\int_\forall u_j(\underline{x}, t)e^{-i\underline{k}\cdot\underline{x}}\underline{dx}$$

Inner product = Volume average

$\forall = L^3$

Where the operator $\mathcal{F}_k\{\ \ \}$ is defined as

$$\mathcal{F}_k\{\, g(\underline{x})\} = \langle g(\underline{x}), e^{-i\underline{k}\cdot\underline{x}}\rangle \quad (2)$$

Note that $e^{i\underline{\kappa}\cdot\underline{x}} =$ constant=1 for $\underline{\kappa}\ \cdot \underline{x} = 0, \rightarrow \underline{\kappa} \perp \underline{x}$

The Fourier modes are orthogonal:

$$\langle e^{i\underline{\kappa}\cdot\underline{x}}, e^{-i\underline{\kappa\prime}\cdot\underline{x}}\rangle = \delta_{\underline{\kappa},\underline{\kappa\prime}} = \begin{cases} 1, & \text{if} \quad \underline{\kappa} = \underline{\kappa}', \\ 0, & \text{if} \quad \underline{\kappa} \neq \underline{\kappa}' \end{cases}$$

Inner product

$$\langle f, g\rangle = \int_\forall f(\underline{x})g^*(\underline{x})\, d\underline{x}$$

Since $\underline{u}(\underline{x}, t)$ is real,

$$\underline{u}(\underline{x}, t) = \underline{u}^*(\underline{x}, t)$$

Where an asterisk denotes the complex conjugate.

Therefore,

$$\underline{u}(\underline{x},t) = \sum_{\underline{\kappa}} e^{i\underline{k}\cdot\underline{x}}\,\underline{\hat{u}}(\underline{\kappa},t) = \sum_{\underline{\kappa}} e^{-i\underline{k}\cdot\underline{x}}\,\underline{\hat{u}}^*(\underline{\kappa},t) = \sum_{\underline{\kappa}} e^{i\underline{k}\cdot\underline{x}}\,\underline{\hat{u}}^*(-\underline{\kappa},t)$$

Since the first and second equalities are true [$\underline{u}(\underline{x},t)$ real] then the last equality follows by substitution $-\underline{\kappa}$ for $\underline{\kappa}$ which is possible since the Fourier series is symmetric about $\underline{\kappa} = 0$ [$\underline{u}(\underline{x},t)$ real] therefore:

$$\sum_{\underline{\kappa}} \left[\underline{\hat{u}}^*(-\underline{\kappa},t) - \underline{\hat{u}}(\underline{\kappa},t)\right] e^{i\underline{k}\cdot\underline{x}} = 0$$

$$\underline{\hat{u}}^*(-\underline{\kappa},t) = \underline{\hat{u}}(\underline{\kappa},t), \text{ i.e., conjugate symmetry}$$

One of the principal reasons for invoking the Fourier representation is the form taken by derivatives. Using Eq. (1) and taking derivative with respect to x_j:

$$\frac{\partial \underline{u}(\underline{x},t)}{\partial x_j} = \frac{\partial}{\partial x_j}\sum_{\underline{\kappa}} e^{i\underline{k}\cdot\underline{x}}\,\underline{\hat{u}}(\underline{\kappa},t)$$

$$\frac{\partial \underline{u}(\underline{x},t)}{\partial x_j} = ik_j \sum_{\underline{\kappa}} e^{i\underline{k}\cdot\underline{x}}\,\underline{\hat{u}}(\underline{\kappa},t) = ik_j \sum_{\underline{\kappa}} e^{-i\underline{k}\cdot\underline{x}}\,\underline{\hat{u}}^*(\underline{\kappa},t)$$

$$\mathcal{F}_k\left\{\frac{\partial \underline{u}(\underline{x},t)}{\partial x_j}\right\} = \langle ik_j \sum_{\underline{\kappa}'} e^{-i\underline{k}'\cdot\underline{x}}\,\underline{\hat{u}}^*(\underline{\kappa}',t), e^{-i\underline{k}\cdot\underline{x}}\rangle$$

Inner product

$$= ik_j \sum_{\underline{\kappa}'} \underline{\hat{u}}^*(\underline{\kappa}',t)\,\langle e^{-i\underline{k}'\cdot\underline{x}}, e^{-i\underline{k}\cdot\underline{x}}\rangle$$

$$\mathcal{F}_k\left\{\frac{\partial \underline{u}(\underline{x},t)}{\partial x_j}\right\} = ik_j \sum_{\underline{\kappa}'} \underline{\hat{u}}^*(\underline{\kappa}',t)\,\delta_{\underline{\kappa},\underline{\kappa}'} = ik_j\underline{\hat{u}}^*(\underline{\kappa},t) = ik_j\underline{\hat{u}}(\underline{\kappa},t) \quad (3)$$

Since $\underline{u}$ is real

$$\underline{\hat{u}}^*(\underline{\kappa},t) = \underline{\hat{u}}(\underline{\kappa},t)$$

$$\underline{u}(\underline{x},t) = \sum_{\underline{\kappa}} \cos(\underline{k}\cdot\underline{x})\,\underline{\hat{u}}(\underline{\kappa},t) = \sum_{\underline{\kappa}} cos(\underline{k}\cdot\underline{x})\,\underline{\hat{u}}^*(\underline{\kappa},t)$$

Differentiation with respect to x_j in physical space corresponds to multiplication by $i\kappa_j$ in wave number space.

The Evolution of Fourier modes

Divergence of velocity in wave number space

$$\mathcal{F}_k\{u_{i,j}\} = i\kappa_j \hat{u}_j = i\underline{\kappa} \cdot \underline{\hat{u}}$$

$$\nabla \cdot \underline{u} = 0 \rightarrow \mathcal{F}_k\left\{\frac{\partial u_i}{\partial x_i}\right\} = ik_i\hat{u}_i = \underline{\kappa} \cdot \underline{\hat{u}} = 0 \rightarrow \underline{\kappa} \perp \underline{\hat{u}}$$

Consider an arbitrary vector $\hat{G}$, it can always be decomposed into a component parallel to $\underline{\kappa}$ and a component normal to $\underline{\kappa}$

$$\underline{\hat{G}} = \underline{\hat{G}^{||}} + \underline{\hat{G}^{\perp}}$$

And considering $\hat{e} = \underline{\kappa}/\kappa$ the unit vector in the direction of $\underline{\kappa}$, we have

$$\underline{\hat{G}^{||}} = \hat{e}(\hat{e} \cdot \hat{G}) = \underline{\kappa}(\underline{\kappa} \cdot \underline{\hat{G}})/\kappa^2$$

Or using index notation

$$\hat{G}^{||}{}_j = \frac{\kappa_j \kappa_k}{\kappa^2} \hat{G}_k$$

$$\left(\kappa_1 \widehat{G_1} + \kappa_2 \widehat{G_2} + \kappa_3 \widehat{G_3}\right)(\kappa_1, \kappa_2, \kappa_3)/\kappa^2 = \underline{\hat{G}^{||}}$$

$$\left[(\underline{\kappa} \cdot \underline{\hat{G}})\kappa_1, (\underline{\kappa} \cdot \underline{\hat{G}})\kappa_2, (\underline{\kappa} \cdot \underline{\hat{G}})\kappa_3\right]/\kappa^2 = \underline{\hat{G}^{||}}$$

For the perpendicular component

$$\underline{\hat{G}^{\perp}} = \underline{\hat{G}} - \underline{\hat{G}^{||}} = \underline{\hat{G}} - \underline{\kappa}\left(\underline{\kappa} \cdot \underline{\hat{G}}\right)/\kappa^2 = P_{jk}\hat{G}_k$$

Where the projection tensor $P_{jk}\left(\underline{\kappa}\right)$ is

$$P_{jk} \equiv \delta_{jk} - \frac{\kappa_j \kappa_k}{\kappa^2}$$

Which determines $\underline{\hat{G}^{\perp}}$ to be the projection of $\underline{\hat{G}}$ onto the plane normal to $\underline{\kappa}$.

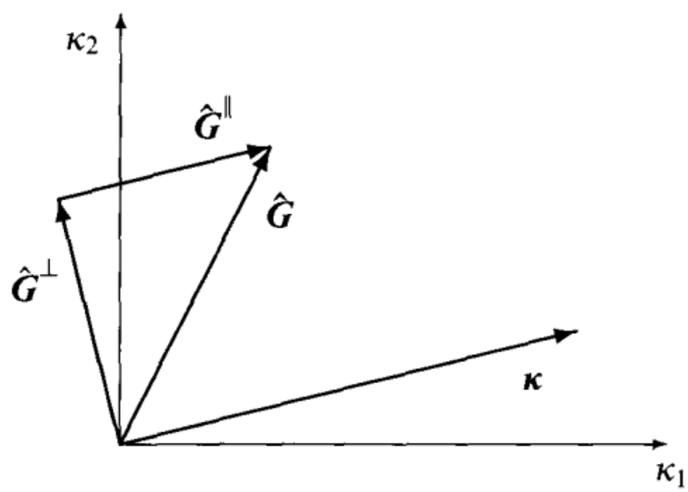


Fig. 6.9. A sketch (in two-dimensional wavenumber space) showing the decomposition of any vector $\hat{\boldsymbol{G}}$ into a component $\hat{\boldsymbol{G}}^{\|}$ parallel to $\boldsymbol{\kappa}$, and a component $\hat{\boldsymbol{G}}^{\perp}$ perpendicular to $\boldsymbol{\kappa}$.

Navier-Stokes in conservative form:

$$\frac{\partial u_j}{\partial t} + \frac{\partial (u_j u_k)}{\partial x_k} = \nu \frac{\partial^2 u_j}{\partial x_k x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_j}$$

Apply the operator $\mathcal{F}_k\{\ \}$ to NS:

$$\mathcal{F}_k\left\{\frac{\partial u_j}{\partial t}\right\} = \frac{d\hat{u}_j}{dt}(\underline{\kappa}, t)$$

Partial derivative in time in physical space becomes ordinary derivative in wavenumber space.

$$\mathcal{F}_k\left\{\nu \frac{\partial^2 u_j}{\partial x_k x_k}\right\} = -\nu\kappa^2 \hat{u}_j(\underline{\kappa}, t)$$

$$\mathcal{F}_k\left\{-\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right\} = -i\kappa_j \hat{p}$$

$$\hat{p}(\underline{\kappa}, t) \equiv \mathcal{F}_k\left\{\frac{p(\underline{x}, t)}{\rho}\right\}$$

Where previous derivation showing differentiation with respect to x_j in physical space corresponds to multiplication by $i\kappa_j$ in wave number space was used and

$$\frac{p(\underline{x}, t)}{\rho} = \sum_{\underline{\kappa}} e^{i\underline{k}\cdot\underline{x}} \hat{p}(\underline{\kappa}, t)$$

The nonlinear convection term is written as

$$\mathcal{F}_k\left\{\frac{\partial (u_j u_k)}{\partial x_k}\right\} = \hat{G}_j(\underline{\kappa}, t)$$

And its Fourier coefficients will be defined later.

Thus, NS becomes:

$$\frac{\partial \hat{u}_j}{\partial t}(\underline{\kappa}, t) + \nu\kappa^2\hat{u}_j(\underline{\kappa}, t) = -i\kappa_j\hat{p}(\underline{\kappa}, t) - \hat{G}_j(\underline{\kappa}, t)$$

Multiply by κ_j such that LHS=0, since

$$\kappa_j\hat{u}_j = 0 \quad (4a) \text{ (continuity equation)}$$[1]

and multiply by i to obtain

$$\kappa^2\hat{p} = i\kappa_j\hat{G}_j \quad (4b)$$

Eq. (4b) can be shown to be equivalent to the pressure Poisson equation in Fourier space and to show that the pressure and convection terms can be combined using the projection tensor.

1) In wave number space, the Poisson equation for pressure is obtained by taking the Fourier transform of the divergence of the NS equations:

$$\mathcal{F}_k\left\{-\nabla^2\left(\frac{p}{\rho}\right)\right\} = \mathcal{F}_k\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial(u_j u_k)}{\partial x_k}\right]\right\}$$

$$\mathcal{F}_k\left\{-\nabla^2\left(\frac{p}{\rho}\right)\right\} = \mathcal{F}_k\left\{-\frac{\partial^2}{\partial x_j^2}\left(\frac{p}{\rho}\right)\right\} = -i^2k_j^2\hat{p} = k^2\hat{p}$$

$$\mathcal{F}_k\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial(u_j u_k)}{\partial x_k}\right]\right\} = i\kappa_j\mathcal{F}_k\left\{\frac{\partial(u_j u_k)}{\partial x_k}\right\} = i\kappa_j\hat{G}_j(\underline{\kappa}, t)$$

In both cases using the property of $\frac{\partial}{\partial x_j}$ in wave number space = $i\kappa_j$, as per Eq. (3).

Thus:

$$k^2\hat{p} = i\kappa_j\hat{G}_j(\underline{\kappa}, t)$$

[1] Eq. (4a) shows that in incompressible flow $\hat{u}_j$ is perpendicular to $\underline{\kappa}$, i.e., for any value of $\underline{\kappa}$, $\hat{u}_j$ is oriented tangent to the surface of the sphere of radius $|\underline{\kappa}|$ centered at the origin.

2) By using $j = k$ in Eq. (4b) and multiplying by $-i\kappa_j$

$$-i\kappa_j\kappa^2\hat{p} = \kappa_j\kappa_k\hat{G}_k$$

Dividing by κ^2

$$-i\kappa_j\hat{p} = \frac{\kappa_j\kappa_k}{\kappa^2}\hat{G}_k = \hat{G}^{||}{}_j$$

i.e., the pressure term $-i\kappa_j\hat{p}$ exactly balances $\underline{\hat{G}}^{||}$, the component of $\underline{\hat{G}}$ in direction of $\underline{\kappa}$.

The NS equations can be re-written as

$$\frac{\partial\hat{u}_j}{\partial t} + \nu\kappa^2\hat{u}_j = \hat{G}^{||}{}_j - \hat{G}_j \quad (5a)$$

$$= -\hat{G}^{\perp}{}_j$$

$$= -P_{jk}\hat{G}_k$$

→ Combines pressure and convection terms.

Consider the final period of decay of isotropic turbulence in which Re is so low, that convection is negligible relative to the effects of viscosity such that the RHS of the above equation is zero. Then, for a specified initial condition $\underline{\hat{u}}(\underline{\kappa}, 0)$, the solution of the NS in wave number space is:

$$\underline{\hat{u}}(\underline{\kappa}, t) = \underline{\hat{u}}(\underline{\kappa}, 0)e^{-\nu\kappa^2 t}$$

Thus, each Fourier mode evolves and decays exponentially with t at rate $\nu\kappa^2$, independently from the other modes. High wave number modes (small λ) decay more rapidly than low wave numbers (large λ), as per $E(\kappa, t_0)$ Part 4 Eq. (16).

Expressed in terms of $\underline{\hat{u}}(\underline{\kappa})$, the nonlinear convective term is:

$$\hat{G}_k(\underline{\kappa}, t) = \mathcal{F}_k\left\{\frac{\partial(u_k u_l)}{\partial x_l}\right\} = i\kappa_l \mathcal{F}_k\{u_k u_l\}$$

As per Eq. (3)

$$= i\kappa_l \mathcal{F}_k\left\{\left(\sum_{\underline{\kappa}'} \hat{u}_k(\underline{\kappa}') e^{i\underline{\kappa}'\cdot\underline{x}}\right)\left(\sum_{\underline{\kappa}''} \hat{u}_l(\underline{\kappa}'') e^{i\underline{\kappa}''\cdot\underline{x}}\right)\right\}$$

$$= i\kappa_l \sum_{\underline{\kappa}'}\sum_{\underline{\kappa}''} \hat{u}_k(\underline{\kappa}')\hat{u}_l(\underline{\kappa}'')\,\langle e^{i(\underline{\kappa}'+\underline{\kappa}'')\cdot\underline{x}}, e^{-i\underline{\kappa}\cdot\underline{x}}\rangle$$

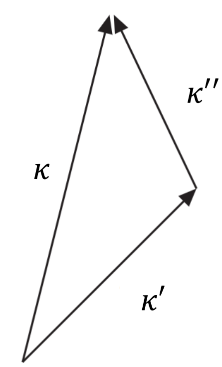
Inner product

$$= i\kappa_l \sum_{\underline{\kappa}'}\sum_{\underline{\kappa}''} \hat{u}_k(\underline{\kappa}')\hat{u}_l(\underline{\kappa}'')\,\delta_{\underline{\kappa},\underline{\kappa}'+\underline{\kappa}''}$$

$$\underline{\kappa} = \underline{\kappa}' + \underline{\kappa}''$$

$$\underline{\kappa} - \underline{\kappa}' = \underline{\kappa}''$$

$$= i\kappa_l \sum_{\underline{\kappa}'} \hat{u}_k(\underline{\kappa}')\hat{u}_l\underbrace{(\underline{\kappa}-\underline{\kappa}')}_{\underline{\kappa}''}$$



Triads of wavenumbers.

Thus, the incompressible flow NS equations in wave number space:

$$\left(\frac{d}{dt} + \nu\kappa^2\right)\hat{u}_j(\underline{\kappa}, t)$$
$$= -P_{jk}(\underline{\kappa}) i\kappa_l \sum_{\underline{\kappa}'} \hat{u}_k(\underline{\kappa}', t)\hat{u}_l\left(\underbrace{\underline{\kappa}-\underline{\kappa}'}_{\underline{\kappa}''}, t\right) \quad (5b)$$

The LHS involves $\underline{\hat{u}}$ only at $\underline{\kappa}$. In contrast, the RHS involves $\underline{\hat{u}}$ at $\underline{\kappa}'$ and $\underline{\kappa}''$, such that $\underline{\kappa}' + \underline{\kappa}'' = \underline{\kappa}$, and the contributions from $\underline{\kappa}' = \underline{\kappa}$ and $\underline{\kappa}'' = \underline{\kappa}$ are zero.

In wave number space, the convection term is nonlinear and non-local, involving the interaction of *wave number triads*, $\underline{\kappa}, \underline{\kappa}'$ and $\underline{\kappa}''$, such that $\underline{\kappa}' + \underline{\kappa}'' = \underline{\kappa}$, i.e., responsible for inter-scale interactions.

Eq. (5) is a deterministic (for a truncated series) coupled set of nonlinear ODEs for $\hat{u}_j(\underline{\kappa}, t)$ for each $\underline{\kappa}$, i.e., three equations and three unknowns. Based continuity $\kappa_j\hat{u}_j(\underline{\kappa}, t) = 0$ LHS perpendicular $\underline{\kappa}$ and so also is the RHS, as per Eq. (5a).

The kinetic energy of Fourier modes (Pope)

Under the assumption of homogeneous flow, the mean velocity $\overline{U}(\underline{x},t)$ is zero, such that its Fourier coefficients are also zero. Therefore, the instantaneous velocity field corresponds to the fluctuating velocity, having zero mean. To describe the turbulence statistically, now consider homogeneous flow and higher order statistics.

The two-point two-velocity correlation $\mathcal{R}_{ij}(\underline{x}, \underline{x}+\underline{r}, t)$ can be represented in physical space and wave number space:

$$\mathcal{R}_{ij}(\underline{x}, \underline{x}+\underline{r}, t) = \langle u_i(\underline{x},t) u_j(\underline{x}+\underline{r},t)\rangle \leftarrow \text{Ensemble average}$$

$$\hat{\mathcal{R}}_{ij}(\underline{\kappa}, \underline{\kappa}', t) = \langle \mathcal{F}_k\{u_i(\underline{x},t)\} \mathcal{F}_{k'}\{u_j(\underline{x}',t)\}\rangle$$

$$= \langle \hat{u}_i(\underline{k},t) \hat{u}_j(\underline{k}',t)\rangle$$

The dependence from $\underline{x}$ and $\underline{x}' = \underline{x}+\underline{r}$ in physical space is transformed into dependence from $\underline{\kappa}$ and $\underline{\kappa}'$ in wave number space.

Recall for homogeneous turbulence $\mathcal{R}_{ij}(\underline{x}, \underline{x}+\underline{r}, t) = \mathcal{R}_{ij}(\underline{r}, t)$ and equivalently in wave number space, $\hat{u}_i(\underline{k},t)$ and $\hat{u}_j(\underline{k}',t)$ are uncorrelated, unless $\underline{\kappa}' + \underline{\kappa} = 0$, i.e., $\underline{\kappa}' = -\underline{\kappa} \rightarrow \hat{u}_i$ and $\hat{u}_j$ only $f(\underline{k},t)$, as per Appendix A.1. This relates the vector $\underline{r}$ in physical space, with an equivalent vector $\underline{\kappa}$ in wave number space.

Thus, all the covariance information is contained in:

$$\hat{\mathcal{R}}_{ij}(\underline{\kappa},t) = \langle \hat{u}_i(\underline{\kappa},t) \hat{u}_j(\underline{\kappa},t)\rangle = \langle \hat{u}_i(-\underline{\kappa},t) \hat{u}_j(\underline{\kappa},t)\rangle = \langle \hat{u}_i^*(\underline{\kappa},t) \hat{u}_j(\underline{\kappa},t)\rangle = \langle \hat{u}_i^*(-\underline{\kappa},t) \hat{u}_j(\underline{\kappa},t)\rangle$$

Real signal — Conjugate symmetry — Real signal

And $\hat{\mathcal{R}}_{ij}(\underline{\kappa},t)$ represent the Fourier coefficients of the two-point velocity correlation: $\hat{\mathcal{R}}_{ij}(\underline{\kappa},t) = \mathcal{F}_k\{\mathcal{R}_{ij}(\underline{x},t)\}$. Appendix A.1

In homogeneous turbulence, the Fourier representation of $\mathcal{R}_{ij}$ becomes

$$\mathcal{R}_{ij}(\underline{r}\,,t) = \sum_{\underline{\kappa}} \hat{\mathcal{R}}_{ij}(\underline{\kappa},t)e^{i\,\underline{\kappa}\cdot\underline{r}}$$

The kinetic energy of the Fourier mode $\hat{E}(\underline{\kappa},t)$ is defined as:

$$\hat{E}(\underline{\kappa},t) = \frac{1}{2}\overline{\hat{u}_i^*(\underline{\kappa},t)\hat{u}_i(\underline{\kappa},t)} = \frac{1}{2}\hat{\mathcal{R}}_{ii}(\underline{\kappa},t) \quad (6a)$$

The TKE is:

$$k(t) = \frac{1}{2}\overline{u_i u_i} = \sum_{\underline{\kappa}} \frac{1}{2}\hat{\mathcal{R}}_{ii}(\underline{\kappa},t) = \sum_{\underline{\kappa}} \hat{E}(\underline{\kappa},t)$$

The dissipation rate $\varepsilon(t)$ is also related to $\hat{E}(\underline{\kappa},t)$, by

$$\varepsilon(t) = -\nu \lim_{r\to 0}\frac{\partial^2}{\partial r_k^2}\mathcal{R}_{jj}(\underline{r},t) \quad \text{(6b)}$$

$$= -\nu \lim_{r\to 0}\sum_{\underline{\kappa}} e^{i\underline{k}\cdot\underline{r}}(-\kappa_k\kappa_k)\,\hat{\mathcal{R}}_{jj}(\underline{\kappa},t)$$

$$= \sum_{\underline{\kappa}} 2\nu\kappa^2\hat{E}(\underline{\kappa},t)$$

Thus, $\hat{E}(\underline{\kappa},t)$ and $2\nu\kappa^2\hat{E}(\underline{\kappa},t)$ are the contributions to TKE and ε from Fourier mode $\underline{\kappa}$, i.e., over spherical shell of radius $|\underline{\kappa}|$.

A dynamical equation for the discrete energy spectrum [Eq. (6a)] may be derived by taking the average of the sum of Eq. (5b) times $\hat{u}_j^*(\underline{\kappa},t)$ and the complex conjugate of Eq. (5b) times $\hat{u}_j(\underline{\kappa},t)$. The result is:

$$\frac{d}{dt}\hat{E}(\underline{\kappa},t) = \hat{T}(\underline{\kappa},t) - 2\nu\kappa^2\hat{E}(\underline{\kappa},t) \quad (7)$$

Appendix A.2

$$\hat{T}(\underline{\kappa},t) = \kappa_l P_{jk}(\underline{\kappa})\Re\left\{ i\sum_{\underline{\kappa}'}\langle\hat{u}_j(\underline{\kappa})\hat{u}_k^*(\underline{\kappa}')\hat{u}_l^*\big(\underbrace{\underline{\kappa}-\underline{\kappa}'}_{\underline{\kappa}''}\big)\rangle\right\}$$

And $\Re\{\,\}$ denotes the real part.

Comparing Eq. (7) with Part 4 Eq. (4) and since the time derivative and dissipation terms are the same suggests that the transfer terms are also equivalent; however, the former is in continuous form, whereas the latter in discrete form. Note that Eq. (7) derived from NS, whereas Part 4 Eq. (4) from $\mathcal{R}_{ij}$ equation

$$\frac{dE}{dt}(\kappa,t) = T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t) \quad (Part\ 4, Eq.4)$$

$$T(\kappa,t) = \frac{1}{2}\int_{|\underline{\kappa}|=\kappa} T_{ii}(\underline{\kappa},t)\,\kappa^2 d\Omega = 2\pi\kappa^2 T_{ii}(\kappa,t) \quad (Part\ 4, Eq.5)$$

$$T_{ii}(\underline{\kappa},t) = \frac{1}{(2\pi)^3}\int_{\forall} S_{ii}(\underline{r},t)e^{i\underline{\kappa}\cdot\underline{r}}\underline{dr} \quad (Part\ 4, Eq.2\ j=1)$$

$$S_{ii}(\underline{r},t) = \frac{\partial S_{ik,i}}{\partial r_k}(-\underline{r},t) + \frac{\partial S_{ik,i}}{\partial r_k}(\underline{r},t) \quad (Part\ 4, Eq.3\ j=1)$$

$$S_{ik,i}(\underline{r},t) = \overline{u_i(\underline{x},t)u_k(\underline{x},t)u_i(\underline{x}+\underline{r},t)}$$

and in both cases under the assumption of homogeneous turbulence.

Summing over all $\underline{\kappa}$, LHS of Eq. (7) becomes dk/dt, while the last term on the right-hand side sums to $-\varepsilon$, such that the sum of $\hat{T}(\underline{\kappa},t)$ is zero: $\sum_{\underline{\kappa}}\hat{T}(\underline{\kappa},t) = 0$ Thus, the term $\hat{T}(\underline{\kappa},t)$ represents the transfer of energy between modes.

Eq. (7) has a direct correspondence with K-H equation, but has the advantage of providing clear quantification of the energy at different scales of motion and an explicit expression for the energy-transfer rate → $\hat{T}(\underline{\kappa},t)$, which plays a central role in the energy cascade and involves the wave number triplets $\underline{\kappa'} + \underline{\kappa''} = \underline{\kappa}$. Triad interactions allow energy of different scales to give rise to new scales. These in turn have triad interactions and this chain branching gives rise to chaotic behavior.

The terms $\hat{E}(\underline{\kappa},t)$ and $-2\nu\kappa^2\hat{E}(\underline{\kappa},t)$ in Eq. (7) can be related to the two-point two-velocity correlation in wave number space $\hat{\mathcal{R}}_{ij}(\underline{\kappa},t)$ via Eq. (6a,b). Eq. (7) and $Part\ 4, Eq.4$ have the same assumptions and in discrete and continuous form, respectively, which suggest correspondence between wave number triplets and Fourier transform of $S_{ii}(\underline{r},t)$, which is related to $S_{ik,i}(\underline{r},t)$.

The kinetic energy of Fourier modes (Bernard)

Recall previous derivation discrete NS as per Pope

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_j(\underline{\kappa},t) = -i\kappa_l P_{jk}\sum_{\underline{\kappa'}}\hat{u}_k(\underline{\kappa'},t)\hat{u}_l(\underline{\kappa}-\underline{\kappa'},t) \quad (5)$$

Can be transformed to Bernard form by setting $j=i,\ l=m,\ k=j,\ \underline{\kappa'}=\underline{l}$

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i(\underline{\kappa},t) = -i\kappa_m P_{ij}\sum_{\underline{l}}\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t) \quad (8)$$

An equivalent form of Eq. (8) is given by:

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i(\underline{\kappa},t) = M_{ijm}(\underline{\kappa})\sum_{\underline{l}}\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t) \quad (9)$$

Appendix A.3

$$M_{ijm} = -\frac{i}{2}\left(\kappa_m P_{ij}(\underline{\kappa}) + \kappa_j P_{im}(\underline{\kappa})\right)$$

Which can be obtained noting that the RHS of Eq. (8) is left unchanged if the dummy indices j and m are switched, and summation on $\underline{l}$ is replaced by the equivalent summation on $\underline{l'}=\underline{\kappa}-\underline{l}$.

Applying the same steps used to go from Eq. (5) to (7), i.e., taking the average of the sum of Eq. (9) times $\hat{u}_i^*(\underline{\kappa},t)$ and the complex conjugate of Eq. (9) times $\hat{u}_i(\underline{\kappa},t)$ gives a dynamical equation for the discrete energy spectrum in the form:

Appendix A.4

$$\frac{d}{dt}\hat{E}(\underline{\kappa},t) + 2\nu\kappa^2\hat{E}(\underline{\kappa},t)$$

$$= \frac{1}{2}M_{ijm}\sum_{\underline{\kappa'}}\left(\overline{\widehat{u_i}(-\underline{\kappa})\hat{u}_j(\underline{l})\widehat{u_m}(\underline{\kappa}-\underline{l})} - \overline{\widehat{u_i}(\underline{\kappa})\hat{u}_j(\underline{l})\widehat{u_m}(-\underline{\kappa}-\underline{l})}\right) \quad (10)$$

Where $\underline{l}$ has been replaced with $-\underline{l}$ in the second term for later convenience. The RHS accounts for the energy transfer between wave numbers. The triadic nature of such exchanges is evident in these expressions.

Equivalency of Eq. (10) and Eq. (7) transfer terms needs to be shown.

Limit of Infinite Space (In progress)

Consider now limit of Eq. (10) as $L \to \infty$.

Define

$$E_{ij}^{L}(\underline{\kappa}, t) = \left(\frac{L}{2\pi}\right)^3 \overline{\hat{u}_i(\underline{\kappa}, t)\hat{u}_j(-\underline{\kappa}, t)} \quad (11)$$

And using the following:

$$\hat{u}_j^*(\underline{\kappa}, t) = \hat{u}_j(-\underline{\kappa}, t)$$

$$\overline{\hat{u}_i(\underline{\kappa}, t)\hat{u}_j^*(\underline{\kappa}, t)} = \frac{1}{L^3}\int_{\forall} \mathcal{R}_{ij}(\underline{r}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{r}$$

Eq. (11) becomes:

$$E_{ij}^{L}(\underline{\kappa}, t) = \left(\frac{L}{2\pi}\right)^3 \overline{\hat{u}_i(\underline{\kappa}, t)\hat{u}_j^*(\underline{\kappa}, t)} = \left(\frac{1}{2\pi}\right)^3 \int_{\forall} \mathcal{R}_{ij}(\underline{r}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{r}$$

In the limit as $L \to \infty$, RHS becomes the Fourier transform or $\mathcal{R}_{ij} \to \mathcal{E}_{ij}$

$$\lim_{L\to\infty} E_{ij}^{L}(\underline{\kappa}, t) = \mathcal{E}_{ij}(\underline{\kappa}, t)$$

During this process, $\underline{\kappa}$ values become closer and closer, transforming from a discrete distribution to a continuous vector.

A similar reasoning can be applied in the case of the two-point triple velocity correlation. Thus, define:

$$T_{ijn}^{L}(\underline{\kappa}, \underline{l}, t) = \left(\frac{L}{2\pi}\right)^6 \overline{\hat{u}_i(\underline{\kappa}, t)\hat{u}_j(\underline{l}, t)\hat{u}_n(-\underline{\kappa} - \underline{l}, t)} \quad (12)$$

Where the fact that

$$\overline{\hat{u}_i(\underline{\kappa}, t)\hat{u}_j(\underline{l}, t)\hat{u}_n(\underline{m}, t)} = 0$$

Unless $\underline{\kappa} + \underline{l} + \underline{m} = 0$ is used.

Substituting the Fourier components according to

$$\hat{u}_i(\underline{\kappa}, t) = \frac{1}{L^3} \int_{\forall} u_i(\underline{x}, t)\, e^{-i\underline{\kappa}\cdot\underline{x}} d\underline{x} \quad (13)$$

Transforms Eq. (12) into

$$T_{ijn}^{L}(\underline{\kappa}, \underline{l}, t) = \left(\frac{L}{2\pi}\right)^6 \frac{1}{L^9} \int_{\forall}\int_{\forall}\int_{\forall} \overline{u_i(\underline{x}, t) u_j\left(\underline{y}, t\right) u_n(\underline{z}, t)}\, e^{-i\underline{\kappa}\cdot(\underline{x}-\underline{z}) - i\underline{l}\cdot\left(\underline{y}-\underline{z}\right)} d\underline{x}d\underline{y}d\underline{z} \quad (14)$$

For homogeneous turbulence, the triple velocity correlation S_{ijn} depends only on $\underline{r} = \underline{x} - \underline{z}$ and $\underline{s} = \underline{y} - \underline{z} \ \rightarrow\ \underline{x} = \underline{r} + \underline{z},\ \underline{y} = \underline{s} + \underline{z}$.

Therefore,

$$S_{ijn}(\underline{r}, \underline{s}, t) = \overline{u_i(\underline{x}, t) u_j\left(\underline{y}, t\right) u_n(\underline{z}, t)} = \overline{u_i(\underline{r} + \underline{z}, t) u_j(\underline{s} + \underline{z}, t) u_n(\underline{z}, t)} \quad (15)$$

Changing $\underline{x}$ and $\underline{y}$ variables in Eq. (14) with $\underline{r}$ and $\underline{s}$, respectively, and using Eq. (15) gives

$$T_{ijn}^{L}(\underline{\kappa}, \underline{l}, t) = \left(\frac{1}{2\pi}\right)^6 \frac{1}{L^3} \int_{\forall}\int_{\forall}\int_{\forall} \underbrace{\overline{u_i(\underline{r} + \underline{z}, t) u_j(\underline{s} + \underline{z}, t) u_n(\underline{z}, t)}}_{\boxed{S_{ijn}(\underline{r},\underline{s},t)}} e^{-i\underline{\kappa}\cdot\underline{r} - i\underline{l}\cdot\underline{s}} d\underline{r}d\underline{s}d\underline{z}$$

And carrying out the $\underline{z}$ integration

$$T_{ijn}^{L}(\underline{\kappa}, \underline{l}, t) = \left(\frac{1}{2\pi}\right)^6 \int_{\forall}\int_{\forall} S_{ijn}(\underline{r}, \underline{s}, t) e^{-i\underline{\kappa}\cdot\underline{r} - i\underline{l}\cdot\underline{s}} d\underline{r}d\underline{s} \qquad \boxed{\int_{\forall} d\underline{z} = L^3}$$

In the limit as $L \rightarrow \infty$ this becomes

$$T_{ijn}(\underline{\kappa}, \underline{l}, t) = \left(\frac{1}{2\pi}\right)^6 \int\int S_{ijn}(\underline{r}, \underline{s}, t) e^{-i\underline{\kappa}\cdot\underline{r} - i\underline{l}\cdot\underline{s}} d\underline{r}d\underline{s} \quad (16)$$

Which represents the Fourier transform of S_{ijn}.

Now, the tools to consider the limit of Eq. (10) as $L \to \infty$ have been developed.

Multiplying Eq. (10) by $(L/2\pi)^3$ and taking the limit as $L \to \infty$ gives

$$\lim_{L\to\infty}\Bigg[\underbrace{\left(\frac{L}{2\pi}\right)^3 \frac{d}{dt}\hat{E}(\underline{\kappa},t)}_{\boxed{1}} + \underbrace{2\left(\frac{L}{2\pi}\right)^3 \nu\kappa^2\hat{E}(\underline{\kappa},t)}_{\boxed{2}} = \frac{1}{2}\left(\frac{L}{2\pi}\right)^3 M_{ijm}\sum_{\underline{l}}\Bigg(\underbrace{\widehat{u_\iota}(-\underline{\kappa})\hat{u}_j(\underline{l})\widehat{u_m}(\underline{\kappa}-\underline{l})}_{\boxed{3a}} - \underbrace{\widehat{u_\iota}(\underline{\kappa})\hat{u}_j(\underline{l})\widehat{u_m}(-\underline{\kappa}-\underline{l})}_{\boxed{3b}}\Bigg)\Bigg] \quad (17)$$

Now, consider each term separately.

Term 1:

$$\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3 \frac{d}{dt}\hat{E}(\underline{\kappa},t) = \frac{1}{2}\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3 \frac{d}{dt}\overline{\hat{u}_i^*(\underline{\kappa},t)\hat{u}_i(\underline{\kappa},t)}$$

And using Eq. (11) and the fact that $\hat{u}_i^*(\underline{\kappa},t) = \hat{u}_i(-\underline{\kappa},t)$

$$\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3 \frac{d}{dt}\hat{E}(\underline{\kappa},t) = \frac{1}{2}\lim_{L\to\infty}E_{ij}^L(\underline{\kappa},t) = \frac{1}{2}\frac{d}{dt}\mathcal{E}_{ii}(\underline{\kappa},t) \quad (18)$$

Moreover, in Chapter 4 Part 5, the following relation was derived:

$$\mathcal{E}_{ij}(\underline{\kappa},t) = \frac{E(\kappa,t)}{4\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right)$$

And contracting indices gives

$$\mathcal{E}_{ii}(\underline{\kappa},t) = \frac{E(\kappa,t)}{4\pi\kappa^2}\left(\underbrace{\delta_{ii}}_{\boxed{3}} - \underbrace{\frac{\kappa_i\kappa_i}{\kappa^2}}_{\boxed{1}}\right) = \frac{E(\kappa,t)}{2\pi\kappa^2} \quad (19)$$

Substituting Eq. (19) into (18) yields

$$\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3 \frac{d}{dt}\hat{E}(\underline{\kappa},t) = \frac{1}{4\pi\kappa^2}\frac{dE(\kappa,t)}{dt}$$

Term 2:

$$\lim_{L\to\infty} 2\left(\frac{L}{2\pi}\right)^3 \nu\kappa^2\hat{E}(\underline{\kappa},t) = \lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3 \nu\kappa^2\overline{\hat{u}_i^*(\underline{\kappa},t)\hat{u}_i(\underline{\kappa},t)}$$

Using similar steps shown for Term 1:

$$\lim_{L\to\infty} 2\left(\frac{L}{2\pi}\right)^3 \nu\kappa^2\hat{E}(\underline{\kappa},t) = 2\lim_{L\to\infty}\nu\kappa^2\mathcal{E}_{ii}(\underline{\kappa},t) = \frac{\nu}{2\pi}E(\kappa,t)$$

Term 3b:

$$\lim_{L\to\infty}\frac{1}{2}\left(\frac{L}{2\pi}\right)^3 M_{ijm}\sum_{\underline{l}}\left(\widehat{u_i}(\underline{\kappa})\hat{u}_j(\underline{l})\widehat{u_m}(-\underline{\kappa}-\underline{l})\right) \quad (20)$$

Recall relation between triad of wave numbers:

$$\underline{\kappa}+\underline{l}+\underline{m}=0$$

Using Eq. (12) with $i=j,\ j=m,\ m=i$ and $\underline{\kappa}=\underline{l},\ \underline{l}=\underline{\kappa}$

$$\lim_{L\to\infty}\frac{1}{2}\left(\frac{L}{2\pi}\right)^3\sum_{\underline{l}}\left(\widehat{u_j}(l)\hat{u}_m(\underline{k}-\underline{l})\widehat{u_i}(-\underline{\kappa})\right)$$

$$=\frac{1}{2}\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3\sum_{\underline{l}}\left(\frac{2\pi}{L}\right)^6 T_{jmi}^L(\underline{l},\underline{\kappa}-\underline{l},t)$$

$$=\frac{1}{2}\lim_{L\to\infty}\left(\frac{2\pi}{L}\right)^3\sum_{\underline{l}} T_{jmi}^L(\underline{l},\underline{\kappa}-\underline{l},t)$$

Therefore

$$\lim_{L\to\infty}\frac{1}{2}\left(\frac{L}{2\pi}\right)^3 M_{ijm}\sum_{\underline{l}}\left(\widehat{u_i}(\underline{\kappa})\hat{u}_j(\underline{l})\widehat{u_m}(\underline{\kappa}-\underline{l})\right) = \frac{1}{2}M_{ijm}\int T_{jmi}(\underline{l},\underline{\kappa}-\underline{l},t)\,d\underline{l}$$

Where the last equality derives from the fact that $\lim_{L\to\infty} T_{jmi}^{L} = T_{jmi}$, as shown in Eq. (16), and that $\left(\frac{2\pi}{L}\right)^3$ represents the volume surrounding each wave number vectors in the sum, since $\kappa = 2\pi \underline{n}/L$.

Term 3a:

Same steps as Term 3b give:

$$\lim_{L\to\infty} \frac{1}{2}\left(\frac{L}{2\pi}\right)^3 M_{ijm} \sum_{\underline{l}} \left(\hat{u}_i(\underline{\kappa})\hat{u}_j(\underline{l})\widehat{u_m}(-\underline{\kappa}-\underline{l})\right) = \frac{1}{2} M_{ijm} \int T_{jmi}(\underline{l}, -\underline{\kappa}-\underline{l}, t)\, d\underline{l}$$

Therefore Eq. (17) becomes:

$$\frac{1}{4\pi\kappa^2}\frac{dE(\kappa,t)}{dt} + \frac{\nu}{2\pi}E(\kappa,t) = \frac{1}{2} M_{ijm}(\underline{\kappa}) \int T_{jmi}(\underline{l}, \underline{\kappa}-\underline{l}, t) - T_{jmi}(\underline{l}, -\underline{\kappa}-\underline{l}, t)\, d\underline{l} \quad (21)$$

And using homogeneity properties of T_{jmi}, it can be shown that:

$$T_{jmi}(\underline{l}, -\underline{\kappa}-\underline{l}, t) = -T_{jmi}(\underline{l}, \underline{\kappa}-\underline{l}, t)$$

And Eq. (21) becomes:

$$\frac{1}{4\pi\kappa^2}\frac{dE(\kappa,t)}{dt} + \frac{\nu}{2\pi}E(\kappa,t) = M_{ijm} \int T_{jmi}(\underline{l}, \underline{\kappa}-\underline{l}, t)\, d\underline{l}$$

Finally, multiplying by $4\pi\kappa^2$

$$\frac{dE(\kappa,t)}{dt} + 2\nu\kappa^2 E(\kappa,t) = 4\pi\kappa^2 M_{ijm} \int T_{jmi}(\underline{l}, \underline{\kappa}-\underline{l}, t)\, d\underline{l} \quad (22)$$

$$M_{ijm} = -\frac{i}{2}\left(\kappa_m P_{ij}(\underline{\kappa}) + \kappa_j P_{im}(\underline{\kappa})\right)$$

This represents an alternative form of the equation for the energy spectrum that can be compared with

$$\frac{dE}{dt}(\kappa, t) = T(\kappa, t) - 2\nu\kappa^2 E(\kappa, t) \quad (Part\ 4, Eq. 4)$$

$$T(\kappa, t) = \frac{1}{2}\int_{|\underline{\kappa}|=\kappa} T_{ii}(\underline{\kappa}, t)\, \kappa^2 d\Omega = 2\pi\kappa^2 T_{ii}(\kappa, t) \quad (Part\ 4, Eq. 5)$$

For homogeneous turbulence and

$$\frac{dE}{dt}(\kappa, t) + 2\nu\kappa^2 E(\kappa, t) = \frac{u_{rms}^3}{\pi}\int_0^\infty \kappa[(3 - \kappa^2 r^2)\sin\kappa r - 3\kappa r\cos\kappa r]k(r, t)dr \quad (23)$$

obtained in Part 4; however, also subject assumption of isotropy, whereas Eq. (22) only assume homogeneity. In both expressions, the RHS represents the rate of transfer of energy between scales.

Eq. (22) clearly shows the interaction between the wave number triads that are responsible for the transfer of energy between scales.

Eq. (23), on the other hand, shows the role of the two-point three-velocity correlation in the transfer process.

It would be useful to understand the relationship and physics of the transfer terms in triad vs. two-point three-velocity and its correlation forms.

Appendix A

A.1

$$\mathcal{R}_{ij}(\underline{x}, \underline{x} + \underline{r}\,, t) = \langle u_i(\underline{x}, t) u_j(\underline{x} + \underline{r}, t)\rangle$$

$$\hat{\mathcal{R}}_{ij}(\underline{\kappa}, \underline{\kappa}', t) = \langle \mathcal{F}_k\{u_i(\underline{x}, t)\} \mathcal{F}_{k'}\{u_j(\underline{x}', t)\}\rangle$$

$$= \langle \hat{u}_i(\underline{k}, t) \hat{u}_j(\underline{k}', t)\rangle$$

$$= \langle \underbrace{\langle u_i(\underline{x}, t), e^{-i\underline{\kappa}\cdot\underline{x}}\rangle}_{\boxed{inner\ product}} \underbrace{\langle u_j(\underline{x}', t), e^{-i\underline{\kappa'}\cdot\underline{x'}}\rangle}_{\boxed{inner\ product}}\rangle$$

$$\langle \hat{u}_i(\underline{k}, t) \hat{u}_j(\underline{k}', t)\rangle = \frac{1}{L^6}\int_0^{\mathcal{L}}\int_0^{\mathcal{L}} \underbrace{\langle u_i(\underline{x}, t) u_j(\underline{x}', t)\rangle}_{\boxed{average}} e^{-i(\underline{\kappa}\cdot\underline{x} + \underline{\kappa}'\cdot\underline{x}')} d\underline{x} d\underline{x}'$$

Substituting $\underline{x}' = \underline{x} + \underline{r}$ and using the fact that in homogeneous turbulence $\mathcal{R}_{ij}(\underline{x}, \underline{x} + \underline{r}\,, t) = \mathcal{R}_{ij}(\underline{r}\,, t)$

$$\underbrace{\langle u_i(\underline{k}, t) u_j(\underline{k}', t)\rangle}_{\boxed{average}} = \frac{1}{L^6}\int_0^{\mathcal{L}}\int_0^{\mathcal{L}} \mathcal{R}_{ij}(\underline{r}\,, t) e^{-i\underline{x}\cdot(\underline{\kappa} + \underline{\kappa}')} e^{-i\underline{\kappa}'\cdot\underline{r}} d\underline{x} d\underline{x}'$$

Using the fact that $d\underline{x}' = d\underline{r}$

$$= \frac{1}{L^3}\int_0^{\mathcal{L}} e^{-i\underline{x}\cdot(\underline{\kappa} + \underline{\kappa}')} d\underline{x} \frac{1}{L^3}\int_0^{\mathcal{L}} \mathcal{R}_{ij}(\underline{r}\,, t) e^{-i\underline{\kappa}'\cdot\underline{r}} d\underline{r}$$

$$= \underbrace{\langle e^{-i\underline{x}\cdot\underline{k}}, e^{-i\underline{x}\cdot\underline{k}'}\rangle}_{\boxed{inner\ product}} \underbrace{\langle \mathcal{R}_{ij}(\underline{r}\,, t), e^{-i\underline{\kappa}'\cdot\underline{r}}\rangle}_{\boxed{inner\ product}}$$

$$= \delta_{\underline{\kappa}, -\underline{\kappa}'} \underbrace{\langle \mathcal{R}_{ij}(\underline{r}\,, t), e^{-i\underline{\kappa}'\cdot\underline{r}}\rangle}_{\boxed{inner\ product}}$$

And using the definition of the Fourier coefficients

$$\underbrace{\langle \hat{u}_i(\underline{k}, t) \hat{u}_j(\underline{k}', t)\rangle}_{\boxed{average}} = \mathcal{F}_k\{\mathcal{R}_{ij}(\underline{r}\,, t)\}\delta_{\underline{\kappa}, -\underline{\kappa}'}$$

Substituting $\underline{\kappa}' = -\underline{\kappa}$

$$\hat{\mathcal{R}}_{ij}(\underline{k}\,, t) = \underbrace{\langle \hat{u}_i(\underline{k}, t) \hat{u}_j(-\underline{k}, t)\rangle}_{\boxed{average}} = \mathcal{F}_k\{\mathcal{R}_{ij}(\underline{x}\,, t)$$

A.2

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_j(\underline{\kappa},t) = -i\kappa_l P_{jk}\sum_{\underline{k'}}\widehat{u_k}(\underline{k}',t)\widehat{u_l}(\underline{\kappa}-\underline{k}',t) \quad (1A)$$

The conjugate of Eq. (1A) is equal to

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_j^*(\underline{\kappa},t) = -i\kappa_l P_{jk}\sum_{\underline{k'}}\widehat{u_k^*}(\underline{k}',t)\widehat{u_l^*}(\underline{\kappa}-\underline{k}',t) \quad (2A)$$

Multiplying Eq. (1A) by $\hat{u}_j^*(\underline{\kappa},t)$ gives

$$\left(\frac{d\hat{u}_j(\underline{\kappa},t)}{dt}+\nu\kappa^2\hat{u}_j(\underline{\kappa},t)\right)\hat{u}_j^*(\underline{\kappa},t) = -i\kappa_l P_{jk}\sum_{\underline{k'}}\hat{u}_j^*(\underline{\kappa},t)\widehat{u_k}(\underline{k}',t)\widehat{u_l}(\underline{\kappa}-\underline{k}',t) \quad (3A)$$

Multiplying Eq. (2A) by $\hat{u}_i(\underline{\kappa},t)$ gives

$$\left(\frac{d\hat{u}_j^*(\underline{\kappa},t)}{dt}+\nu\kappa^2\hat{u}_j^*(\underline{\kappa},t)\right)\hat{u}_j(\underline{\kappa},t) = -i\kappa_l P_{jk}\sum_{\underline{k'}}\hat{u}_j(\underline{\kappa},t)\widehat{u_k^*}(\underline{k}',t)\widehat{u_l^*}(\underline{\kappa}-\underline{k}',t) \quad (4A)$$

Taking the sum of Eq. (3A) and (4A) yields

$$\left(\frac{d\hat{u}_j(\underline{\kappa},t)}{dt}\hat{u}_j^*(\underline{\kappa},t)+\frac{\partial\hat{u}_j^*(\underline{\kappa},t)}{\partial t}\hat{u}_j(\underline{\kappa},t)\right)$$

$$+\left(\nu\kappa^2\hat{u}_j(\underline{\kappa},t)\hat{u}_j^*(\underline{\kappa},t)+\nu\kappa^2\hat{u}_j^*(\underline{\kappa},t)\hat{u}_j(\underline{\kappa},t)\right)$$

$$=-i\kappa_l P_{jk}\sum_{\underline{k'}}\hat{u}_j^*(\underline{\kappa},t)\hat{u}_k(\underline{k}',t)\hat{u}_l(\underline{\kappa}-\underline{k}',t)$$

$$-i\kappa_l P_{jk}\sum_{\underline{k'}}\hat{u}_j(\underline{\kappa},t)\hat{u}_k^*(\underline{k}',t)\hat{u}_l^*(\underline{\kappa}-\underline{k}',t)\quad(5A)$$

Now, it is necessary to take the average of Eq. (5A) and analyze each term.

1)

$$\overline{\left(\frac{d\hat{u}_j(\underline{\kappa},t)}{dt}\hat{u}_j^*(\underline{\kappa},t)+\frac{\partial\hat{u}_j^*(\underline{\kappa},t)}{\partial t}\hat{u}_j(\underline{\kappa},t)\right)}=\frac{\partial\left[\overline{\hat{u}_j(\underline{\kappa},t)\hat{u}_j^*(\underline{\kappa},t)}\right]}{\partial t}\quad(6A)$$

Recall definition of the discrete energy spectrum

$$\hat{E}(\underline{\kappa},t)=\frac{1}{2}\overline{\hat{u}_j^*(\underline{\kappa},t)\hat{u}_j(\underline{\kappa},t)}$$

And substitute it into Eq. (6A)

$$\frac{d\left[\overline{\hat{u}_j(\underline{\kappa},t)\hat{u}_j^*(\underline{\kappa},t)}\right]}{dt}=2\frac{d\left[\overline{\frac{1}{2}\hat{u}_j(\underline{\kappa},t)\hat{u}_j^*(\underline{\kappa},t)}\right]}{dt}=2\frac{d\hat{E}(\underline{\kappa},t)}{dt}$$

2)

$$\overline{\left(\nu\kappa^2\hat{u}_j(\underline{\kappa},t)\hat{u}_j^*(\underline{\kappa},t)+\nu\kappa^2\hat{u}_j^*(\underline{\kappa},t)\hat{u}_j(\underline{\kappa},t)\right)}=2\nu\kappa^2\overline{\hat{u}_j(\underline{\kappa},t)\hat{u}_j^*(\underline{\kappa},t)}$$

$$=4\nu\kappa^2\frac{1}{2}\overline{\hat{u}_j(\underline{\kappa},t)\hat{u}_j^*(\underline{\kappa},t)}=4\nu\kappa^2\hat{E}(\underline{\kappa},t)$$

3)

$$-i\kappa_l P_{jk}\sum_{\underline{k'}}\overline{\hat{u}_j^*(\underline{\kappa},t)\hat{u}_k(\underline{k}',t)\hat{u}_l(\underline{\kappa}-\underline{k}',t)}\ -i\kappa_l P_{jk}\sum_{\underline{k'}}\overline{\hat{u}_j(\underline{\kappa},t)\hat{u}_k^*(\underline{k}',t)\hat{u}_l^*(\underline{\kappa}-\underline{k}',t)}$$

Recall for real valued function:

$$\hat{u}_i^*(\underline{\kappa},t)=\hat{u}_i(-\underline{\kappa},t)=\hat{u}_i(\underline{\kappa},t)$$

$$-i\kappa_l P_{jk}\sum_{\underline{k'}}\overline{\hat{u}_j(\underline{\kappa},t)\hat{u}_k^*(\underline{k}',t)\hat{u}_l{}^*(\underline{\kappa}-\underline{k}',t)}\ -i\kappa_l P_{jk}\sum_{\underline{k'}}\overline{\hat{u}_j(\underline{\kappa},t)\hat{u}_k^*(\underline{k}',t)\hat{u}_l^*(\underline{\kappa}-\underline{k}',t)}$$

$$=-2i\kappa_l P_{jk}\sum_{\underline{k'}}\langle\hat{u}_j(\underline{\kappa},t)\hat{u}_k^*(\underline{k}',t)\hat{u}_l{}^*(\underline{\kappa}-\underline{k}',t)\rangle$$

Therefore, Eq. (5A) becomes:

$$2\frac{d\hat{E}(\underline{\kappa},t)}{dt}+4\nu\kappa^2\hat{E}(\underline{\kappa},t)=-2i\kappa_l P_{jk}\sum_{\underline{k'}}\langle\hat{u}_j(\underline{\kappa},t)\hat{u}_k^*(\underline{k}',t)\hat{u}_l{}^*(\underline{\kappa}-\underline{k}',t)\rangle$$

Or equivalently

$$\frac{d\hat{E}(\underline{\kappa},t)}{dt}=-2\nu\kappa^2\hat{E}(\underline{\kappa},t)-i\kappa_l P_{jk}\sum_{\underline{k'}}\langle\hat{u}_j(\underline{\kappa},t)\hat{u}_k^*(\underline{k}',t)\hat{u}_l{}^*(\underline{\kappa}-\underline{k}',t)\rangle$$

And since $\hat{E}(\underline{\kappa},t)$ is real, it can be rewritten as

$$\frac{d\hat{E}(\underline{\kappa},t)}{dt}=-2\nu\kappa^2\hat{E}(\underline{\kappa},t)+\kappa_l P_{jk}\Re\left\{-i\sum_{\underline{\kappa'}}\langle\hat{u}_j(\underline{\kappa})\hat{u}_k^*(\underline{\kappa}')\hat{u}_l^*(\underline{\kappa}-\underline{\kappa}')\rangle\right\}$$

A.3

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i\left(\underline{\kappa},t\right)=-i\kappa_m P_{ij}\sum_{\underline{l}}\hat{u}_j\left(\underline{l},t\right)\widehat{u_m}\left(\underline{\kappa}-\underline{l},t\right)\quad(7A)$$

Switching the dummy indices j and m gives

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i\left(\underline{\kappa},t\right)=-i\kappa_j P_{im}\sum_{\underline{l}}\widehat{u_m}\left(\underline{l},t\right)\hat{u}_j\left(\underline{\kappa}-\underline{l},t\right)$$

and replacing the summation on $\underline{l}$ with the equivalent summation on $\underline{l}'=\underline{\kappa}-\underline{l}$:

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i\left(\underline{\kappa},t\right)=-i\kappa_j P_{im}\sum_{\underline{k}-\underline{l}}\widehat{u_m}\left(\underline{k}-\underline{l}',t\right)\hat{u}_j\left(\underline{l}',t\right)$$

$$\left(\frac{d}{dt}+\nu\kappa^2\right)=-i\kappa_j P_{im}\sum_{\underline{l}'}\hat{u}_j\left(\underline{l}',t\right)\widehat{u_m}\left(\underline{k}-\underline{l}',t\right)\quad(8A)$$

Taking the average of the sum of the RHS of Eq. (7A) and (8A), i.e.,

$$(RHS_{1A}+RHS_{2A})/2$$

Gives

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i\left(\underline{\kappa},t\right)=\underbrace{-\frac{i}{2}\left(\kappa_m P_{ij}+\kappa_j P_{im}\right)}_{\boxed{M_{ijm}}}\sum_{\underline{l}}\hat{u}_j\left(\underline{l},t\right)\widehat{u_m}\left(\underline{\kappa}-\underline{l},t\right)$$

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i\left(\underline{\kappa},t\right)=M_{ijm}\left(\underline{k}\right)\sum_{\underline{l}}\hat{u}_j\left(\underline{l},t\right)\widehat{u_m}\left(\underline{\kappa}-\underline{l},t\right)$$

A.4

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i(\underline{\kappa},t)=M_{ijm}(\underline{\kappa})\sum_{\underline{l}}\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t) \quad (9A)$$

The conjugate of Eq. (9A) is equal to

$$\left(\frac{d}{dt}+\nu\kappa^2\right)\hat{u}_i^*(\underline{\kappa},t)=M_{ijm}(\underline{\kappa})\sum_{\underline{l}}\hat{u}_j^*(\underline{l},t)\hat{u}_m^*(\underline{\kappa}-\underline{l},t) \quad (10A)$$

Multiplying Eq. (9A) by $\hat{u}_i^*(\underline{\kappa},t)$ gives

$$\left(\frac{d\hat{u}_i(\underline{\kappa},t)}{dt}+\nu\kappa^2\hat{u}_i(\underline{\kappa},t)\right)\hat{u}_i^*(\underline{\kappa},t)$$
$$=M_{ijm}(\underline{\kappa})\sum_{\underline{l}}\hat{u}_i^*(\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t) \quad (11A)$$

Multiplying Eq. (10A) by $\hat{u}_i(\underline{\kappa},t)$ gives

$$\left(\frac{d\hat{u}_i^*(\underline{\kappa},t)}{dt}+\nu\kappa^2\hat{u}_i^*(\underline{\kappa},t)\right)\hat{u}_i(\underline{\kappa},t)$$
$$=M_{ijm}(\underline{\kappa})\sum_{\underline{l}}\hat{u}_i(\underline{\kappa},t)\hat{u}_j^*(\underline{l},t)\hat{u}_m^*(\underline{\kappa}-\underline{l},t) \quad (12A)$$

Taking the sum of Eq. (11A) and (12A) and averaging yields

$$\left(\frac{d\hat{u}_i(\underline{\kappa},t)}{dt}\hat{u}_i^*(\underline{\kappa},t)+\frac{d\hat{u}_i^*(\underline{\kappa},t)}{dt}\hat{u}_i(\underline{\kappa},t)\right)$$
$$+\Big(\nu\kappa^2\hat{u}_i(\underline{\kappa},t)\hat{u}_i^*(\underline{\kappa},t)$$
$$+\nu\kappa^2\hat{u}_i^*(\underline{\kappa},t)\hat{u}_i(\underline{\kappa},t)\Big)=M_{ijm}(\underline{\kappa})\left(\sum_{\underline{l}}\hat{u}_i^*(\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t)\right.$$
$$\left.+\sum_{\underline{l}}\hat{u}_i(\underline{\kappa},t)\hat{u}_j^*(\underline{l},t)\hat{u}_m^*(\underline{\kappa}-\underline{l},t)\right)$$

First two terms follow same steps as shown in Appendix A.2.

$$2\frac{d\hat{E}(\underline{\kappa},t)}{dt}+4\nu\kappa^2\hat{E}(\underline{\kappa},t)$$
$$=M_{ijm}(\underline{\kappa})\left(\sum_{\underline{l}}\hat{u}_i^*(\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t)\right.$$
$$\left.+\sum_{\underline{l}}\hat{u}_i(\underline{\kappa},t)\hat{u}_j^*(\underline{l},t)\hat{u}_m^*(\underline{\kappa}-\underline{l},t)\right)$$

$$\frac{d\hat{E}(\underline{\kappa},t)}{dt}+2\nu\kappa^2\hat{E}(\underline{\kappa},t)$$
$$=\frac{1}{2}M_{ijm}(\underline{\kappa})\left(\sum_{\underline{l}}\hat{u}_i(-\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t)\right.$$
$$\left.+\sum_{\underline{l}}\hat{u}_i(\underline{\kappa},t)\hat{u}_j(-\underline{l},t)\hat{u}_m(-\underline{\kappa}+\underline{l},t)\right)$$

Replacing $\underline{l}$ with $-\underline{l}$ in the second summation:

$$\frac{d\hat{E}(\underline{\kappa},t)}{dt}+2\nu\kappa^2\hat{E}(\underline{\kappa},t)$$
$$=\frac{1}{2}M_{ijm}(\underline{\kappa})\left(\sum_{\underline{l}}\hat{u}_i(-\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t)\right.$$
$$\left.+\sum_{-\underline{l}}\hat{u}_i(\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(-\underline{\kappa}-\underline{l},t)\right)$$

However, summation indices are dummy variables, such that the second summation can have either $-\underline{l}$ or $\underline{l}$ as the summation index without changing the result:

$$\frac{d\hat{E}(\underline{\kappa},t)}{dt} + 2\nu\kappa^2\hat{E}(\underline{\kappa},t) = \frac{1}{2}M_{ijm}(\underline{\kappa})\left(\sum_{\underline{l}}\hat{u}_i(-\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(\underline{\kappa}-\underline{l},t) + \sum_{\underline{l}}\hat{u}_i(\underline{\kappa},t)\hat{u}_j(\underline{l},t)\hat{u}_m(-\underline{\kappa}-\underline{l},t)\right)$$