Derivation -10/7 and -6/5 decay laws

Fourier transform pair for energy spectrum tensor and two-point two-velocity correlation tensor for homogeneous turbulence:

$$\mathcal{E}_{ij}(\underline{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{\forall} e^{i\underline{\kappa}\cdot\underline{r}}\, \mathcal{R}_{ij}(\underline{r}, t)\, d\underline{r}$$

$$\mathcal{R}_{ij}(\underline{r}, t) = \int_{\forall} e^{-i\underline{\kappa}\cdot\underline{r}}\, \mathcal{E}_{ij}(\underline{\kappa}, t)\, d\underline{\kappa}$$

Additionally for isotropic turbulence:

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}\frac{df}{dr}\right]$$

And focusing on $\mathcal{R}_{ii}$:

$$\mathcal{R}_{ii}(r) = \overline{u^2}[3f + rf']$$

Such that (t implied):

$$\mathcal{E}_{ii}(\kappa) = \frac{1}{(2\pi)^3} \int_{\forall} e^{i\underline{\kappa}\cdot\underline{r}}\, \mathcal{R}_{ii}(r)\, d\underline{r}$$

Shifting to spherical coordinates:

Problem 4.7 Carry out the θ and ϕ integration in Eq. (4.70) to then derive Eq. (4.71).

Solution:

Without loss of generality, assume that $\mathbf{k} = k\mathbf{e}_3$ and note that in spherical coordinates $\mathbf{r} = r(\sin\theta\cos\phi\ \mathbf{e}_1 + \sin\theta\sin\phi\ \mathbf{e}_2 + \cos\theta\ \mathbf{e}_3)$. It follow that

$$\begin{aligned}
\mathcal{E}_{ii}(k) &= \frac{1}{(2\pi)^3}\int \mathcal{R}_{ii}(r)e^{\imath\mathbf{k}\cdot\mathbf{r}}d\mathbf{r} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty dr\mathcal{R}_{ii}(r)\int_0^{2\pi} d\phi \int_0^\pi d\theta e^{\imath kr\cos\theta} r^2\sin(\theta) \\
&= \frac{2}{(2\pi)^2}\int_0^\infty dr\mathcal{R}_{ii}(r) r^2 \frac{\sin(kr)}{kr}
\end{aligned}$$

where the integral in θ in the second equation is readily evaluated since

$$\frac{d}{d\theta}e^{\imath kr\cos\theta} = -\imath kr\sin\theta e^{\imath kr\cos\theta}.$$

$$\mathcal{E}_{ii}(\kappa) = \frac{2}{(2\pi)^2}\int_0^\infty \frac{sinkr}{kr} r^2 \mathcal{R}_{ii}(r)\, dr$$

$$\mathcal{E}_{ij}(\underline{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right) = \frac{E(\kappa)}{4\pi\kappa^2} P_{ij}(\kappa)$$

$$E(\kappa) = 2\pi\kappa^2 \mathcal{E}_{ii}(\kappa)$$

$$E(\kappa) = \frac{1}{\pi}\int_0^\infty krsinkr\mathcal{R}_{ii}(r)\ dr$$

For small k:

$$sinkr = kr - \frac{(kr)^3}{3!} + \frac{(kr)^5}{5!}$$

Thus, to third order in $sinkr$:

$$E(\kappa) = \frac{k^2}{\pi}\int_0^\infty r^2 \mathcal{R}_{ii}(r)\ dr - \frac{(k)^4}{3!\,\pi}\int_0^\infty r^4 \mathcal{R}_{ii}(r)\ dr$$

$$\mathcal{R}_{ii}(r) = \overline{u^2}[3f + rf'] = \frac{\overline{u^2}}{r^2}\frac{d}{dr}(r^3 f)$$

$$E(\kappa) = \frac{2k^2}{\pi}\int_0^\infty r^2 \frac{\overline{u^2}}{2r^2}\frac{d}{dr}(r^3 f)\ dr - \frac{(k)^4}{3!\,\pi}\int_0^\infty r^4 \overline{u^2}[3f + rf']\ dr$$

$$E(\kappa) = \frac{2k^2}{\pi}\int_0^\infty r^2 \frac{\overline{u^2}}{2r^2}\frac{d}{dr}(r^3 f)\ dr - \frac{(k)^4}{3!\,\pi}\int_0^\infty r^4 \overline{u^2}[3f + rf']\ dr$$

Other power laws can be deduced on the assumption that certain invariants of the Kármán–Howarth equations exist. The most important invariants are related to the spectral behaviour near the wavenumber origin (or the behaviour of correlation functions for large r). We may write the three-dimensional energy spectrum at low wavenumbers in the form [1]

$$E(\kappa,t) = \frac{2}{\pi} I_{BS}(t)\kappa^2 + \frac{1}{3\pi} I_L(t)\kappa^4 + \cdots, \tag{2.5a}$$

where I_S and I_L, known, respectively, as the Birkhoff-Saffman [48,49] and Loitsiansky [50] invariants, are defined as

$$I_{BS} = u'^2 \int_0^\infty r^2 \frac{1}{2r^2} \frac{\partial}{\partial r}[r^3 f(r,t)] \mathrm{d}r \tag{2.5b}$$

and

$$I_L = u'^2 \int_0^\infty r^4 f(r,t) \mathrm{d}r. \tag{2.5c}$$

In (2.5b) and (2.5c), f is the longitudinal correlation function that depends solely on the scalar value of the separation distance, r. Two classical theories for decaying turbulence based on the existence of one or the invariants have been proposed. If $I_{BS} = 0$ [1], we have $E(\kappa) \propto \kappa^4$ near the origin; in this case, I_L is the invariant of motion, interpreted as the conservation of angular momentum [51]. However, for certain initial conditions where the correlation does not decay faster than r^{-3} at large scales, Birkhoff [48] and Saffman [49] independently argued that I_L diverges and I_{BS} is the invariant of motion. This corresponds to the case $E(\kappa) \propto \kappa^2$ at the origin, and is interpreted as a result of conservation of linear momentum [51].

These constraints on the dynamical motion of turbulence can be exploited to obtain the power law relation for u'^2 and L. The existence of these invariants can be shown [10,48,49,52]) to lead, in each case, to:

$$\text{Birkhoff-Saffman: } n = \frac{6}{5}, m = \frac{2}{5}$$

and

$$\text{Loitsiansky: } n = \frac{10}{7}, m = \frac{2}{7}.$$

$$\mathrm{I_{BS}} = \overline{u^2} \int_0^\infty \frac{r^2}{2r^2} \frac{d}{dr}(r^3 f)\ dr$$

For large $r \to \infty$ under the assumption f(r) $\propto r^{-n}$, then for n > 3 say 4, $\mathrm{I_{BS}} = 0$ and E(κ) $\propto \kappa^4$, which is referred to as the Batchelor spectrum. Hinze pp. 206-207 reaches the same conclusion with a more complex argument requiring the incompressible continuity equation and that the energy spectrum tensor is analytic at $\kappa = 0$, i.e., $f'(0)$ exists.[1] Thus,

$$E(\kappa) = -\frac{(k)^4}{3!\,\pi} \int_0^\infty r^4 \overline{u^2}[3f + rf']\ dr$$

Also, if f(r) $\propto r^{-4}$ then $\lim_{r\to\infty}\left(r^4 \frac{\partial f}{\partial r}\right) = 0$ such that,

$$E(\kappa) = -\frac{3(k)^4}{3!\,\pi} \int_0^\infty r^4 \overline{u^2}[f]\ dr$$

[1] Hinze pp. 199-200 and 206-207 should be integrated into present discussions.

Where:

$$\mathrm{I_L} = \overline{u^2}\int_0^\infty r^4 f\ dr$$

Assuming the Loitsiansky integral is invariant (constant), the Kolmogorov/ Loitsiansky -10/7 decay law is obtained, as per B5.4. Thus:

$k \propto t^{-10/7}$ (1.4) and $\varepsilon = -\frac{dk}{dt} \propto\ t^{-17/7}$ and $L =\ k^{3/2}/\ \varepsilon \propto\ t^{2/7}$.

Alternatively, Safffman argued that it was also possible that for large $r \to \infty$ f(r) ∝ r^{-3}, which is referred to as Saffman turbulence, such that $\mathrm{I_L}$ is divergent,[2] and the assumption is made that E(κ) ∝ κ^2 such that

$$E(\kappa) = \frac{2k^2}{\pi}\int_0^\infty r^2\frac{\overline{u^2}}{2r^2}\frac{d}{dr}(r^3 f)\ dr$$

Which is referred to as the Saffman spectrum. B5.5 argues that for $r \to \infty$ $r^3 f$ = constant such that

$\frac{d}{dr}(r^3 f) = 3r^2 f + r^3 f'$ = $r^2(3f + f')$ = 0 requiring that $3f + f'$ = 0 to derive the Saffman -6/5 decay law. Thus:

$k \propto t^{-6/5}$ (1.2) and $\varepsilon = -\frac{dk}{dt} \propto\ t^{-2/5}$ and $L =\ k^{3/2}/\ \varepsilon \propto\ t^{2/5}$.

0th Law Turbulence and Self-Similarity Decay Laws

$$\varepsilon = C_\varepsilon\frac{u_0^3}{l_0}$$

$$\frac{dk}{dt} = -\varepsilon = \frac{3}{2}\frac{d(u_0^2)}{dt} = -C_\varepsilon\frac{u_0^3}{l_0}$$

Assuming $l_0 \propto\ t^0$ then $k \propto t^{-2}$, $\varepsilon =\ t^{-3}$, and $L =\ k^{3/2}/\ \varepsilon \propto\ t^0$.

The self-similarity decay law was derived in Part 2

$k \propto t^{-1}$ and $\varepsilon = -\frac{dk}{dt} \propto\ t^{-2}$ and $L =\ k^{3/2}/\ \varepsilon \propto\ t^{1/5}$.

[2] If $\mathrm{I_L}$ divergent how justify neglect.

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Laws of turbulence decay from direct numerical simulations

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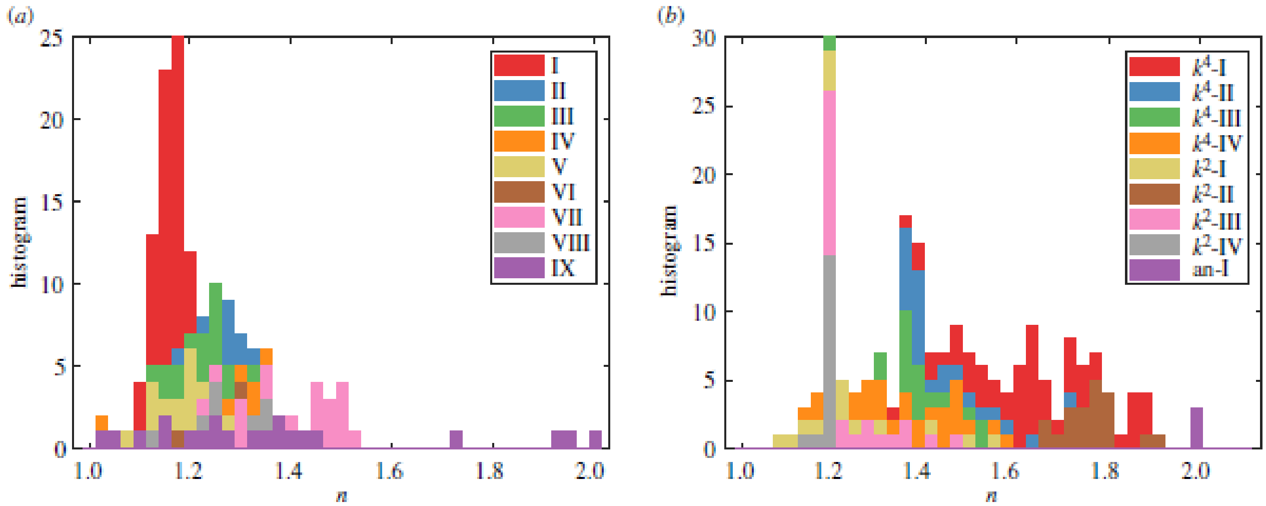

Figure 1. Histogram of decay exponents from the literature: (*a*) experiments, (*b*) simulations. Data in (*a*) are from I [5], II [6–9], III [10], IV [11–14], V [15–19], VI [20], VII [21], VIII [22–24], IX [10,14,22,23,25–28]. The data points IX correspond to fractal or active grids. In (*b*), the initial spectrum with $E(\kappa) \propto \kappa^4$ near the origin correspond to: κ^4-I [29–32], κ^4-II [33], κ^4-III [34–37], κ^4-IV [38]. The initial spectrum with $E(\kappa) \propto \kappa^2$ for small κ corresponds to cases: κ^2-I [39,40], κ^2-II [31], κ^2-III [33], κ^2-IV [35,36,41] an-I[42] corresponds to decay of an anisotropically forced turbulence. For the case [37], the simulations were compressible. Simulations here include DNS, LES using different numerical methods, and EDQNM. Some experiments and simulations are no doubt more thorough than others, but we cannot *a priori* discard any of them on the basis of available information. It should not be inferred that the 'correct' exponent is necessarily the most frequently observed one. (Online version in colour.)

The decay laws range from the 0^{th} law to Kolmogorov to Saffman to similarity, i.e., -2, -1.4, -1.2, and -1. Saffman is closest to the experiments, whereas both Saffman and Kolmogorov are closest to the simulations.

Both experiments and the simulations show large scatter indicating that realization of homogeneous isotropic turbulence (HIT) decay is an idealization, which is a disappointing paradox since HIT is a fundamental building block of turbulence theory and modeling. The experiments are for different approaches to grid turbulence, whereas the simulations are for box turbulence with different $E(\kappa)$ initial conditions. Although many factors affect both the experiments and simulations, it's clear that how the energy is initiated/injected in both is likely the most important. For the experiments there is no clear correlation to the invariant assumptions, whereas for the simulations its clear the initial condition $E(\kappa) \propto \kappa^4$ correlates with Kolmogorov, whereas the initial condition $E(\kappa) \propto \kappa^2$ correlates with the Saffman, which provides credence for their turbulence assumptions and invariants. Davidson, Turbulence, Section 6.3 provides detailed discussion of the physics of the Loitsiansky and Saffman invariants.

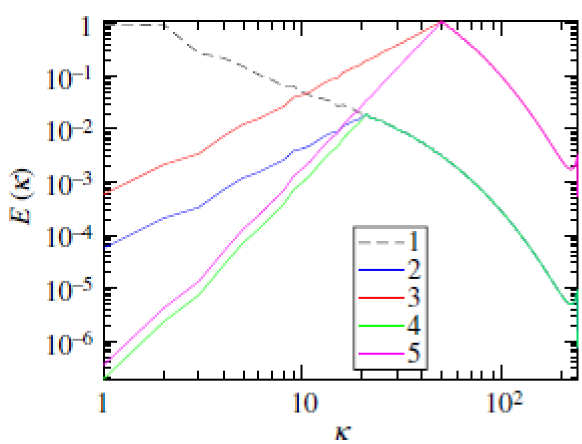

Figure 2. Initial spectra. (1) Dashed black line: energy spectra from fully developed forced turbulence simulation; (2) Blue line: modified initial energy spectra with κ^2 scaling and $\kappa_p = 21$; (3) Red line: modified initial energy spectra with κ^2 scaling and $\kappa_p = 50$, translated upwards to increase initial Reynolds number; (4) Green line: modified initial energy spectra with κ^4 scaling and $\kappa_p = 21$; (5) Magenta line: modified initial energy spectra with κ^4 scaling and $\kappa_p = 50$ translated upwards to increase the initial Reynolds number. See text for explanations. (Online version in colour.)