Chapter 5: Energy Decay in Isotropic Turbulence

Part 4: Energy Spectrum Equation

In Part 3, an equation for $\mathcal{R}_{ij}(\underline{r}, t)$ was derived by taking the average of $u_i(\underline{x}, t)$ times NS_j at $\underline{y}$ + average of $u_j\left(\underline{y}, t\right)$ times NS_i at $\underline{x}$ subject assumptions of homogeneous turbulence:

$$\frac{\partial \mathcal{R}_{ij}}{\partial t}(\underline{r}, t) = \frac{\partial S_{jk,i}}{\partial r_k}(-\underline{r}, t) + \frac{\partial S_{ik,j}}{\partial r_k}(\underline{r}, t) - \frac{1}{\rho}\frac{\partial \mathcal{K}_i}{\partial r_j}(\underline{r}, t) - \frac{1}{\rho}\frac{\partial \mathcal{K}_j}{\partial r_i}(-\underline{r}, t) + 2\nu \frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k^2}(\underline{r}, t)$$

Where:

$$\mathcal{K}_i\left(\underline{x}, \underline{y}, t\right) = \overline{u_i(\underline{x}, t) p\left(\underline{y}, t\right)}$$

is the two-point pressure-velocity correlation vector.

Recall Fourier transform definitions of the velocity-spectrum $\mathcal{E}_{ij}(\underline{\kappa}, t)$ and $\mathcal{R}_{ij}(\underline{r}, t)$ tensors:

$$\mathcal{E}_{ij}(\underline{\kappa}, t) = \frac{1}{(2\pi)^3}\int_{\forall} \mathcal{R}_{ij}(\underline{r}, t) e^{i\underline{\kappa}\cdot\underline{r}} \underline{dr}$$

$$\mathcal{R}_{ij}(\underline{r}, t) = \int_{\forall} \mathcal{E}_{ij}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} \underline{d\kappa}$$

$$\underline{dr} = dr_1 dr_2 dr_3$$

$$\underline{d\kappa} = d\kappa_1 d\kappa_2 d\kappa_3$$

These two equations provide a means of decomposing turbulence correlations into contributions from a continuous range of scales as represented by Fourier components $e^{i\underline{\kappa}\cdot\underline{r}}$.

Fourier transform of the $\mathcal{R}_{ij}$ equation gives,

$$\frac{\partial \mathcal{E}_{ij}}{\partial t}\left(\underline{\kappa}, t\right) = T_{ij}\left(\underline{\kappa}, t\right) + P_{ij}\left(\underline{\kappa}, t\right) - 2\nu\kappa^2\mathcal{E}_{ij}\left(\underline{\kappa}, t\right) \quad (1)$$

Which is a 2nd order tensor equation, where:

$$T_{ij}\left(\underline{\kappa}, t\right) = \frac{1}{(2\pi)^3}\int_{\forall} S_{ij}\left(\underline{r}, t\right)e^{i\underline{\kappa}\cdot\underline{r}}\underline{dr} \quad (2)$$

Is the Fourier transform of

$$S_{ij}\left(\underline{r}, t\right) = \frac{\partial S_{jk,i}}{\partial r_k}\left(-\underline{r}, t\right) + \frac{\partial S_{ik,j}}{\partial r_k}\left(\underline{r}, t\right) \quad (3)$$

i.e., $S_{ij}\left(\underline{r}, t\right) = \int_{\forall} T_{ij}\left(\underline{\kappa}, t\right)e^{-i\underline{\kappa}\cdot\underline{r}}\underline{d\kappa}$

$T_{ij}\left(\underline{\kappa}, t\right)$ = rate of transfer of energy (gains or losses) between different scales of turbulent motion due to vortex stretching and re-orientation.

$$P_{ij}\left(\underline{\kappa}, t\right) = -\frac{1}{(2\pi)^3}\int_{\forall}\left[\frac{1}{\rho}\frac{\partial \mathcal{K}_i}{\partial r_j}\left(\underline{r}, t\right) + \frac{1}{\rho}\frac{\partial \mathcal{K}_j}{\partial r_i}\left(-\underline{r}, t\right)\right]e^{i\underline{\kappa}\cdot\underline{r}}\underline{dr}$$

is the Fourier transformed pressure velocity term.

$P_{ij}\left(\underline{\kappa}, t\right) = 0$ for isotropic turbulence = influence of pressure field on bringing anisotropic turbulence to an isotropic state.

The viscous dissipation term is evaluated as follows:

$$\frac{2\nu}{(2\pi)^3}\int_{\forall}\frac{\partial^2\mathcal{R}_{ij}}{\partial r_k^2}\left(\underline{r}, t\right)e^{i\underline{\kappa}\cdot\underline{r}}\underline{dr} = \frac{2\nu}{(2\pi)^3}\int_{\forall} i^2\kappa^2\mathcal{R}_{ij}e^{i\underline{\kappa}\cdot\underline{r}}\underline{dr} = -2\nu\kappa^2\mathcal{E}_{ij}\left(\underline{\kappa}, t\right)$$

Contracting indices in Eq. (1), the pressure term drops out and integrating over a spherical shell converts $\mathcal{E}_{ij}$ to the energy spectrum $E(\kappa,t) = \frac{1}{2}\int_{|\underline{\kappa}|=\kappa} \mathcal{E}_{ii}(\underline{\kappa},t)\,\kappa^2 d\Omega$ with $K = \int_0^\infty E(\kappa,t)d\kappa$ to obtain the energy spectrum equation:

$$\frac{\partial E}{\partial t}(\kappa,t) = T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t) \quad (4)$$

Which is a scalar equation, where:

$$T(\kappa,t) = \frac{1}{2}\int_{|\underline{\kappa}|=\kappa} T_{ii}(\underline{\kappa},t)\,\kappa^2 d\Omega = 2\pi\kappa^2 T_{ii}(\kappa,t) \quad (5)$$

$d\Omega$ =elemental solid angle $dS(\kappa) = \kappa^2 \mathrm{d}\Omega$ $\oint dS(\kappa) = 4\pi\kappa^2$

is the transfer term and $T_{ii} = \mathcal{F}_k\{S_{ii}(\underline{r},t)\}$, as will be shown.

Next assume in addition to homogeneous also isotropic turbulence and substitute the isotropic form of $S_{ik,j}$, obtained in Chapter 4 Part 2, into Eq. (3) which gives,

$$S_{ii}(r,t) = u_{rms}^3 \frac{1}{r^2}\frac{d}{dr}\left(r^3 \frac{dk}{dr}(r,t) + 4r^2 k(r,t)\right) \quad (6)$$

Appendix A.1

Where $k(r,t) = S_{111}(r\hat{e}_1)/u_{rms}^3$ is the previously defined (Chapter 4 Part 2) correlation function, not to be confused with wave number κ. Note that $S_{ii} = f(r)$ only in isotropic turbulence.

Using Eq. (6) and contracting indices, Eq. (2) becomes [Bernard Eq. (5.121)],

$$T_{ii}(\kappa,t) = \frac{1}{(2\pi)^2}\int_0^\infty S_{ii}(r,t) r^2 \frac{\sin \kappa r}{\kappa r} dr = f(\kappa) \neq f(\underline{\kappa}) \quad (7)$$

Appendix A.2

Substituting Eq. (7) into Eq. (5) gives,

$$T(\kappa,t) = \frac{1}{2\pi}\int_0^\infty S_{ii}(r,t)\kappa r \sin\kappa r \, dr = 2\pi\kappa^2 T_{ii}(\kappa,t) \quad (8)$$

Which can be interpreted as the Fourier transform of S_{ii} such that the inverse transform is,

$$S_{ii}(r,t) = 2\int_0^\infty T(\kappa,t)\frac{\sin\kappa r}{\kappa r}d\kappa = 2\int_0^\infty (2\pi\kappa^2 T_{ii}(\kappa,t))\frac{\sin\kappa r}{\kappa r}d\kappa \quad (9)$$

Substituting Eq. (6) into Eq. (8) and integrating by parts twice gives

$$T(\kappa,t) = \frac{u_{rms}^3}{\pi}\int_0^\infty \kappa\,[(3-\kappa^2 r^2)\sin(\kappa r) - 3\kappa r\cos(\kappa r)]k(r,t)dr \quad (10)$$ Appendix A.3

Which shows that $k(r,t)$ determines the rate of energy transfer between the scales of turbulence, as shown later.

Integrating Eq. (4) between 0 and ∞ and using

$$K(t) = \int_0^\infty E(\kappa,t)d\kappa$$

$$\varepsilon = 2\nu\int_0^\infty \kappa^2 E(\kappa,t)d\kappa$$

Gives

$$\frac{dK(t)}{dt} = \int_0^\infty T(\kappa,t)d\kappa - \varepsilon$$

And in isotropic turbulence

$$\frac{dK}{dt} = -\varepsilon$$

Therefore, in isotropic turbulence (alternate derivation to follow)

$$\int_0^\infty T(\kappa,t)d\kappa = 0$$

i.e., net energy transfer between scales equals zero or in other words gains and losses are conserved.

When $f(r,t)$ is known, $E(\kappa,t)$ can be evaluated using:

$$E(\kappa) = \frac{\overline{u^2}}{\pi}\int_0^\infty \big(3f(r) + rf'(r)\big)\kappa r \sin(\kappa r)dr \qquad (11)$$ Appendix A.4

For the final decay, $f(r,t) = e^{-\frac{r^2}{2\lambda_g^2}}$, such that $E(\kappa)$ for the final decay is given by:

$$E(\kappa,t) = \frac{u_{rms}^2\lambda_g}{\sqrt{2\pi}}\left(\kappa\lambda_g\right)^4 e^{-\frac{1}{2}\left(\kappa\lambda_g\right)^2} \qquad (12)$$ Appendix A.5

Substituting Eq. (12) into Eq. (4) yields

$$T(\kappa,t) = E(\kappa,t)\frac{u_{rms}}{\lambda_g}\left(\left(\kappa\lambda_g\right)^2 - 5\right)$$

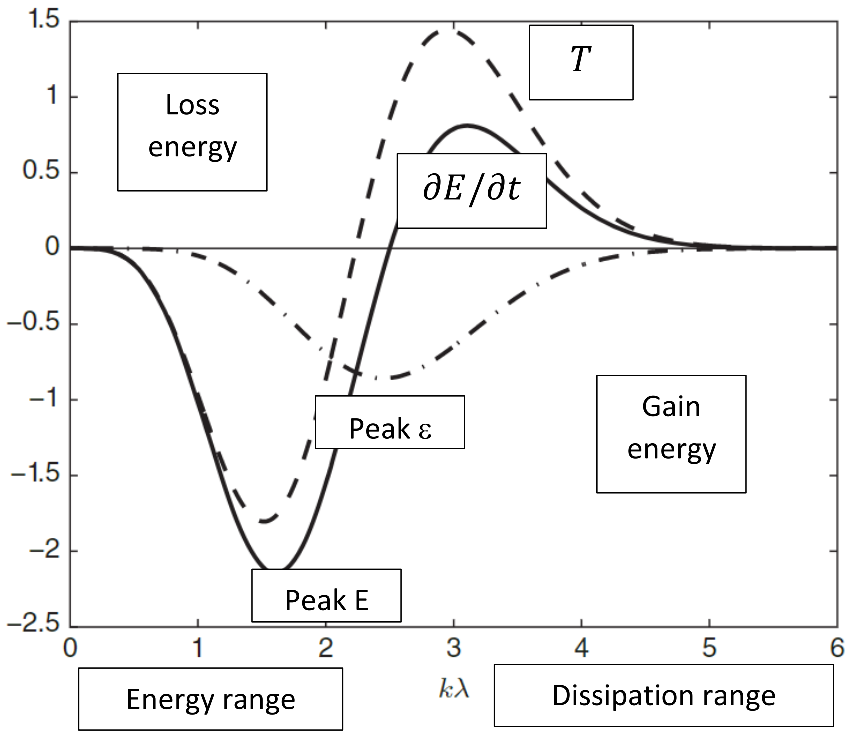

Figure 5.8 Energy spectrum budget in final period. —, $\partial E/\partial t$; – –, transfer term; – · –, dissipation. In this illustration $R_\lambda = 10$, with λ denoting λ_g.

Transfer term shows that in final decay scales for which $\kappa < \sqrt{5}/\lambda_g$ lose energy to those for which $\kappa > \sqrt{5}/\lambda_g$. $\sqrt{5} = 2.2$

$\kappa_e = 2/\lambda_g$ peak E

See Bernard Problem 4.1

$\kappa_d = \sqrt{6}/\lambda_g$ peak dissipation $\sqrt{6} = 2.4$

In this example, κ_e and κ_d not well separated due to low R_λ.

No inertial range for above form of $E(\kappa, t)$ equation. As λ_g rises the balance in Fig. 5.8 shifts to smaller wave number, i.e., larger scale → higher wave numbers lose all their energy before lower wave numbers.

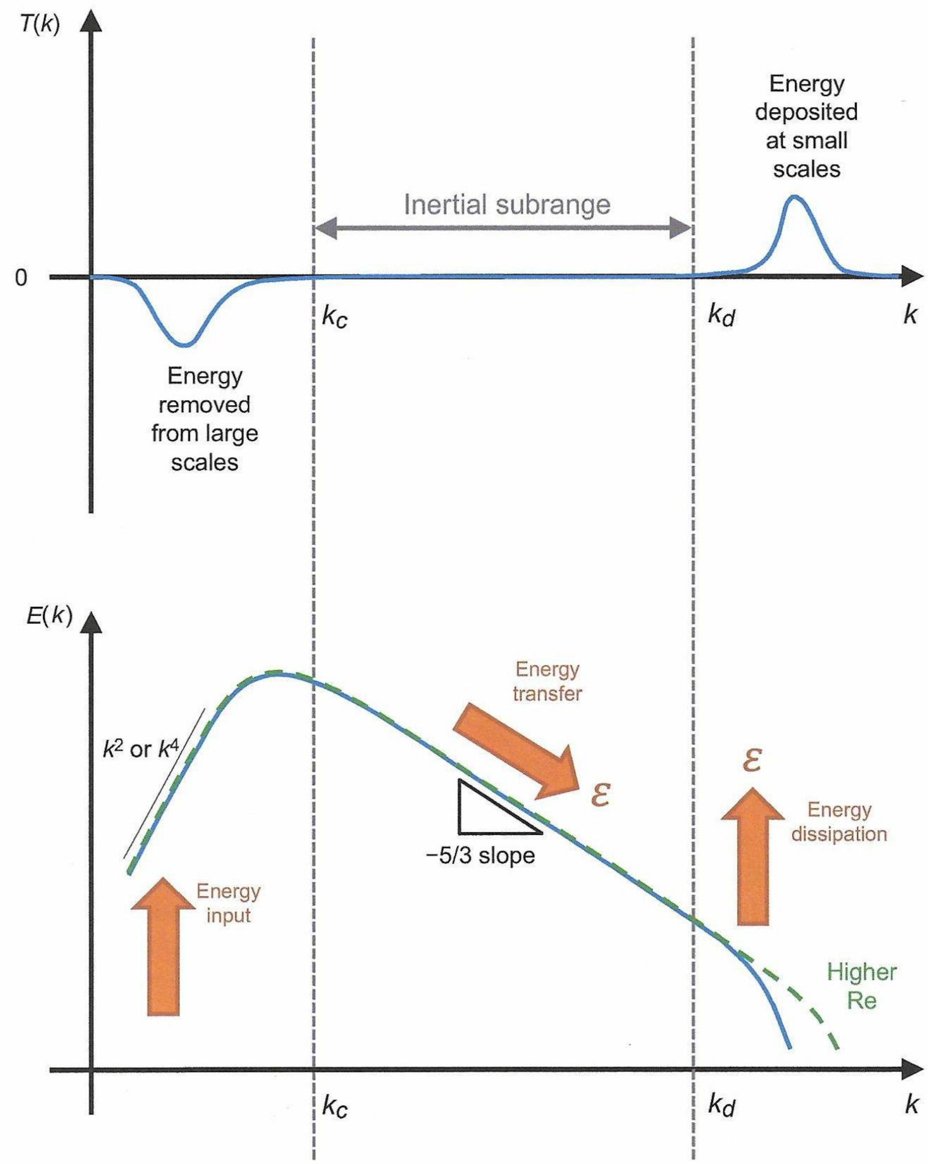

Fig. 5.3 Schematics of variation of (top) $T(k)$ and (bottom) $E(k)$ with k on logarithmic axes for homogeneous isotropic turbulence at sufficiently high Re. The occurrence of the $-5/3$ power law at intermediate wavenumbers in the $E(k)$ spectrum will be discussed in Section 5.5.

Although the expression for the dissipation rate, $\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$, includes viscosity, its magnitude is set by the large-scale energy production rate, which is independent of viscosity.

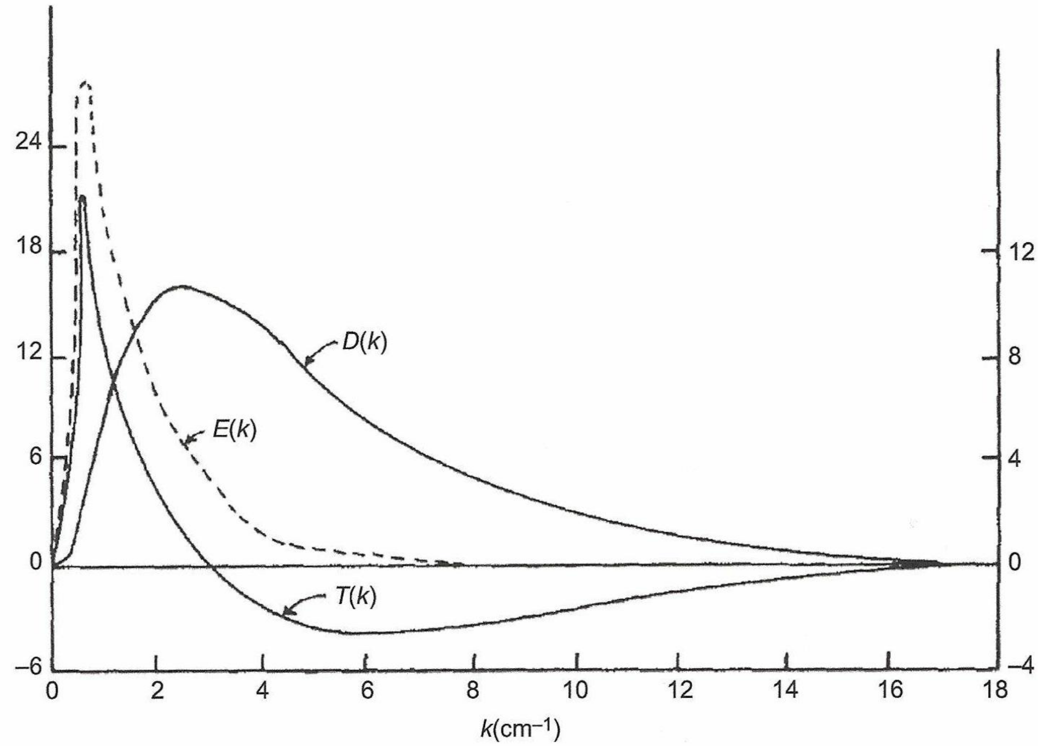

Fig. 5.4 Variation of $E(k)$, $D(k)$, and $T(k)$ on linear axes from numerical simulation of homogeneous isotropic turbulence. (See (5.8) for the definition of $D(k)$.) Note that $T(k)$ is plotted with the opposite sign. In this simulation, $T(k)$ does not plateau at zero at intermediate wavenumbers since the Reynolds number of these pioneering simulations is relatively low ($Re_\lambda \approx 40$), and $E(k)$ does not have a well-defined inertial subrange. (Image credit: Clark, Ferziger and Reynolds (1979), figure 3)

Shows that $T(k) \approx \varepsilon$ in the inertial subrange.

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Consider the integrated form of Eq. (4)

$$\frac{d}{dt}\int_0^{\kappa} E(\kappa,t)d\kappa = \int_0^{\kappa} T(\kappa,t)d\kappa - 2\nu\int_0^{\kappa} \kappa^2 E(\kappa,t)d\kappa \quad (13)$$

If, as previously done, the upper limit of the integral is increased to $\kappa = \infty$

$$\int_0^{\infty} T(\kappa,t)d\kappa = 0 \quad (14)$$

i.e., net energy transfer between scales is zero or in other words gains and losses are conserved.

An alternative derivation for Eq. (14) can be obtained starting from Eq. (9). Specifying $r = 0$ in Eq. (9) recovers Eq. (14)

$$\int_0^{\infty} T(\kappa,t) \underbrace{\lim_{r\to 0}\frac{\sin\kappa r}{\kappa r}}_{\boxed{1}} d\kappa = \frac{1}{2}S_{ii}(0,t)\text{=}0$$

If $S_{ii}(0,t) = 0$.

An alternative form of Eq. (6) for $S_{ii}(r,t)$ is:

$$S_{ii}(r,t) = u_{rms}^3\left(7\frac{dk}{dr}(r,t) + r\frac{d^2k}{dr^2}(r,t) + \frac{8}{r}k(r,t)\right) \quad (15)$$ Appendix A.6

Substituting the Taylor expansion of $k(r,t) = \frac{r^3}{3!}k'''(0,t) + \cdots$ into Eq. (15) gives

$$S_{ii}(r,t) = \frac{35}{6}r^2k'''(0,t)$$ Appendix A.7

Which shows that $S_{ii}(r,t)$ behaves like r^2 for small values of r. Consequently,

$$S_{ii}(0,t) = 0$$

And Eq. (14) must hold.

Therefore, for $\kappa = \infty$, Eq. (13) becomes,

$$\frac{d}{dt}\int_0^\infty E(\kappa,t)d\kappa = -2\nu\int_0^\infty \kappa^2E(\kappa,t)d\kappa$$

$$\frac{dK(t)}{dt} = -\varepsilon$$

The LHS shows the change of total kinetic energy of turbulence and since there are no external energy sources, LHS must equal the dissipation caused by viscous effects.

$T(\kappa, t)$ is the Fourier transform of $S_{ii}(r, t)$, which is related to $k(r, t)$. In the K-H equation, it was previously stated that the term:

$$u_{rms}^3 \left[\frac{\partial k}{\partial r} + \frac{4}{r} k\right]$$

represents inertial processes. However, it can also be interpreted as a "convective" action in the transport of $f(r, t)$, caused by the interaction of eddies of different sizes.

Similarly, the term $\int_0^\kappa T(\kappa, t) d\kappa$ can be interpreted as the interaction of eddies of different wave numbers, transferring energy by inertial effects to or from the eddies in region 0 to κ, which is the reason $T(\kappa, t)$ is referred to as the energy-transfer-spectrum function.

Neglecting the interaction of eddies in Eq. (4) gives

$$\frac{\partial E}{\partial t}(\kappa, t) = -2\nu\kappa^2 E(\kappa, t)$$

And integrating

$$E(\kappa, t) = E(\kappa, t_0) \exp[-2\nu\kappa^2 (t - t_0)] \quad (16)$$

Comparing this expression with Eq. (12)

$$E(\kappa,t) = \frac{u_{rms}^2 \lambda_g}{\sqrt{2\pi}} \left(\kappa\lambda_g\right)^4 e^{-\frac{1}{2}\left(\kappa\lambda_g\right)^2} \quad (12)$$

The two exponential forms are equivalent for $t \gg t_0 \rightarrow t - t_0 \approx t$ since (Hinze p.210, for final decay)

$$\lambda_f = \sqrt{8\nu t} \;\; \Rightarrow \lambda_g = \sqrt{4\nu t}$$

$$\boxed{\lambda_f = \sqrt{2}\lambda_g}$$

$$\exp[-2\nu\kappa^2 t] = \exp\left[-\frac{1}{2}\kappa^2\sqrt{4\nu t}\sqrt{4\nu t}\right] = \exp -\frac{1}{2}\left(\kappa\lambda_g\right)^2$$

Moreover, assuming IC

$$E(\kappa,t_0) = \frac{u_{rms}^2 \lambda_g}{\sqrt{2\pi}} \left(\kappa\lambda_g\right)^4$$

Eq. (12) and Eq. (15) become identical.

Eq. (12) shows that the decrease of kinetic energy with time occurs at a higher rate for large wave number eddies and $E(\kappa,t)$ increases very rapidly, proportional to κ^4 for small wave numbers, and decreases monotonously to zero as κ increases.

Appendix A

A.1

$$S_{ijl}(\underline{r}) = u_{rms}^3 \left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_j r_l}{2r^3} - \frac{k}{2}\delta_{ij}\frac{r_l}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

For $j = k$ and $l = i$

$$S_{iki}(\underline{r}) = u_{rms}^3 \left[\left(k - r\frac{dk}{dr}\right)\frac{r_k}{2r} - \frac{k}{2}\delta_{ik}\frac{r_i}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{ii}\frac{r_k}{r} + \delta_{ik}\frac{r_i}{r}\right)\right]$$

$$= u_{rms}^3 \left[\left(k - r\frac{dk}{dr}\right)\frac{r_k}{2r} - \frac{k}{2}\frac{r_k}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(4\frac{r_k}{r}\right)\right]$$

$$= u_{rms}^3 \left[-\frac{dk}{dr}\frac{r_k}{2} + \frac{r_k}{r^2}\frac{d(kr^2)}{dr}\right]$$

$$= u_{rms}^3 \left[\frac{r_k}{2}\frac{dk}{dr} + 2k\frac{r_k}{r}\right]$$

Taking a derivative with respect to r_k yields

$$\frac{\partial S_{iki}}{\partial r_k}(\underline{r}) = u_{rms}^3 \left[\frac{r}{2}k''(r) + \frac{7}{2}k'(r) + \frac{4k(r)}{r}\right] \quad (1A)$$

As shown in Chapter 5 Part 1 Appendix A.2.

Similarly,

$$\frac{\partial S_{iki}}{\partial r_k}(-\underline{r}) = u_{rms}^3 \left[-\frac{r}{2}k''(-r) + \frac{7}{2}k'(-r) - \frac{4k(-r)}{r}\right] \quad (2A)$$

And using the following relations

$$k(r) = -k(-r)$$

$$k'(r) = k'(-r)$$

$$k''(r) = -k''(-r)$$

Into Eq. (2A) gives

$$\frac{\partial S_{iki}}{\partial r_k}(-\underline{r}) = u_{rms}^3\left[\frac{r}{2}k''(r) + \frac{7}{2}k'(r) + \frac{4k(r)}{r}\right] \quad (3A)$$

Now, defining

$$S_{ii}(\underline{r},t) = \frac{\partial S_{ik,i}}{\partial r_k}(-\underline{r},t) + \frac{\partial S_{ik,i}}{\partial r_k}(\underline{r},t)$$

And using Eqs. (1A) and (3A) yields

$$S_{ii}(\underline{r},t) = u_{rms}^3\left[rk''(r) + 7k'(r) + \frac{8k(r)}{r}\right]$$

Multiplying and dividing by r^2 results in

$$S_{ii}(\underline{r},t) = \frac{1}{r^2}u_{rms}^3[r^3k'' + 7r^2k' + 8kr]$$

Or equivalently

$$S_{ii}(r,t) = u_{rms}^3\frac{1}{r^2}\frac{d}{dr}(r^3k' + 4r^2k)$$

A.2

$$T_{ij}(\underline{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{\forall} S_{ij}(\underline{r}, t) e^{i\underline{\kappa}\cdot\underline{r}} \underline{dr} \qquad (4A)$$

Converting to spherical coordinates

$$\underline{r} = r(\sin\theta \cos\phi\, \hat{e}_1 + \sin\theta \sin\phi\, \hat{e}_2 + \cos\theta\, \hat{e}_3)$$

Also, for isotropic turbulence

$$S_{ii}(r,t) = u_{rms}^3 \frac{1}{r^2} \frac{d}{dr}\left(r^3 \frac{dk}{dr}(r,t) + 4r^2 k(r,t)\right)$$

Such that S_{ii} is only a function of r, a scalar quantity.

Assume that,

$$\underline{\kappa} = \kappa \hat{e}_3$$

Which is possible due to the isotropy hypothesis, i.e., invariance under rotation and reflection.

Therefore, Eq. (4A) for T_{ii} becomes,

$$T_{ii}(\kappa) = \frac{1}{(2\pi)^3} \int_{\forall} S_{ii}(r) e^{i\underline{\kappa}\cdot\underline{r}} \underline{dr}$$

$$= \frac{1}{(2\pi)^3} \int_0^\infty dr S_{ii}(r) \int_0^{2\pi} d\phi \int_0^\pi d\theta e^{i\kappa r \cos\theta} r^2 \sin(\theta)$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty r^2 dr S_{ii}(r) \int_0^\pi d\theta e^{i\kappa r \cos\theta} \sin(\theta)$$

Multiply and divide by $-i\kappa r$

$$T_{ii}(\kappa) = -\frac{1}{i\kappa r (2\pi)^2} \int_0^\infty r^2 dr S_{ii}(r) \int_0^\pi -i\kappa r e^{i\kappa r \cos\theta} \sin(\theta)\, d\theta$$

And use the relation:

$$\frac{d}{d\theta} e^{i\kappa r \cos\theta} = -i\kappa r \sin(\theta) e^{i\kappa r \cos\theta}$$

Gives

$$T_{ii}(\kappa) = -\frac{1}{i\kappa r(2\pi)^2}\int_0^\infty r^2 dr S_{ii}(r) \int_0^\pi \frac{d}{d\theta} e^{i\kappa r\cos\theta}\, d\theta$$

$$= -\frac{1}{i\kappa r(2\pi)^2}\int_0^\infty r^2 dr S_{ii}(r) \left[e^{i\kappa r\cos\theta}\right]_0^\pi$$

$$= \frac{2}{\kappa r(2\pi)^2}\int_0^\infty r^2 S_{ii}(r) \underbrace{\frac{e^{i\kappa r} - e^{-i\kappa r}}{2i}}_{\boxed{\sin(\kappa r)}} dr$$

$$T_{ii}(\kappa) = \frac{2}{(2\pi)^2}\int_0^\infty r^2 S_{ii}(r)\ \frac{\sin(\kappa r)}{\kappa r} dr$$

A.3

$$T_{ii}(\kappa,t)=\frac{1}{(2\pi)^2}\int_0^\infty S_{ii}(r,t)r^2\frac{\sin\kappa r}{\kappa r}dr \qquad (5A)$$

$$S_{ii}(r,t)=u_{rms}^3\frac{1}{r^2}\frac{d}{dr}\left(r^3\frac{dk}{dr}(r,t)+4r^2k(r,t)\right) \qquad (6A)$$

$$T(\kappa,t)=\frac{1}{2}\int_{|\underline{\kappa}|=\kappa}T_{ii}\left(\underline{\kappa},t\right)\kappa^2 d\Omega=2\pi\kappa^2T_{ii}(\kappa,t) \qquad (7A)$$

$$T(\kappa,t)=\frac{u_{rms}^3}{\pi}\int_0^\infty \kappa\,[(3-\kappa^2r^2)\sin(\kappa r)-3\kappa r\cos(\kappa r)]k(r,t)dr$$

Substituting Eq. (6A) into (5A) gives

$$T_{ii}(\kappa,t)=\frac{1}{2\pi^2}\int_0^\infty u_{rms}^3\frac{d}{dr}(r^3k'(r,t)+4r^2k(r,t))\frac{\sin\kappa r}{\kappa r}dr$$

$$=\frac{u_{rms}^3}{2\pi^2}\int_0^\infty \underbrace{\frac{d}{dr}\left(r^3k'(r,t)+4r^2k(r,t)\right)}_{\boxed{dv}}\underbrace{\frac{\sin\kappa r}{\kappa r}}_{\boxed{u}}\,dr$$

Integration by parts

$$\int u dv=uv-\int v du$$

$$T_{ii}(\kappa,t)=\left[\frac{u_{rms}^3}{2\pi^2\kappa}\left(r^2k'(r,t)+4rk(r,t)\right)\sin\kappa r\right]_0^\infty$$
$$-\frac{u_{rms}^3}{2\pi^2}\int_0^\infty\left(r^3k'(r,t)+4r^2k(r,t)\right)\frac{d}{dr}\left(\frac{\sin\kappa r}{\kappa r}\right)dr \qquad (8A)$$

Where:

$$\left[\frac{u_{rms}^3}{2\pi^2\kappa}\left(r^2k'(r,t)+4rk(r,t)\right)\sin\kappa r\right]_0^\infty$$
$$=\frac{u_{rms}^3}{2\pi^2\kappa}\left[\left(r^2k'(r,t)+4rk(r,t)\right)\sin\kappa r\right]_\infty$$
$$-\frac{u_{rms}^3}{2\pi^2\kappa}\cancel{\left[\left(0^2k'(0,t)+4\cdot 0k(0,t)\right)\sin 0\right]}$$

And using the fact that at large r, the triple correlation $k(r,t)$ behaves like r^{-4} (see Part 3)

$$\lim_{r\to\infty} r^2k'(r,t)=\lim_{r\to\infty} rk(r,t)=0$$

Therefore,

$$\left[\frac{u_{rms}^3}{2\pi^2\kappa}\left(r^2k'(r,t)+4rk(r,t)\right)\sin\kappa r\right]_0^\infty=0$$

And Eq. (8A) becomes:

$$T_{ii}(\kappa,t)=-\frac{u_{rms}^3}{2\pi^2}\int_0^\infty\left(r^3k'^{(r,t)}+4r^2k(r,t)\right)\frac{d}{dr}\left(\frac{\sin\kappa r}{\kappa r}\right)dr \qquad (9A)$$

Evaluating the derivative of $\sin\kappa r/\kappa r$ as

$$\frac{d}{dr}\left(\frac{\sin\kappa r}{\kappa r}\right)=\frac{\cos\kappa r}{r}-\frac{\sin\kappa r}{\kappa r^2}$$

And substituting into Eq. (9A) gives

$$T_{ii}(\kappa,t) = -\frac{u_{rms}^3}{2\pi^2}\int_0^\infty \left(r^3k'(r,t) + 4r^2k(r,t)\right)\left[\frac{\cos\kappa r}{r} - \frac{\sin\kappa r}{\kappa r^2}\right]dr$$

$$= -\frac{u_{rms}^3}{2\pi^2}\int_0^\infty \left(r^2k'(r,t)\cos\kappa r + 4rk(r,t)\cos\kappa r - \frac{r}{\kappa}k'(r,t)\sin\kappa r - \frac{4}{\kappa}k(r,t)\sin\kappa r\right)dr$$

Using the Product Rule of derivatives

$$T_{ii}(\kappa,t) = -\frac{u_{rms}^3}{2\pi^2}\int_0^\infty \left(\frac{d\left(r^2k(r,t)\right)}{dr} - 2rk(r,t)\right)\cos\kappa r + 4rk(r,t)\cos\kappa r - \frac{1}{\kappa}\left(\frac{d\left(rk(r,t)\right)}{dr} - k(r,t)\right)\sin\kappa r - \frac{4}{\kappa}k(r,t)\sin\kappa r\, dr$$

$$= -\frac{u_{rms}^3}{2\pi^2}\int_0^\infty \left(\frac{d\left(r^2k(r,t)\right)}{dr} + 2rk(r,t)\right)\cos\kappa r - \frac{1}{\kappa}\left(\frac{d\left(rk(r,t)\right)}{dr} + 3k(r,t)\right)\sin\kappa r\, dr$$

Grouping terms depending on $k(r,t)$ and integrating by parts again

$$
\begin{aligned}
T_{ii}(\kappa,t) = &-\frac{u_{rms}^3}{2\pi^2}\left(\int_0^\infty 2rk(r,t)\cos\kappa r - \frac{3k(r,t)}{\kappa}\sin\kappa r\, dr\right) \\
&- \frac{u_{rms}^3}{(2\pi)^2}\int_0^\infty \underbrace{\left(\frac{d\left(r^2k(r,t)\right)}{dr}\right)}_{\boxed{dv}} \underbrace{\cos\kappa r}_{\boxed{u}}\, dr \\
&+ \frac{u_{rms}^3}{\kappa(2\pi)^2}\int_0^\infty \underbrace{\left(\frac{d\left(rk(r,t)\right)}{dr}\right)}_{\boxed{dv}} \underbrace{\sin\kappa r}_{\boxed{u}}\, dr
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{u_{rms}^3}{2\pi^2}\left(\int_0^\infty 2rk(r,t)\cos\kappa r - \frac{3k(r,t)}{\kappa}\sin\kappa r\, dr\right) \\
&- \frac{u_{rms}^3}{2\pi^2}\left[\cancel{r^2k(r,t)\cos\kappa r|_0^\infty} + \int_0^\infty \kappa r^2k(r,t)\sin\kappa r\, dr\right] \\
&+ \frac{u_{rms}^3}{2\pi^2\kappa}\left[\cancel{rk(r,t)\sin\kappa r|_0^\infty} - \int_0^\infty \kappa rk(r,t)\cos\kappa r\, dr\right]
\end{aligned}
$$

Where the fact that at large r, the triple correlation $k(r,t)$ behaves like r^{-4} was invoked.

$$
\begin{aligned}
T_{ii} = -\frac{u_{rms}^3}{2\pi^2}\int_0^\infty &\left[3rk(r,t)\cos\kappa r - \frac{3k(r,t)}{\kappa}\sin\kappa r\right. \\
&\left. + \kappa r^2k(r,t)\sin\kappa r\right] dr \quad (10A)
\end{aligned}
$$

Substituting Eq. (10A) into (7A) gives

$$T(\kappa,t) = \frac{\kappa^2 u_{rms}^3}{\pi}\int_0^\infty \left[-3rk(r,t)\cos\kappa r + \frac{3k(r,t)}{\kappa}\sin\kappa r - \kappa r^2 k(r,t)\sin\kappa r\right] dr$$

$$= \frac{u_{rms}^3}{\pi}\int_0^\infty \left[-3\kappa^2 rk(r,t)\cos\kappa r + 3\kappa k(r,t)\sin\kappa r - \kappa^3 r^2 k(r,t)\sin\kappa r\right] dr$$

$$T(\kappa,t) = \frac{u_{rms}^3}{\pi}\int_0^\infty \kappa\left[(3-\kappa^2 r^2)\sin(\kappa r) - 3\kappa r\cos(\kappa r)\right]k(r,t)dr$$

A.4

In Chapter 2, the velocity spectrum tensor was defined as

$$\mathcal{E}_{ij}(\underline{\kappa}, t) = \frac{1}{(2\pi)^3}\int_{\forall} \mathcal{R}_{ij}(\underline{r}, t) e^{i\underline{\kappa}\cdot\underline{r}} \underline{dr}$$

Using a contraction of indices

$$\mathcal{E}_{ii}(\underline{\kappa}, t) = \frac{1}{(2\pi)^3}\int_{\forall} \mathcal{R}_{ii}(\underline{r}, t) e^{i\underline{\kappa}\cdot\underline{r}} \underline{dr}$$

And for isotropic turbulence, as shown in Chapter 4 Part 2

$$\mathcal{R}_{ii}(r, t) = 3f(r) + rf'(r) \quad \text{(11A1)}$$

Proves that $\mathcal{R}_{ii}$ is only a function of r, a scalar quantity.

Following the same steps taken for $T_{ii}(\kappa)$ in Appendix A.2 to convert from vector to scalar equation with time dependence implied, the following result is obtained,

$$\mathcal{E}_{ii}(\kappa) = \frac{2}{(2\pi)^2}\int_0^{\infty} r^2 \mathcal{R}_{ii}(r)\, \frac{\sin(\kappa r)}{\kappa r} dr \quad (11A2)$$

It is also possible to relate $\mathcal{E}_{ii}(\kappa)$ to $E(\kappa)$, as shown for relation between 1D and 3D spectra [Chapter 4 Part 2 Eq. (10)], obtaining the equation:

$$E(\kappa) = 2\pi\kappa^2 \mathcal{E}_{ii}(\kappa) \quad (12A)$$

Substituting Eq. (11A) into (12A) gives

$$E(\kappa) = \kappa^2 \frac{1}{\pi}\int_0^{\infty} r^2 \mathcal{R}_{ii}(r)\, \frac{\sin(\kappa r)}{\kappa r} dr \quad (13A)$$

And substituting Eq. (11A1) into (13A) yields

$$E(\kappa) = \frac{\overline{u^2}}{\pi}\int_0^{\infty} \big(3f(r) + rf'(r)\big)\kappa r \sin(\kappa r) dr$$

A.5

$$f(r,t) = e^{-\frac{r^2}{2\lambda_g^2}} \quad (14A)$$

Evaluate 1D spectrum E_{11} using (Chapter 4 Part 5 Eq. (14))

$$E_{11}(\kappa_1,t) = \frac{2}{\pi}\overline{u^2}\int_0^\infty f(r,t)\cos\kappa_1 r\,dr \quad (15A)$$

Substituting Eq. (14A) into (15A) gives

$$E_{11}(\kappa_1,t) = \frac{2}{\pi}\overline{u^2}\int_0^\infty e^{-\frac{r^2}{2\lambda_g^2}}\cos\kappa_1 r\,dr$$

This integral can be reconducted to a differential equation. Differentiate with respect to κ_1

$$\frac{dE_{11}(\kappa_1,t)}{d\kappa_1} = -\frac{2}{\pi}\overline{u^2}\int_0^\infty re^{-\frac{r^2}{2\lambda_g^2}}\sin\kappa_1 r\,dr$$

Use the substitution:

$$de^{-\frac{r^2}{2\lambda_g^2}} = -\frac{r}{\lambda_g^2}e^{-\frac{r^2}{2\lambda_g^2}}dr$$

To obtain

$$\frac{dE_{11}(\kappa_1,t)}{d\kappa_1} = \frac{2}{\pi}\overline{u^2}\lambda_g^2\int_0^\infty \sin\kappa_1 r\,de^{-\frac{r^2}{2\lambda_g^2}}$$

Integrate by parts:

$$\frac{dE_{11}(\kappa_1,t)}{d\kappa_1} = \frac{2}{\pi}\overline{u^2}\lambda_g^2\left\{\cancel{\left[\sin(\kappa_1 r)e^{-\frac{r^2}{2\lambda_g^2}}\right]_0^\infty} - \kappa_1\int_0^\infty \cos\kappa_1 r\,e^{-\frac{r^2}{2\lambda_g^2}}dr\right\}$$

$$\frac{dE_{11}(\kappa_1,t)}{d\kappa_1} = -\frac{2}{\pi}\overline{u^2}\lambda_g^2\kappa_1\int_0^\infty \cos\kappa_1 r\,e^{-\frac{r^2}{2\lambda_g^2}}dr$$

Which is equal to:

$$\frac{dE_{11}(\kappa_1,t)}{d\kappa_1} = -\lambda_g^2\kappa_1 E_{11}(\kappa_1,t)$$

And using separation of variables yields

$$\frac{dE_{11}(\kappa_1,t)}{E_{11}(\kappa_1,t)} = -\lambda_g^2\kappa_1 d\kappa_1$$

$$E_{11}(\kappa_1,t) = C\exp\left(-\frac{\lambda_g^2\kappa_1^2}{2}\right)$$

Where the constant C is found by evaluating

$$E_{11}(0,t) = \frac{2}{\pi}\overline{u^2}\int_0^\infty f(r,t)dr = \frac{2}{\pi}\overline{u^2}\int_0^\infty e^{-\frac{r^2}{2\lambda_g^2}}dr = \frac{\sqrt{2}}{\sqrt{\pi}}\overline{u^2}\lambda_g$$

Such that

$$E_{11}(\kappa_1,t) = \frac{\sqrt{2}}{\sqrt{\pi}}\overline{u^2}\lambda_g\exp\left(-\frac{\lambda_g^2\kappa_1^2}{2}\right) \quad (16A)$$

Using the relation between 1D and 3D spectra (Chapter 4 Part 5 Appendix A.2)

$$E(\kappa_1) = \frac{\kappa_1^2}{2}\frac{d^2E_{11}}{d\kappa_1^2} - \frac{\kappa_1}{2}\frac{dE_{11}}{d\kappa_1}$$

And substituting Eq. (16A) gives

$$E(\kappa_1) = \frac{\sqrt{2}}{\sqrt{\pi}}\overline{u^2}\lambda_g\left[\frac{\kappa_1^2}{2}\left(\lambda_g^4\kappa_1^2 - \lambda_g^2\right)\exp\left(-\frac{\lambda_g^2\kappa_1^2}{2}\right) + \frac{\kappa_1}{2}\kappa_1\lambda_g^2\exp\left(-\frac{\lambda_g^2\kappa_1^2}{2}\right)\right]$$

$$E(\kappa_1) = \frac{\overline{u^2}\lambda_g}{\sqrt{2\pi}}\left(\lambda_g\kappa_1\right)^4\exp\left(-\frac{\lambda_g^2\kappa_1^2}{2}\right)$$

A.6

$$S_{ii}(r,t) = u_{rms}^3 \frac{1}{r^2}\frac{d}{dr}\left(r^3 \frac{dk}{dr}(r,t) + 4r^2 k(r,t)\right)$$

$$= u_{rms}^3 \frac{1}{r^2}\frac{d}{dr}\left(3r^2 \frac{dk}{dr}(r,t) + r^3 \frac{d^2k}{dr^2}(r,t) + 8rk(r,t) + 4r^2 \frac{dk}{dr}(r,t)\right)$$

$$S_{ii}(r,t) = u_{rms}^3 \left(7\frac{dk}{dr}(r,t) + r\frac{d^2k}{dr^2}(r,t) + \frac{8}{r}k(r,t)\right)$$

A.7

$$S_{ii}(r,t) = u_{rms}^3 \left(7\frac{dk}{dr}(r,t) + r\frac{d^2k}{dr^2}(r,t) + \frac{8}{r}k(r,t) \right) \quad (17A)$$

Taylor expansion for $k(r,t)$ and its derivatives

$$k(r,t) \approx \frac{r^3}{3!}k'''(0,t)$$

$$\frac{dk}{dr}(r,t) \approx 3\frac{r^2}{3!}k'''(0,t) = \frac{r^2}{2}k'''(0,t)$$

$$\frac{d^2k}{dr^2}(r,t) \approx 2\frac{r}{2}k'''(0,t) = rk'''(0,t)$$

Substituting these expressions into Eq. (17A) gives

$$S_{ii}(r,t) = u_{rms}^3 \left(7\frac{r^2}{2}k'''(0,t) + r^2k'''(0,t) + \frac{4}{3}r^2k'''(0,t) \right)$$
$$= \frac{35}{6}u_{rms}^3 r^2 k'''(0,t)$$