Chapter 5: Energy Decay in Isotropic Turbulence

Part 3: Equation for Two-Point Correlations & Self-Preservation and the Karman-Howarth Equation

5.5 Equation for Two-Point Correlations

The analysis of isotropic decay carried out in the previous section concentrates on tracing the history of K and ϵ as they change in time. Only minimal information about the flow structure was needed, in fact, just the skewness and palenstrophy coefficient that are related to the two-point correlation functions. To proceed to a more extensive analysis of the decay problem that includes analyzing the time dependence of G^* and S_K^* it is necessary to include dynamical information about multi-point correlations. This means introducing an equation for the time history of the two-point velocity correlation tensor $\mathcal{R}(\mathbf{r}, t)$ and then considering its form during isotropic decay. From such an analysis it is also possible to consider the spectral properties of the turbulence during the decay process.

An equation governing $\mathcal{R}_{ij}(\mathbf{x}, \mathbf{y}, t)$ for arbitrary incompressible flow is derived by taking the average of $u_i(\mathbf{x}, t)$ times the jth component of the Navier–Stokes equation in Eq. (2.2) at $\mathbf{y}$ and adding to this the same quantity with i and j and $\mathbf{x}$ and $\mathbf{y}$ reversed. The result is

$$\mathcal{R}_{ij}(\underline{r}, t) = \overline{u_i(\underline{x}, t) u_j(\underline{x} + \underline{r}, t)}$$

$$\underline{r} = \underline{y} - \underline{x}$$

$$\begin{aligned}
&\overline{\rho u_i(\mathbf{x}, t)\frac{\partial \mathcal{U}_j}{\partial t}(\mathbf{y}, t)} + \overline{\rho u_j(\mathbf{y}, t)\frac{\partial \mathcal{U}_i}{\partial t}(\mathbf{x}, t)} \\
&+ \overline{\rho u_i(\mathbf{x}, t)\mathcal{U}_k(\mathbf{y}, t)\frac{\partial \mathcal{U}_j}{\partial y_k}(\mathbf{y}, t)} + \overline{\rho u_j(\mathbf{y}, t)\mathcal{U}_k(\mathbf{x}, t)\frac{\partial \mathcal{U}_i}{\partial x_k}(\mathbf{x}, t)} = \\
&- \overline{u_i(\mathbf{x}, t)\frac{\partial p}{\partial y_j}(\mathbf{y}, t)} - \overline{u_j(\mathbf{y}, t)\frac{\partial p}{\partial x_i}(\mathbf{x}, t)} \\
&+ \overline{\mu u_i(\mathbf{x}, t)\nabla^2 \mathcal{U}_j(\mathbf{y}, t)} + \overline{\mu u_j(\mathbf{y}, t)\nabla^2 \mathcal{U}_i(\mathbf{x}, t)}.
\end{aligned} \tag{5.83}$$

Using the definition of $\mathcal{R}_{ij}$ given in Eq. (2.30) it follows that the first two terms on the left-hand side of Eq. (5.83) may be written as

$$\overline{\rho u_i(\mathbf{x}, t)\frac{\partial \mathcal{U}_j}{\partial t}(\mathbf{y}, t)} + \overline{\rho u_j(\mathbf{y}, t)\frac{\partial \mathcal{U}_i}{\partial t}(\mathbf{x}, t)} = \rho \frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{x}, \mathbf{y}, t) \tag{5.84}$$

since terms such as $\overline{u_i(\mathbf{x}, t)\partial \overline{U}_j(\mathbf{y}, t)/\partial t} \equiv 0$. The next two terms in Eq. (5.83), coming from the advection term, give

$$\begin{aligned}
&\overline{\rho u_i(\mathbf{x}, t)\mathcal{U}_k(\mathbf{y}, t)\frac{\partial \mathcal{U}_j}{\partial y_k}(\mathbf{y}, t)} + \overline{\rho u_j(\mathbf{y}, t)\mathcal{U}_k(\mathbf{x}, t)\frac{\partial \mathcal{U}_i}{\partial x_k}(\mathbf{x}, t)} = \\
&\overline{\rho u_i(\mathbf{x}, t)u_k(\mathbf{y}, t)}\frac{\partial \overline{U}_j}{\partial y_k}(\mathbf{y}, t) + \overline{\rho u_j(\mathbf{y}, t)u_k(\mathbf{x}, t)}\frac{\partial \overline{U}_i}{\partial x_k}(\mathbf{x}, t) + \\
&\rho \overline{U}_k(\mathbf{y}, t)\overline{u_i(\mathbf{x}, t)\frac{\partial u_j}{\partial y_k}(\mathbf{y}, t)} + \rho \overline{U}_k(\mathbf{x}, t)\overline{u_j(\mathbf{y}, t)\frac{\partial u_i}{\partial x_k}(\mathbf{x}, t)} + \\
&\overline{\rho u_i(\mathbf{x}, t)u_k(\mathbf{y}, t)\frac{\partial u_j}{\partial y_k}(\mathbf{y}, t)} + \overline{\rho u_j(\mathbf{y}, t)u_k(\mathbf{x}, t)\frac{\partial u_i}{\partial x_k}(\mathbf{x}, t)}.
\end{aligned} \tag{5.85}$$

The first two terms on the right-hand side of Eq. (5.85) are equal to

$$\rho \mathcal{R}_{ik}(\mathbf{x}, \mathbf{y}, t)\frac{\partial \overline{U}_j}{\partial y_k}(\mathbf{y}, t) + \rho \mathcal{R}_{jk}(\mathbf{y}, \mathbf{x}, t)\frac{\partial \overline{U}_i}{\partial x_k}(\mathbf{x}, t). \tag{5.86}$$

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Furthermore, differentiation of Eq. (2.30) gives

$$\frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x},\mathbf{y},t) = \overline{u_i(\mathbf{x},t)\frac{\partial u_j}{\partial y_k}(\mathbf{y},t)} \tag{5.87}$$

and similarly for x_k derivatives, so that the third and fourth terms on the right-hand side of Eq. (5.85) take the form of convection terms

$$\rho\overline{U}_k(\mathbf{y},t)\frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x},\mathbf{y},t) + \rho\overline{U}_k(\mathbf{x},t)\frac{\partial \mathcal{R}_{ij}}{\partial x_k}(\mathbf{x},\mathbf{y},t). \tag{5.88}$$

As far as the last two terms on the right-hand side of Eq. (5.85) are concerned, they may be written using Eq. (2.31) as

$$\rho\overline{u_i(\mathbf{x},t)u_k(\mathbf{y},t)\frac{\partial u_j}{\partial y_k}(\mathbf{y},t)} = \rho\frac{\partial S_{jk,j}}{\partial y_k}(\mathbf{y},\mathbf{x},t) \tag{5.89}$$

and

$$\rho\overline{u_j(\mathbf{y},t)u_k(\mathbf{x},t)\frac{\partial u_i}{\partial x_k}(\mathbf{x},t)} = \rho\frac{\partial S_{ik,j}}{\partial x_k}(\mathbf{x},\mathbf{y},t) \tag{5.90}$$

where the fact that

$$\overline{u_i(\mathbf{x},t)\frac{\partial u_j}{\partial x_j}(\mathbf{x},t)u_k(\mathbf{y},t)} = 0 \tag{5.91}$$

has been used as implied by incompressibility.

To treat the contribution to Eq. (5.83) from the terms containing pressure, introduce the two-point pressure-velocity correlation vector

$$\mathcal{K}_i(\mathbf{x},\mathbf{y},t) = \overline{u_i(\mathbf{x},t)p(\mathbf{y},t)} \tag{5.92}$$

and see that

$$\overline{u_i(\mathbf{x},t)\frac{\partial p}{\partial y_j}(\mathbf{y},t)} + \overline{u_j(\mathbf{y},t)\frac{\partial p}{\partial x_i}(\mathbf{x},t)} = \frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x},\mathbf{y},t) + \frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y},\mathbf{x},t). \tag{5.93}$$

Finally, the viscous terms in Eq. (5.83) become

$$\mu\overline{u_i(\mathbf{x},t)\nabla^2 U_j(\mathbf{y},t)} + \mu\overline{u_j(\mathbf{y},t)\nabla^2 U_i(\mathbf{x},t)} = \mu\frac{\partial^2 \mathcal{R}_{ij}}{\partial y_k^2}(\mathbf{x},\mathbf{y},t) + \mu\frac{\partial^2 \mathcal{R}_{ij}}{\partial x_k^2}(\mathbf{x},\mathbf{y},t). \tag{5.94}$$

Putting the above results together it has been shown that Eq. (5.83) becomes

$$\begin{aligned}\frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{x},\mathbf{y},t) &+ \overline{U}_k(\mathbf{y},t)\frac{\partial \mathcal{R}_{ij}}{\partial y_k}(\mathbf{x},\mathbf{y},t) + \overline{U}_k(\mathbf{x},t)\frac{\partial \mathcal{R}_{ij}}{\partial x_k}(\mathbf{x},\mathbf{y},t)\\ &= -\mathcal{R}_{ik}(\mathbf{x},\mathbf{y},t)\frac{\partial \overline{U}_j}{\partial y_k}(\mathbf{y},t) - \mathcal{R}_{jk}(\mathbf{y},\mathbf{x},t)\frac{\partial \overline{U}_i}{\partial x_k}(\mathbf{x},t)\\ &\quad - \frac{\partial S_{jk,i}}{\partial y_k}(\mathbf{y},\mathbf{x},t) - \frac{\partial S_{ik,j}}{\partial x_k}(\mathbf{x},\mathbf{y},t) - \frac{1}{\rho}\frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x},\mathbf{y},t)\\ &\quad - \frac{1}{\rho}\frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y},\mathbf{x},t) + \nu\frac{\partial^2 \mathcal{R}_{ij}}{\partial y_k^2}(\mathbf{x},\mathbf{y},t) + \nu\frac{\partial^2 \mathcal{R}_{ij}}{\partial x_k^2}(\mathbf{x},\mathbf{y},t).\end{aligned}$$

Only hypothesis is incompressibility.

If $\underline{x} = \underline{y}$ recover Reynolds stress equation for incompressible flow

Time derivative of $\mathcal{R}_{ij}$
Convective transport of $\mathcal{R}_{ij}$
Production terms
Flux terms due to $S_{ik,j}$ related to vortex stretching.
Flux terms due to two-point pressure-velocity correlation
Viscous tensorial dissipation

When $\mathbf{x} = \mathbf{y}$, $\mathcal{R}_{ij}(\mathbf{x}, \mathbf{x}, t) = R_{ij}(\mathbf{x}, t)$, and it may be shown that Eq. (5.95) becomes identical to Eq. (3.53). This connection suggests that the first two terms on the right-hand side of Eq. (5.95) are "production" terms. The remaining terms acquire meaning by noting their similarity to the corresponding terms in Eq. (3.53).

No convection or production by the mean flow velocity and velocity gradient, respectively.

The formidable complexity of Eq. (5.95) can be reduced somewhat by applying the relation to the specific case of homogeneous, isotropic turbulence. Since $\overline{U}_k(\mathbf{y}, t) = \overline{U}_k(\mathbf{x}, t)$ in homogeneous turbulence, and using results like Eq. (5.87), it follows that the two convection terms on the left-hand side of Eq. (5.95) sum to zero. Uniformity of $\overline{U}_i$ also implies that the two production terms on the right-hand side of Eq. (5.95) are zero.

The simplification for homogeneous turbulence used in Eq. (4.1) can be generalized to include the statements that

$$\mathcal{S}_{ij,k}(\mathbf{x}, \mathbf{y}, t) = \mathcal{S}_{ij,k}(\mathbf{y} - \mathbf{x}, t) \tag{5.96}$$

and

$$\mathcal{K}_i(\mathbf{x}, \mathbf{y}, t) = \mathcal{K}_i(\mathbf{y} - \mathbf{x}, t), \tag{5.97}$$

where for convenience the same symbols $\mathcal{R}_{ij}$, $\mathcal{S}_{ij,k}$, and $\mathcal{K}_i$ on the right-hand side are adopted; their applicability to homogeneous turbulence is implied by the appearance of one less argument than their more general counterparts. Using these relations, it follows that

$$\frac{\partial \mathcal{S}_{jk,i}}{\partial y_k}(\mathbf{y}, \mathbf{x}, t) = -\frac{\partial \mathcal{S}_{jk,i}}{\partial r_k}(\mathbf{x} - \mathbf{y}, t) \tag{5.98}$$

and

$$\frac{\partial \mathcal{S}_{ik,j}}{\partial x_k}(\mathbf{x}, \mathbf{y}, t) = -\frac{\partial \mathcal{S}_{ik,j}}{\partial r_k}(\mathbf{y} - \mathbf{x}, t), \tag{5.99}$$

and that

$$\frac{\partial \mathcal{K}_i}{\partial y_j}(\mathbf{x}, \mathbf{y}, t) = \frac{\partial \mathcal{K}_i}{\partial r_j}(\mathbf{y} - \mathbf{x}, t) \tag{5.100}$$

and

$$\frac{\partial \mathcal{K}_j}{\partial x_i}(\mathbf{y}, \mathbf{x}, t) = \frac{\partial \mathcal{K}_j}{\partial r_i}(\mathbf{x} - \mathbf{y}, t). \tag{5.101}$$

Putting together the various results, it is found that the two-point velocity correlation tensor in homogeneous turbulence is governed by the equation

$$\begin{aligned}\frac{\partial \mathcal{R}_{ij}}{\partial t}(\mathbf{r}, t) = {} & \frac{\partial \mathcal{S}_{jk,i}}{\partial r_k}(-\mathbf{r}, t) + \frac{\partial \mathcal{S}_{ik,j}}{\partial r_k}(\mathbf{r}, t) \\ & - \frac{1}{\rho}\frac{\partial \mathcal{K}_i}{\partial r_j}(\mathbf{r}, t) - \frac{1}{\rho}\frac{\partial \mathcal{K}_j}{\partial r_i}(-\mathbf{r}, t) + 2\nu \frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k^2}(\mathbf{r}, t).\end{aligned} \tag{5.102}$$

Contracting the indices in Eqs. (5.100) and (5.101), noting the definition of $\mathcal{K}_i$ in Eq. (5.92) and using the incompressibility condition gives in both cases

$$\frac{\partial \mathcal{K}_i}{\partial r_i}(\mathbf{r}, t) = 0. \tag{5.103}$$

Contracting indices the two-point pressure-velocity correlation gradient terms = 0.

Now taking a trace of Eq. (5.102) and using (5.103) gives

$$\frac{\partial \mathcal{R}_{ii}}{\partial t}(\mathbf{r},t) = \frac{\partial S_{ik,i}}{\partial r_k}(-\mathbf{r},t) + \frac{\partial S_{ik,i}}{\partial r_k}(\mathbf{r},t) + 2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\mathbf{r},t), \tag{5.104}$$

which shows that the time rate of change of the trace of the two-point velocity correlation tensor depends on a balance between viscous diffusion, given in the last term, and the two terms depending on the two-point triple velocity correlation tensor. The latter represent the process by which vortex stretching brings energy to small dissipative scales.

Note that the $\mathcal{R}_{ij}$ and $\mathcal{R}_{ii}$ equations are not closed as they contain the two-point triple velocity correlation terms and if equations derived for $S_{ik,i}$ they would contain fourth order velocity correlation terms, i.e., conundrum of the RANS turbulence closure problem and paradox.

Next impose isotropy by using isotropic tensor form of $\mathcal{R}_{ii}$ and $S_{ik,i}$ to obtain the K-H equation.

Self-Preservation and the Karman-Howarth Equation

The $\mathcal{R}_{ii}$ equation is transformed to the Karman-Howarth equation under the assumptions of homogeneous and isotopic turbulence.

$$\underbrace{\frac{\partial \mathcal{R}_{ii}}{\partial t}(\underline{r},t)}_{\boxed{1}} - \underbrace{\frac{\partial S_{ik,i}}{\partial r_k}(-\underline{r},t)}_{\boxed{2}} + \underbrace{\frac{\partial S_{ik,i}}{\partial r_k}(\underline{r},t)}_{\boxed{3}} = \underbrace{2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r},t)}_{\boxed{4}} \qquad (1)$$

Time derivative of a scalar	Divergence of a vector	Laplacian of a scalar

This is a scalar equation, where each term is function of $\underline{r}$ and t, in the most general case.

Combining Eq. (1) with the Chapter 4 Part 2 isotropic expressions for $\mathcal{R}_{ij}$ and $S_{ij,l}$

$$\mathcal{R}_{ij}(\underline{r},t) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}\frac{df}{dr}\right]$$

$$S_{ijl}(\underline{r},t) = u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_j r_l}{2r^3} - \frac{k}{2}\delta_{ij}\frac{r_l}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

it is possible to analyze each term in Eq. (1) separately.

Term 1:

$$\mathcal{R}_{ii}(\underline{r},t) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\underbrace{\delta_{ii}}_{\boxed{3}} - \frac{r_i r_i}{r^2}\frac{r}{2}\frac{df}{dr}\right] = \overline{u^2}\left(3f + r\frac{df}{dr}\right)$$

$$3f + r\frac{df}{dr} = \frac{1}{r^2}\left(3fr^2 + r^3\frac{df}{dr}\right) = \frac{1}{r^2}\left(3fr^2 + \frac{d(r^3 f)}{dr} - 3fr^2\right) = \frac{1}{r^2}\frac{d(r^3 f)}{dr}$$

$$\times\frac{r^2}{r^2}$$

$$\frac{\partial\mathcal{R}_{ii}}{\partial t}(\underline{r},t) = \frac{\partial}{\partial t}\left[\overline{u^2}\frac{1}{r^2}\frac{d(r^3 f)}{dr}\right]$$

Terms 2 and 3:

$$S_{ik,i}(\underline{r},t) = u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_k r_i}{2r^3} - \frac{k}{2}\delta_{ik}\frac{r_i}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{ii}\frac{r_k}{r} + \delta_{ki}\frac{r_i}{r}\right)\right]$$

$$= u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_k}{2r} - \frac{k}{2}\frac{r_k}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(3\frac{r_k}{r} + \frac{r_k}{r}\right)\right]$$

$$= u_{rms}^3\left[-\frac{dk}{dr}\frac{r_k}{2} + \frac{r_k}{r^2}\frac{d(kr^2)}{dr}\right]$$

$$= u_{rms}^3\left[-\frac{dk}{dr}\frac{r_k}{2} + \frac{r_k}{r^2}\left(r^2\frac{dk}{dr} + 2kr\right)\right]$$

$$= u_{rms}^3\left[\frac{r_k}{2}\frac{dk}{dr} + 2k\frac{r_k}{r^2}\right]$$

Now, taking a derivative with respect to r_k

$$\frac{\partial S_{ik,i}}{\partial r_k}(\underline{r},t) = u_{rms}^3 \frac{\partial}{\partial r_k}\left[2k\frac{r_k}{r^2} + \frac{r_k}{2}\frac{dk}{dr}\right]$$

$$= u_{rms}^3\left[\left(\frac{6}{r}k + 2r_k\frac{r_k}{r}\left(-\frac{1}{r^2}\right) + 2\frac{r_k}{r}\frac{r_k}{r}\frac{\partial k}{\partial r}\right) + \left(\frac{3}{2}\frac{dk}{dr} + \frac{1}{2}r_k\frac{r_k}{r}\frac{\partial^2 k}{\partial r^2}\right)\right]$$

$$= u_{rms}^3\left[\left(\frac{4}{r}k + 2\frac{\partial k}{\partial r}\right) + \left(\frac{3}{2}\frac{dk}{dr} + \frac{1}{2}r\frac{\partial^2 k}{\partial r^2}\right)\right]$$

$$= \frac{1}{2}u_{rms}^3\left[r\frac{\partial^2 k}{\partial r^2} + 7\frac{\partial k}{\partial r} + \frac{8}{r}k\right]$$

$$= \frac{1}{2r^2}u_{rms}^3\left[r^3\frac{\partial^2 k}{\partial r^2} + 7r^2\frac{\partial k}{\partial r} + 8rk\right]$$

$$= \frac{1}{2r^2}u_{rms}^3\left[\frac{\partial}{\partial r}\left(r^3\frac{\partial k}{\partial r}\right) - 3r^2\frac{\partial k}{\partial r} + 7r^2\frac{\partial k}{\partial r} + 8rk\right]$$

$$\frac{\partial S_{ik,i}}{\partial r_k}(\underline{r},t) = \frac{1}{2r^2}u_{rms}^3\left[\frac{\partial}{\partial r}\left(r^3\frac{\partial k}{\partial r}\right) + 4\left(\frac{\partial}{\partial r}(r^2k) - 2rk\right)r^2\frac{\partial k}{\partial r} + 8rk\right]$$

$S_{ik,i}(\underline{r},t) = -S_{ik,i}(-\underline{r},t)$ since $k(r) = -k(-r)$

$$\frac{\partial S_{ik,i}}{\partial r_k}(\pm\underline{r},t) = \pm\frac{1}{2r^2}u_{rms}^3\frac{\partial}{\partial r}\left[r^3\frac{\partial k}{\partial r} + 4r^2k\right]$$

Term 4:

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r},t) = \overline{u^2}\left[7\frac{\partial^2 f}{\partial r^2} + \frac{8}{r}\frac{\partial f}{\partial r} + r\frac{\partial^3 f}{\partial r^3}\right]$$

Chapter 4 Part 3 Eq. (7)

$$\times\frac{r^2}{r^2} \qquad = \frac{\overline{u^2}}{r^2}\left[7r^2\frac{\partial^2 f}{\partial r^2} + 8r\frac{\partial f}{\partial r} + r^3\frac{\partial^3 f}{\partial r^3}\right]$$

$$= \frac{\overline{u^2}}{r^2}\left[7\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - 14r\frac{\partial f}{\partial r} + 8r\frac{\partial f}{\partial r} + \frac{\partial}{\partial r}\left(r^3\frac{\partial^2 f}{\partial r^2}\right) - 3r^2\frac{\partial^2 f}{\partial r^2}\right]$$

$$= \frac{\overline{u^2}}{r^2}\left[7\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - \cancel{6r\frac{\partial f}{\partial r}} + \frac{\partial}{\partial r}\left(r^3\frac{\partial^2 f}{\partial r^2}\right) - 3\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \cancel{6r\frac{\partial f}{\partial r}}\right]$$

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r},t) = \frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]$$

Therefore, Eq. (1) becomes,

$$\underbrace{\frac{\partial}{\partial t}\left[\overline{u^2}\frac{1}{r^2}\frac{d(r^3 f)}{dr}\right]}_{\boxed{1}} = \underbrace{2\frac{1}{2r^2}u_{rms}^3\frac{\partial}{\partial r}\left[r^3\frac{\partial k}{\partial r} + 4r^2 k\right]}_{\boxed{2+3}} + \underbrace{2\nu\frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]}_{\boxed{4}}$$

$$\cancel{\frac{1}{r^2}}\frac{\partial}{\partial t}\left[\overline{u^2}\frac{d(r^3 f)}{dr}\right] = \cancel{\frac{1}{r^2}}u_{rms}^3\frac{\partial}{\partial r}\left[r^3\frac{\partial k}{\partial r} + 4r^2 k\right] + 2\nu\cancel{\frac{1}{r^2}}\overline{u^2}\frac{\partial}{\partial r}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]$$

Integrate over r

$$\frac{\partial}{\partial t}\left[\overline{u^2}r^3 f\right] = u_{rms}^3\left[r^3\frac{\partial k}{\partial r} + 4r^2 k\right] + 2\nu\overline{u^2}\left[4r^2\frac{\partial f}{\partial r} + r^3\frac{\partial^2 f}{\partial r^2}\right]$$

Divide by r^3

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = u_{rms}^3\left[\frac{\partial k}{\partial r} + \frac{4}{r}k\right] + 2\nu\overline{u^2}\left[\frac{4}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\right] \quad (2)$$

Karman-Howarth Equation

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (3)$$

Alternative version Pope 6.75 Appendix A.1

The Karman-Howarth equation relates $f(r,t), k(r,t)$ and $u_{rms}(t)$. However, as with the $\mathcal{R}_{ij}$ equation the K-H equation is not closed, as if considered an equation for $f(r,t)$, it contains an additional unknown, i.e., the triple velocity correlation term $k(r,t)$.

Using a Taylor expansion for f and k

$$f(r,t) = \underbrace{f(0,t)}_{\boxed{1}} + f''(0,t)\frac{r^2}{2!} + f^{IV}(0,t)\frac{r^4}{4!} + \ \ldots$$

$$k(r,t) = \underbrace{k'(0,t)^1}_{\boxed{=0}} r + k'''(0,t)\frac{r^3}{3!} + \cdots$$

and substituting into Eq. (3) gives two equations by gathering terms depending on like powers of r.

Appendix A.2

The r^0 equation gives,

$$\frac{d\overline{u^2}}{dt} = 10\nu\overline{u^2}f''(0,t) \quad (4)$$

And using the definitions of turbulent kinetic energy

[1] Chapter 4 Part 2 pg.9: for homogeneous turbulence.

$$k = \frac{3}{2}\overline{u^2}$$

and Taylor microscale

$$\lambda_f^2 = -\frac{2}{f''(0,t)}$$

in Eq. (4) yields

$$\frac{dk}{dt} = \frac{3}{2}10\nu\overline{u^2}\left(-\frac{2}{\lambda_f^2}\right) = -30\frac{\nu\overline{u^2}}{\lambda_f^2}$$

And using the TKE equation for homogeneous isotropic turbulence

$$\frac{dk}{dt} = -\varepsilon$$

Gives

$$\varepsilon = 30\frac{\nu\overline{u^2}}{\lambda_f^2} = 15\frac{\nu\overline{u^2}}{\lambda_g^2}$$

A result obtained already in Chapter 4.

The r^2 equation gives,

$$\frac{d\varepsilon}{dt} = S_k^* R_T^{1/2} \frac{\varepsilon^2}{k} - G^* \frac{\varepsilon^2}{k}$$

Which, coupled with

$$\frac{dk}{dt} = -\varepsilon$$

Describes the decay of homogeneous isotropic turbulence, as per Part 2. This shows that all the isotropy information in the k and ε equations is contained in the Karman-Howart equation, for which it should be emphasized were derived from the Navier-Stokes equations, as were the k and ε equations (see Chapter 3 Parts 3 and 4).

Assuming self-similarity

$$f(r,t) = \tilde{f}\left(\frac{r}{L(t)} = \eta\right)$$

$$k(r,t) = \tilde{k}\left(\frac{r}{L(t)} = \eta\right)$$

$L(t) = \lambda_g(t)$

The Karman-Howarth equation becomes,

$$2\eta^{-4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right) + \eta\frac{d\tilde{f}}{d\eta}\left(\frac{7}{3}G_0 - 5\right) + 10\tilde{f} = R_\lambda\left(\frac{7}{6}S_{k_0}\eta\frac{d\tilde{f}}{d\eta} - \eta^{-4}\frac{d(\eta^4\tilde{k})}{d\eta}\right) \quad (5)$$

Appendix A.3

Where $\eta = r/\lambda_g$ is a similarity variable.

Eq. (5) represents a single ODE for $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ with $R_\lambda(t)$ acting as a parameter. Note that Eq. (5) contains G_0 and S_{k_0}, which were shown in Part 2 to be constants during self-similar decay and equal to:

$$G = \tilde{f}^{IV}(0)$$

$$-S_k = \tilde{k}'''(0)$$

For self-similarity at all times, $f(r,t) = \tilde{f}(\eta) \neq f(t)$ and $k(r,t) = \tilde{k}(\eta)) \neq f(t)$ and Eq. (5) must be always satisfied regardless of how $R_\lambda(t)$ varies. Consequently, both Eq. (5) RHS and LHS = 0 independently, otherwise LHS multivalued for changes in $R_\lambda(t)$, since $\tilde{f}$, $\tilde{k}$, S_{k_0}, $G_0 \neq f(t)$, i.e., LHS = a constant, which can only be zero.

LHS = 0 gives the confluent hypergeometric equation with solution

$$\tilde{f}(\eta) = M\left(\frac{1}{G_0^* - 1}, \frac{5}{2}, -\frac{5(G_0^* - 1)}{4}\eta^2\right) \quad (6)$$

Where M is the confluent hypergeometric function.

Integration of the RHS=0 of Eq. (5) yields

$$\tilde{k}(\eta) = \frac{7}{6} S_{k_0} \frac{1}{\eta^4} \int_0^\eta s^5 \frac{d\tilde{f}}{ds} ds \quad (7)$$

Which can be solved using $\tilde{f}$ given by Eq. (6). Once G_0 and S_{k_0} are specified so to are $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ and vice versa within the constraint of complete self-similarity.

Recall complete similarity not possible in isotropic decay ∴ in reality $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ must be $f(t)$. However, still useful to examine high and low R_T equilibrium solutions.

1) For small R_λ, which is still $f(t)$ and near $R_{T_\infty} = 0 \rightarrow R_{T_0} \sim 0.1$. RHS of Eq. (5) small such that LHS ≈ 0, which is referred to as the separability condition. Since $G_0^* = 7/5$ for final period

$$\tilde{f}(\eta) = M\left(\frac{5}{2}, \frac{5}{2}, -\frac{\eta^2}{2}\right) = e^{-\frac{\eta^2}{2}} \quad (8)$$

As used in Part 2

$$f(r,t) = e^{-\frac{r^2}{2\lambda_g^2}}$$

Assuming that decay is self-similar near $R_T = 0$, then Eq. (7) holds and solving the integral yields,

$$\tilde{k}(\eta) = \frac{7}{6} S_{k_0} \frac{1}{\eta^4}\left[(\eta^5 + 5\eta^3 + 15\eta)e^{-\eta^2/2} - 15\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right)\right] \quad (9)$$

Where

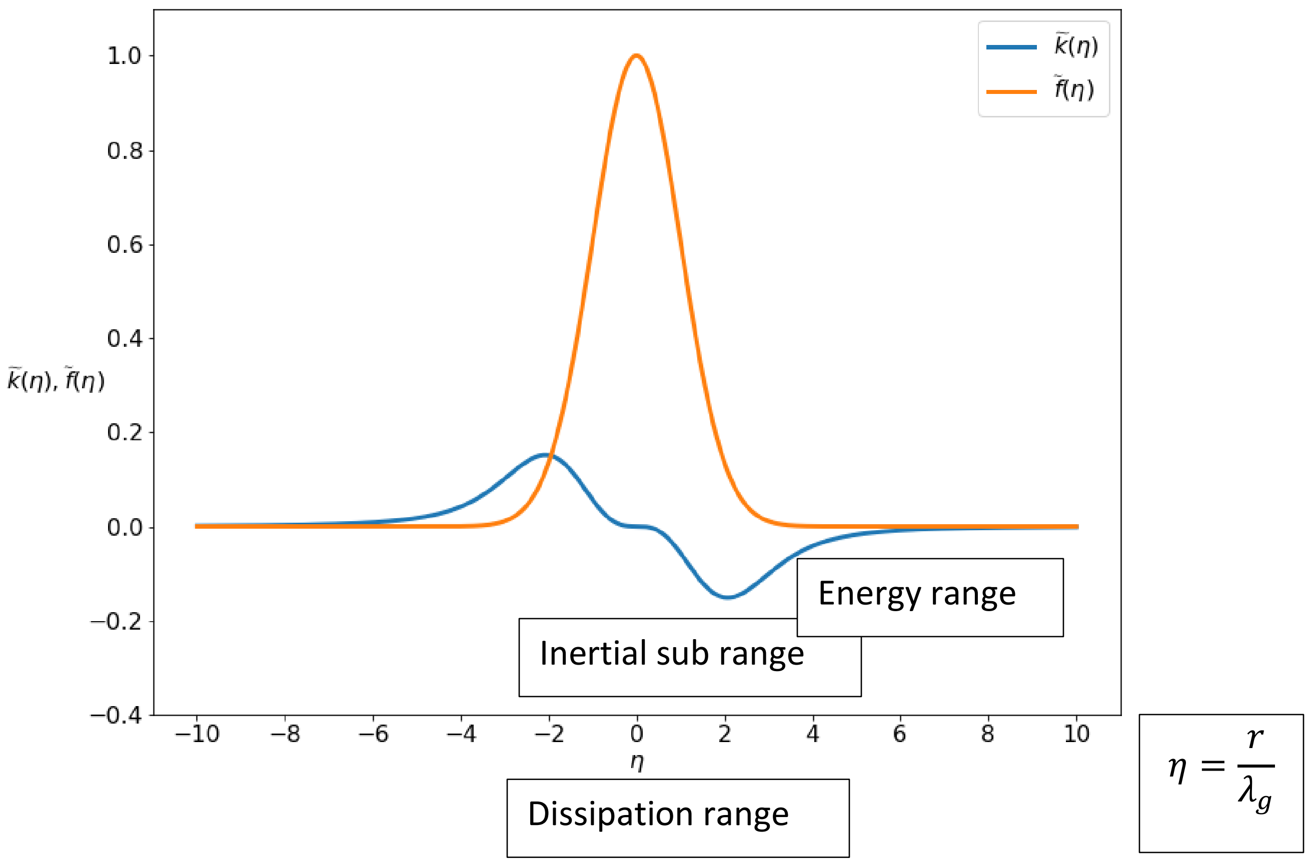
$$\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-s^2} ds$$

is the error function.

Note that G_0^* and S_{k_0} are constants based on assigned values provided by EFD or DNS (see Part 2 pg. 7).

Fig. 5.7 shows a plot of Eq. (9), where $\tilde{k}$ is seen to have a much slower decay for large η than the Gaussian form of $\tilde{f}$, which shows the importance of vortex stretching in the energy cascade process.

$$\tilde{k}(\eta) \rightarrow \eta^{-4} \quad \text{for} \ \eta \rightarrow \infty$$



2) For large Re equilibrium, consider $R_\lambda =$ constant $\neq 0$. If $R_\lambda = f(t)$, LHS would have to be multivalued to satisfy equation. If $R_\lambda =$ constant a solution for and $\tilde{f}(\eta)$ and $\tilde{k}(\eta)$ exists, but the equation is indeterminate (1 eq. 2 unknowns; thus, unlike self-similar k and ε equation, Karman-Howarth equation is not solvable without additional assumptions, i.e., in addition to self-similarity.

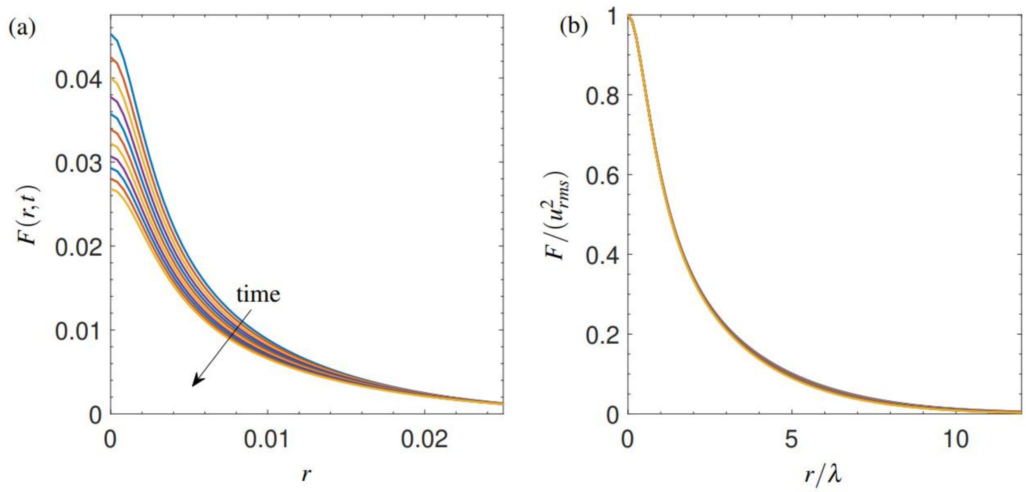


Figure 3. Unscaled (a) and scaled (b) double correlations. Increasing time corresponds to decreasing magnitude in (a), while profiles collapse in (b) under the scaling. All data is Case 2, while Cases 1 and 3 show similar behavior.

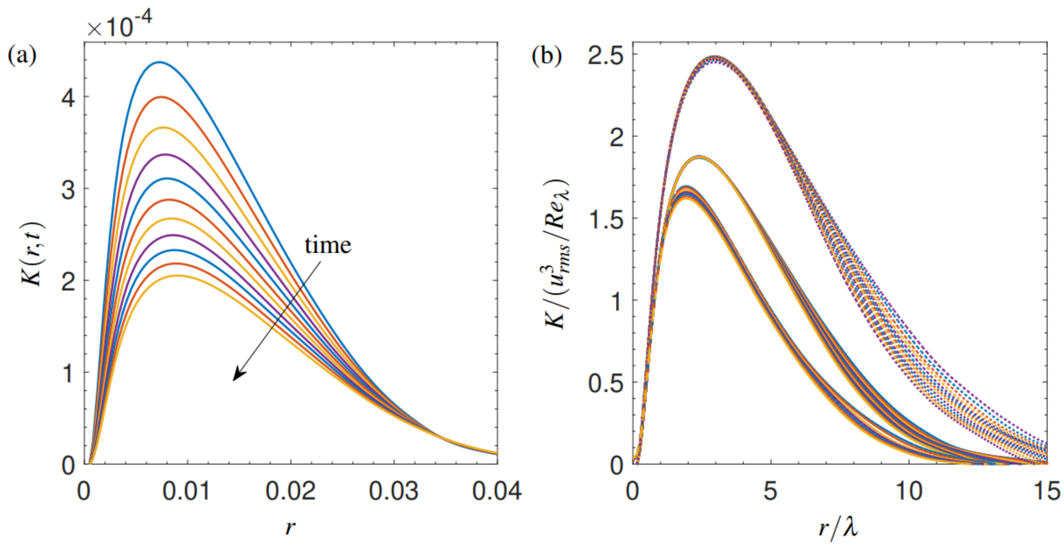


Figure 4. Unscaled (a) and scaled (b) triple correlations. Increasing time corresponds to decreasing magnitude in (a), while profiles collapse in (b) under the scaling, where Case 1 is shown in dashed lines (lowest magnitude), Case 2 in solid, and Case 3 in dotted lines (highest magnitude). The scaling works by accounting for the decreasing Reynolds number in each simulation but not the differences in Reynolds number between cases, which remains a point of ongoing investigation.

Byers, C. P., MacArt, J. F., Mueller, M. E., & Hultmark, M. (2019). Similarity constraints in decaying isotropic turbulence. Paper presented at 11th International Symposium on Turbulence and Shear Flow Phenomena, TSFP 2019, Southampton, United Kingdom.

Additional Discussion Karman-Howarth Equation (Pope pp. 202-205)

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (3)$$

a) Closure problem, i.e., one equation, two unknowns $\tilde{f}(\eta)$ and $\tilde{k}(\eta) \rightarrow$ one could write equation for $\tilde{k}(\eta)$, but it would depend on fourth-order correlation and so on.

b) Terms in k and ν represent inertial and viscous processes, respectively.

c) At $r = 0$, k term $= 0$ since

$$k(r,t) \approx k'''r^3/3! + k^V r^5/5!$$

And homogeneity shows that $k'(0,t) = 0$; also, f is even in r, Eq. (3) becomes,

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right]\Big|_{r=0} = 2\nu\overline{u^2}\left[\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)\right]_{r=0}$$

$$= 2\nu\overline{u^2}\frac{1}{r^4}\left[4r^3\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}\right]_{r=0}$$

$$= 2\nu\overline{u^2}\frac{1}{r^4}\left[4r^4\frac{1}{r}\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}\right]_{r=0} = 2\nu\overline{u^2}\frac{1}{r^4}\left[5r^4\frac{\partial^2 f}{\partial r^2}\right]_{r=0}$$

$$f(0,t) = 1$$

$$\frac{d}{dt}\overline{u^2} = 10\nu\overline{u^2}f''(0,t) = -\frac{10\nu\overline{u^2}}{\lambda_g(t)^2} = -\frac{2}{3}\varepsilon \quad (10)$$

$$\lambda_g(t)^2 = -\frac{1}{f''(0,t)}$$

Pope Ex. 6.6

Where the Taylor expansion for $f(r)$

$$f'(r) = \cancel{f'(0)} + rf''(0) + \cancel{\frac{r^2}{2!}f'''(0)} + \cdots$$

$$\lim_{r\to 0}\frac{f'(r)}{r} = f''(0)$$

was used.

Hence, for $r = 0$, the Karman-Howart equation reduces to $\frac{2}{3}$ times the k equation,

$$\frac{dk}{dt} = -\varepsilon$$

d) Energy cascade for high Re hypothesis is that the energy transfer from larger to smaller scales is an inertial process for $r \gg \eta$, consequently, k term is responsible for this process.

e) If $\underline{u}(\underline{x}, t)$ were a Gaussian field then $k(r, t)$, like all higher order moments, would be zero → energy cascade depends on non-Gaussian aspects of the velocity field. This fact is used in the Quasi-normal approximation method for KH equation.

Skewness of velocity derivative

$$\overline{u^3}k'''(0,t) = \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3} = -S_k\left(\frac{\varepsilon}{15\nu}\right)^{3/2} = -\frac{2}{35}\overline{\omega_i\omega_j\frac{\partial u_i}{\partial x_j}} \text{ Pope (6.84)}$$

Where:

$$S_k = -\frac{\overline{(u_{1,1})^3}}{\overline{(u_{1,1})^2}^{3/2}} \text{ Part 1 Eq. (15)}$$

The velocity-derivative skewness includes – sign as per Bernard. Pope and Hinze define S_k without the − sign. In Bernard definition, S_k is positive, while for Hinze and Pope it is negative. This fact does not change the physical meaning of the equations but could require some sign changes in the derivations. Throughout these notes, Bernard definition is used to be consistent.

∴ connection between S_k, vortex stretching and transfer of energy between different scales, as will be shown in Part 4.

The Kolmogorov 4/5 law (see Chapter 4 Part 8)

The Karman-Howarth equation can be re-expressed in terms of the structure functions $D_{LL}(r,t)$ and $D_{LLL}(r,t)$

$$D_{LLL}(r,t) = \overline{\left[u_1\left(\underline{x} + r\hat{e}_1, t\right) - u_1\left(\underline{x}, t\right)\right]^3}$$

As

$$\frac{\partial}{\partial t} D_{LLL} + \frac{1}{3r^4}\frac{\partial}{\partial r}\left(r^4 D_{LLL}\right) = \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial D_{LLL}}{\partial r}\right) - \frac{4}{5}\varepsilon$$

Pope Ex. 6.9

Integrating

$$\frac{3}{r^5}\int_0^r s^4 \frac{\partial}{\partial t} D_{LLL}(s,t) ds = 6\nu \frac{\partial D_{LLL}}{\partial r} - D_{LLL} - \frac{4}{5}\varepsilon r$$

For isotropic turbulence in the inertial subrange, unsteady term = 0 and viscous term negligible leads to Kolmogorov -4/5 law

$$D_{LLL} = -\frac{4}{5}\varepsilon r$$

Kolmogorov further argued that the structure function skewness,

$$S' \equiv D_{LLL}(r,t)/D_{LL}(r,t)^{3/2}$$

is constant, leading to

$$D_{LL}(r,t) = \left(-\frac{4}{5S'}\right)^{2/3} (\varepsilon r)^{2/3}$$

Which represents Kolmogorov hypothesis, and shows consistency between it and the NS equations, and relates Kolmogorov constant to skewness S'.

The Loitsyanskii integral Appendix A.4

Multiplying Eq. (3) (K-H equation Pope form) by r^4 and integrating between 0 and R, yields

$$\frac{d}{dt}\int_0^R \overline{u^2} r^4 f(r,t)dr = u_{rms}^3 R^4 k(r,t) + 2\nu\overline{u^2} r^4 f'(r,t) \quad (11)$$

Loitsyanskii considered $\lim_{R\to\infty}$ Eq. (11) and assumed $f(r,t)$ and $k(r,t)$ decrease rapidly with r, such that the Loitsyanskii integral,

$$B_2 = \int_0^\infty \overline{u^2} r^4 f(r,t)dr$$

Converges (i.e., constant value), in which case the terms in $k(R,t)$ and $f'(R,t)$ vanish. With these assumptions, $B_2 \neq f(t)$, and became known as Loitsyanskii invariant.

However, these assumptions are incorrect, as shown by Saffman. Depending on how the isotropic turbulence is created, B_2 can be finite or divergent. When it is finite $k(r,t)$ does not vanish as R goes to infinity and B_2 increases in time. Saffman considered the following alternative invariant,

$$C = \int_0^\infty r^2 \mathcal{R}(r)dr$$

Where:

$$\mathcal{R}(r) = \frac{1}{8\pi r^2}\int_{|\underline{r}|=r} \mathcal{R}_{ii}(\underline{r})dA(\underline{r})$$

And for isotropic turbulence, $\mathcal{R}(r) = \frac{1}{2}\overline{u^2}(3f + rf')$.

Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 6.10

Prepared by: Mohammad Mirzadeh *Date:* 2/28/06

We begin by using the *Kárman-Howarth* equation for the final period as,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(u'^2 f\right) = \frac{2\nu u'^2}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)\\
\Rightarrow\quad & f\frac{\mathrm{d}u'^2}{\mathrm{d}t} + u'^2\frac{\partial f}{\partial t} = \frac{2\nu u'^2}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)\\
\Rightarrow\quad & \left(\frac{\mathrm{d}u'^2}{\mathrm{d}t}\frac{1}{u'^2}\right) f + \frac{\partial f}{\partial t} = \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)\\
\Rightarrow\quad & \frac{\mathrm{d}u'^2}{\mathrm{d}t}\frac{1}{u'^2} = \frac{\frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) - \frac{\partial f}{\partial t}}{f} \qquad (1)
\end{aligned}
$$

It is obvious that **LHS** of Eq.1 is only a function of t. Thus Eq. (6.93) can only satisfy Kárman-Howarth equation in the case **RHS** is also only a function of t. Inserting Eq. (6.93) into the **RHS** of Eq.1 yields,

Eq. (6.93) = Eq. (18) below

$$f(r,t) = e^{-\frac{r^2}{8\nu t}}$$

$$
\begin{aligned}
\mathbf{RHS} &= \frac{\frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\left(\frac{-2r}{8\nu t}\right)f\right) - \frac{r^2}{8\nu t^2}f}{f}\\
&= -\frac{\frac{1}{r^4}\left(\frac{5r^4}{2t}f - \frac{r^5}{2t}\frac{2r}{8\nu t}f\right) + \frac{r^2}{8\nu t}f}{f}\\
&= \frac{-5}{2}t \qquad (2)
\end{aligned}
$$

Thus Eq. (6.93) satisfies the Kárman-Howarth equation. We also note that,

$$
\begin{aligned}
&\frac{\mathrm{d}u'^2}{\mathrm{d}t}\frac{1}{u'^2} = \frac{-5}{2}t\\
\Rightarrow\quad & \frac{\mathrm{d}u'^2}{u'^2} = \frac{-5}{2}\frac{\mathrm{d}t}{t}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & u'^2(t) = C\,t^{-5/2}\\
\Rightarrow\quad & k(t) = C'\,t^{-5/2} \qquad (3)
\end{aligned}
$$

Hinze Section 3.3

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (3)$$

Karman-Howarth equation represents one equation in two unknowns $f(r)$ and $k(r)$. Like NS equations leads to closure problem since the number of unknowns is larger than the number of equations. Also, as with NS equations if one obtains higher order velocity correlation equation, they lead to additional unknowns.

Truncation approximation= neglect higher order terms but leads to unphysical solutions.

Quasi-normal approximation= neglect higher order cumulants = assume Gaussian 4th order correlation. However, this implies $S_{ijl} = 0$, which is unacceptable ∴ again leads to unphysical solutions.

† The n-order cumulant tensor of n velocity components is obtained by subtracting from the n-order mean velocity product the various mean products of lower order that can be formed from the n velocity components. For a turbulence with zero mean velocity the cumulant tensor of lowest order is four:

$$\overline{u_iu_ju_ku_l} - \overline{u_iu_j}\,\overline{u_ku_l} - \overline{u_iu_k}\,\overline{u_ju_l} - \overline{u_iu_l}\,\overline{u_ju_k}$$

It is zero when the joint probability distribution is normal.

Direct-interaction approximation= considers interaction of eddies of different sized, including their randomness.

Before these approaches, closure problem also attacked based on physical assumptions for the inertial term.

Consider Eq. (1), i.e., $\mathcal{R}_{ii}$ transport equation for homogeneous turbulence, using the simplification $S_{ik,i}(r,t) = -S_{ik,i}(-r,t)$, i.e., odd function for isotropic turbulence:

$$\frac{\partial \mathcal{R}_{ii}}{\partial t}(\underline{r},t) - 2\frac{\partial S_{ik,i}}{\partial r_k}(-\underline{r},t) = 2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r},t) \quad (12)$$

In Hinze, the following notation is used.

$$\mathcal{R}_{ii} = Q_{i,i} \qquad 2\frac{\partial S_{ik,i}}{\partial r_k}(-r,t) = S_{i,i}$$

1. Taking the second moment of each term in Eq. (12) Saffman invariant

$$\frac{\partial}{\partial t}\int_0^\infty dr\, r^2 \mathcal{R}_{ii} - 2\int_0^\infty dr\, r^2 \frac{\partial S_{ik,i}}{\partial r_k} = 2\nu \left(r^2 \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2} \right)\Bigg|_0^\infty$$

Applying incompressibility yields

$$\int_0^\infty dr\, r^2 \mathcal{R}_{ii} = 0 \text{ and } 2\int_0^\infty dr\, r^2 \frac{\partial S_{ik,i}}{\partial r_k} = 0$$

Consequently

$$\lim_{r\to\infty} r^2 \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2} = 0 \quad (13)$$

Which shows how fast $\mathcal{R}_{ii}$ decreases as r increases, i.e., $\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}$ decreases like r^{-3}.

2. Taking the fourth moment of each term in Eq. (3) Loitsyanskii integral

$$\frac{\partial}{\partial t}\left(\overline{u^2}\int_0^\infty dr\, r^4 f\right) = (u_{rms}^3 r^4 k)|_0^\infty + 2\nu\overline{u^2}\left(r^4 \frac{\partial f}{\partial r}\right)\Bigg|_0^\infty$$

With certain assumptions concerning large-scale structure of turbulence, it is reasonable to expect that,

$$\lim_{r\to\infty}\left(r^4\frac{\partial f}{\partial r}\right)=0 \quad (14)$$

See Hinze pp. 207, 216-218

i.e., $\frac{\partial f}{\partial r}$ decrease like r^{-5}.

$\mathcal{R}_{ii}$ and f have same asymptotic behavior. If we assume that $f(r)$ behaves like r^{-n} for large r, the behavior of $\mathcal{R}_{ii}$ will be:

$$\mathcal{R}_{ii}=\frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}[r^3 f(r)]=\frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}[r^3 r^{-n}]$$
$$=\frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}[r^{3-n}]=\frac{\overline{u^2}}{r^2}r^{2-n}=\overline{u^2}r^{-n}$$

Therefore, Eq. (13) requires:

$$\lim_{r\to\infty} r^2\frac{\partial^2\mathcal{R}_{ii}}{\partial r_k^2}=0$$
$$\to \lim_{r\to\infty}\overline{u^2}r^2\frac{\partial^2 r^{-n}}{\partial r_k^2}=0$$
$$\to \lim_{r\to\infty}\overline{u^2}r^2 n(n+1)r^{-n-2}=0 \to \lim_{r\to\infty}\overline{u^2}n(n+1)r^{-n}=0$$

i.e.., $n>1$. Eq. (14), instead, would require:

$$\lim_{r\to\infty}\left(r^4\frac{\partial f}{\partial r}\right)=0 \to \lim_{r\to\infty}(-r^4 n r^{-n-1}) \to \lim_{r\to\infty}(-n r^{3-n})=0$$

i.e., $n>3$, which is a stronger requirement than Eq. (13). Conclusion: Saffman invariant requires f(r) decay like r^{-2}, whereas Loitsyanskii invariant requires f(r) decay like r^{-4}, which is a stronger requirement.

The term $r^4 k$ has usually been assumed to approach zero for increasing r, and this assumption has been used by Loitsyanskii to obtain,

$$B_2=\int_0^\infty \overline{u^2}r^4 f(r,t)dr$$

must be invariant (i.e., constant value) and not function of time→ Loitsyanskii invariant. However, this is not true and depends on IC of the turbulence (see Saffman).

Consider now limiting case of Eq. (12), where the viscosity effects become predominant → characteristic of final decay,

$$2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(\underline{r}, t) \gg 2\frac{\partial S_{ik,i}}{\partial r_k}(-\underline{r}, t)$$

This hypothesis allows to treat the vector $\underline{r}$ as a scalar r, such that $\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2} = \frac{\partial^2 \mathcal{R}_{ii}}{\partial r}$

$$\frac{\partial \mathcal{R}_{ii}}{\partial t}(r, t) = 2\nu \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \mathcal{R}_{ii}}{\partial r}(r, t)\right] \quad (15)$$

Where the identity

$$2\nu \frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k^2}(r, t) = 2\nu \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \mathcal{R}_{ii}}{\partial r}(r, t)\right]$$

was used.

Assume

$$\mathcal{R}_{ii}(r, t) = \varphi(t)\psi(\chi)$$

Where $\chi = r/\sqrt{8\nu t}$, i.e., separation of variables

Substituting into Eq. (15) leads to two differential equations,

$$\frac{1}{\varphi}\frac{d\varphi}{dt} = -\frac{\alpha}{t}$$

With solution $\varphi = c \times t^{-\alpha}$ and

$$\chi \frac{d^2\psi}{d\chi^2} + 2(\chi^2 + 1)\frac{d\psi}{d\chi} + 4\alpha\chi\psi = 0$$

Assuming $\alpha = (2p + 1)/2$ with p integer, the solution is

$$\psi_p = \frac{1}{\chi}\exp(-\chi^2)H_{2p-1}(\chi)$$

Where $H_n(\chi)$ is the Hermite polynomial

$$H_n(\chi) = (-1)^n \exp(\chi^2) \frac{d^n}{d\chi^n} \exp(-\chi^2)$$

$$H_0 = 1, \qquad H_1 = 2\chi, \qquad H_2 = 4\chi^2 - 2, \qquad H_3 = 8\chi^3 - 12\chi$$

General solution of Eq. (15) is:

$$\mathcal{R}_{ii}(r,t) = \varphi(t)\psi(\chi)$$

$$= \frac{\sqrt{8\nu}}{r} \exp(-r^2/8\nu t) \sum_1^\infty \frac{A_p}{t^p} H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) \qquad (15A)$$

Where the constants A_p must be chosen such that the series converges and that

$$\mathcal{R}_{ii}(0,t) = \overline{u^2}\left(3\underbrace{f(0)}_{\boxed{1}} + r\cancel{\frac{df}{dr}(0)}\right) = 3\overline{u^2}$$

Applying 2nd moment condition

Appendix A.7

$$\int_0^\infty dr\, r^2 \mathcal{R}_{ii} = 0$$

We find that all the terms of Eq. (15A) satisfy this condition except the term where $p = 1$; so A_1 must be zero.

Applying 4th moment condition

Appendix A.8

$$\int_0^\infty dr\, r^4 \mathcal{R}_{ii} = 0$$

We find that only $p = 2$ term $\neq 0$.

Therefore, solution of Eq. (15) for $p = 2$ may be reduced to

$$\mathcal{R}_{ii}(r,t) = -\frac{4A_2}{t^{\frac{5}{2}}}\left(3 - \frac{r^2}{4\nu t}\right) \exp\left(-\frac{r^2}{8\nu t}\right) \quad (16)$$

And applying the condition $\mathcal{R}_{ii}(0,t) = 3\overline{u^2}$ yields:

$$\mathcal{R}_{ii}(0,t) = 3\overline{u^2} = -\frac{4A_2}{t^{5/2}}\left(3 - \frac{0^2}{4\nu t}\right)\exp(-0^2/8\nu t) = -\frac{12A_2}{t^{5/2}}$$

$$\overline{u^2} = -4A_2 t^{-\frac{5}{2}} = c \times t^{-\frac{5}{2}} \quad (17)$$

$$A_2 = -\frac{\overline{u^2}}{4} t^{\frac{5}{2}} \quad (18)$$

And, consequently, from Eq. (16) and (18)

$$\mathcal{R}_{ii}(r,t) = -\frac{4}{t^{\frac{5}{2}}}\frac{\overline{u^2}}{4} t^{\frac{5}{2}}\left(3 - \frac{r^2}{4\nu t}\right)\exp\left(-\frac{r^2}{8\nu t}\right)$$

$$\mathcal{R}_{ii}(r,t) = -\overline{u^2}\left(3 - \frac{r^2}{4\nu t}\right)e^{-\frac{r^2}{8\nu t}} \quad (19)$$

Combining the relation between $\mathcal{R}_{ii}(r,t)$ and $f(r,t)$

$$\mathcal{R}_{ii}(r,t) = \overline{u^2}\left(3f(r,t) + r\frac{\partial f}{\partial r}(r,t)\right) = \frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}[r^3 f(r,t)]$$

With Eq. (19) gives the following differential equation

$$\frac{\overline{u^2}}{r^2}\frac{\partial}{\partial r}[r^3 f(r,t)] = -\overline{u^2}\left(3 - \frac{r^2}{4\nu t}\right)e^{-\frac{r^2}{8\nu t}}$$

With boundary conditions

$$f(0,t) = 1$$

And by integration

$$r^3 f(r,t) = \int r^2 \left(3 - \frac{r^2}{4\nu t}\right) e^{-\frac{r^2}{8\nu t}} = r^3 e^{-\frac{r^2}{8\nu t}} + C$$

Or equivalently

$$f(r,t) = e^{-\frac{r^2}{8\nu t}} \quad (20)$$

Where $C = 0$ from the application of BCs.

Thus, for dominating viscosity effects, Eq. (12) shows the decay law for the turbulence $\Rightarrow -5/2$ decay as shown in Chapter 5 Part 2.

$f(r,t)$ has shape Gaussian curve and remains self-preserving during decay. Shows good agreement with EFD. Moreover, using Eqs. (17) and (20) to evaluate Loitsyanskii integral proves that it is an exact invariant with respect to time, in these conditions.

Same Fig 5.1
Bernard, St
vs r/λ_f

$\lambda_f^2 = \frac{30 \nu u'^2}{\varepsilon}$

using:

$u'^2 = c\, t^{-5/2}$

$\varepsilon = c\, t^{-7/2}$

$\lambda_f = \sqrt{c \nu t}$

$c = 8$

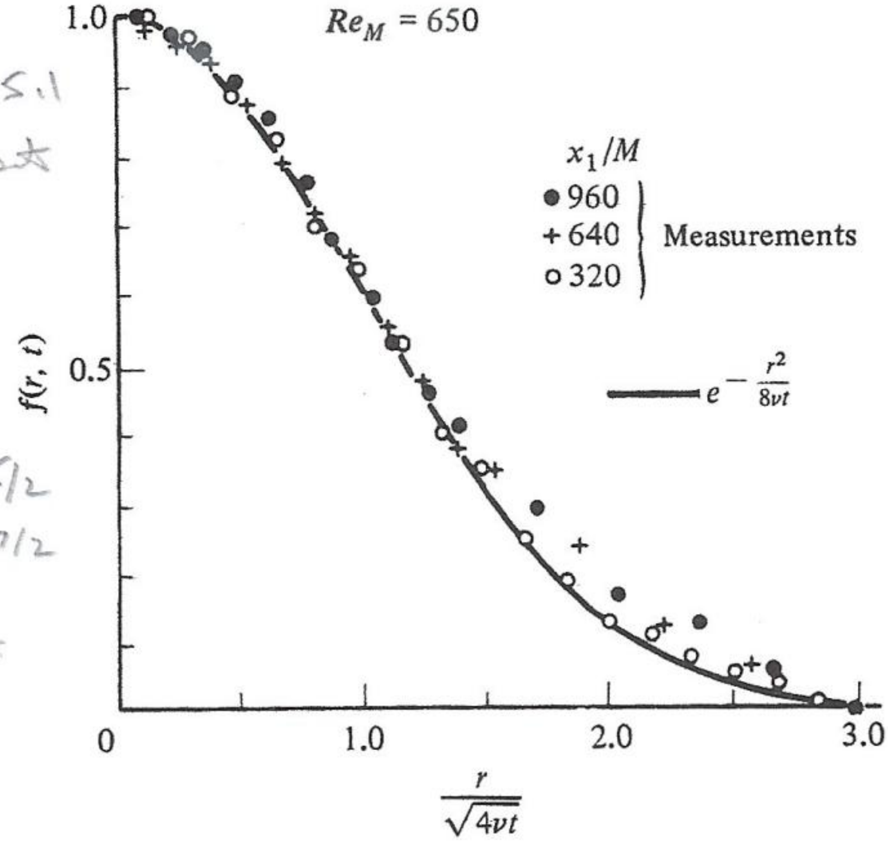


FIGURE 3-11
Longitudinal-velocity-correlation coefficient $f(r,t)$ in the final period of decay. (From: *Batchelor, G. K., and A. A. Townsend,*[7] *by permission of the Royal Society.*)

$k_t = -\varepsilon \qquad \varepsilon_t = (S_2^* R_T^{1/2} - G^*)\,\varepsilon^2/k$

1. High Re Self-Similarity

$S_2^* = (G_0^* - 2)/R_T^{1/2} \qquad \varepsilon_t = -2\,\varepsilon^2/k \qquad k = \frac{k_0}{T_{t_0}} t^{-1}$

$\varepsilon = \frac{\varepsilon_0}{T_{t_0}} t^{-2}$

$\lambda_g^2 = \frac{10 \nu k}{\varepsilon} = \frac{10 \nu k_0}{\varepsilon_0} t \qquad L = \frac{k^{3/2}}{\varepsilon}$

$\lambda_g \propto \sqrt{t} \qquad = \left(\frac{k_0}{T_{t_0}} t^{-1}\right)^{3/2} \Big/ \frac{\varepsilon_0}{T_{t_0}} t^{-2}$

2. Turbulence Modeling

$= \frac{k_0^{3/2} T_{t_0}^{-3/2} t^{-3/2}}{\frac{\varepsilon_0}{T_{t_0}} t^{-2}} = \frac{k_0^{3/2}}{\varepsilon_0 T_{t_0}^{1/2}} t^{1/2}$

$-C_{\varepsilon 2} = (S_2^* R_T^{1/2} - G^*)$

$G^* \sim R_T^{1/2} = a R_T^{1/2} \Rightarrow -C_{\varepsilon 2} = R_T^{1/2}(S_2^* - a)$

G^* = ratio fractional a determines balance

rate of change ε/energy vortex stretching vs dissipation

$= Y_s/s \,/\, \varepsilon/k$

$C_{\varepsilon 2}$	
1.83	
.3	
Homogeneous Shear flow	
1.9	
$k-\varepsilon$	
1.9	

$k = \frac{k_0}{T_{t_0}} t^{1/C_{\varepsilon 2} - 1} \qquad C_{\varepsilon 2}$ sets decay rate

$C_{\varepsilon 2} = 11/6 \Rightarrow t^{-6/5}$ Saffman 1.2

3. Moin: dimensional analysis $= 3/10 \Rightarrow t^{-10/7}$ Kolmogorov 1.4

$k_t = -\varepsilon \qquad \varepsilon = f(k, \varepsilon) \qquad k = \frac{m^2}{s^2} \qquad \varepsilon = \frac{m^2}{s^3}$

$\varepsilon_t = -\alpha\, \varepsilon^2/k$

$\gamma = \beta - 1$

$k = k_0 (1 + ct)^\beta \qquad \varepsilon = \varepsilon_0 (1 + ct)^\gamma \qquad \alpha = \beta - 1/\beta$

$c = -\varepsilon_0/\beta k_0$

EFD: $-1.1 \le \beta \le -1.3$ choose $\beta = -6/5$ Saffman

$L = \frac{k^{3/2}}{\varepsilon} = \frac{k_0^{3/2}}{\varepsilon_0}(1 + ct)^{2/5}$

Example 3.2 A model for the evolution of turbulent kinetic energy and dissipation in isotropic turbulence

Let $k = \frac{1}{2}\overline{u_i'u_i'}$ and $\varepsilon = \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$. For isotropic turbulence in the moving frame, (3.15) simplifies to

$$\frac{dk}{dt} = -\varepsilon. \tag{3.29}$$

We need a closure assumption for ε, which must be expressed in terms of the two variables at hand, k and ε. Since k and ε respectively have dimensions L^2/T^2 and L^2/T^3, we may use dimensional arguments to *model* the temporal evolution of ε as follows:

$$\frac{d\varepsilon}{dt} = -\alpha\frac{\varepsilon^2}{k}, \tag{3.30}$$

where α is a constant to be determined later. We now have two ordinary differential equations for $k(t)$ and $\varepsilon(t)$, which can be solved by assuming power-law solutions of the forms

$$k = k_0(1+ct)^\beta \qquad \text{and} \qquad \varepsilon = \varepsilon_0(1+ct)^\gamma,$$

where k_0 and ε_0 are the initial values of k and ε, respectively, and c, β, and γ are constants. Substitution into the differential equations leads to the following relations for the undetermined coefficients (α, γ, and c) in terms of β and the initial data:

$$\gamma = \beta - 1, \qquad \alpha = \frac{\beta - 1}{\beta}, \qquad c = \frac{-\varepsilon_0}{\beta k_0}.$$

The experimental data of Comte-Bellot and Corrsin (1966) suggests that TKE decays at the rate β in the range -1.1 to -1.3. By choosing $\beta = -6/5$ in the middle of this range, we obtain the remaining coefficients, $\alpha = 11/6$, $\gamma = -11/5$, and $c = (5\varepsilon_0)/(6k_0)$.

Using k and ε, and dimensional arguments, we can estimate the evolution of the turbulent time and spatial scales as turbulence decays in time (or as flow moves downstream in a wind tunnel). For example, the turbulent length scale, defined as $l \sim k^{3/2}/\varepsilon$, evolves in time (or downstream in a wind tunnel) as

$$l \sim \frac{k^{3/2}}{\varepsilon} = \frac{k_0^{3/2}}{\varepsilon_0}(1+ct)^{2/5}.$$

> In the absence of turbulence production, the TKE decays, but the average large-eddy length scale grows.

A plausible explanation for the increase of the average turbulent length scale is that smaller large eddies with faster time scales are short-lived and decay faster, leaving the larger ones to contribute to the average large-eddy length scale or correlation length scale. (The correlation, or integral, length scale was introduced in Section 2.3.1.) The large-eddy Reynolds number, or turbulent Reynolds number, Re_l, can be determined from

$$Re_l = \frac{\sqrt{k}l}{\nu} \propto \frac{k_0^2}{\varepsilon_0\nu}(1+ct)^{-1/5}.$$

If $l \sim k^{3/2}/\varepsilon$ as suggested earlier, and the characteristic time scale $\tau \sim k/\varepsilon$, then we can define another expression for the turbulent Reynolds number based on the velocity, l/τ, and length scale of the large eddies:

$$Re_T = \frac{k^2}{\varepsilon\nu}. \tag{3.31}$$

This is typically smaller than the flow Reynolds number by a factor of 20–100.

Part 1 Eq. (1)

Part 2 Implications Turbulence modeling $\frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k}$ pg. 19

$\beta = -1.2$
$\gamma = -2.2$
$l \sim .4$
l growth $<< k, \varepsilon$ decay

Part 2 pg. 11
$\lambda^2 \sim t$
$\lambda \sim \sqrt{t}$

$l = L = k^{3/2}/\nu$
$Re_L = k^{1/2}L/\nu = \frac{k^2}{\varepsilon\nu}$

While the mean and large-scale flow features vary from flow to flow, the (6) **small scales** are largely independent of the large scales in high-Reynolds-number turbulent flows, and thus **exhibit statistical universality**.

(1) Larger scales small k and vice versa.

(2) E_{max} moves smaller k as t ↑ or smaller eddies die more quickly [than] large eddies.

(3) so L will grow

(4) decay & or per reduction area under spectrum

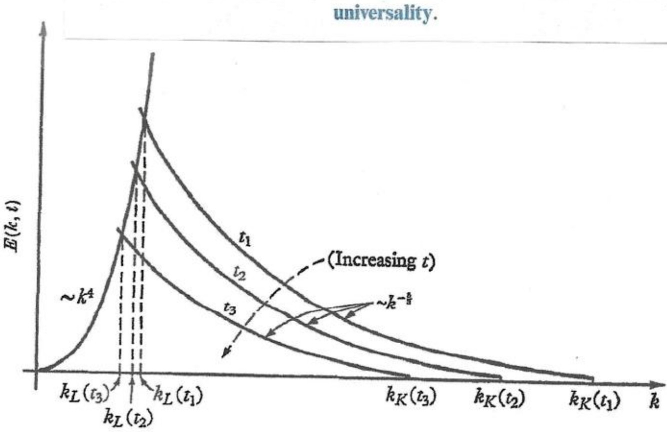


Fig. 4.5 Temporal evolution of a model energy spectrum, such as the von Kármán spectrum, for homogeneous isotropic turbulence. (Image credit: Comte-Bellot and Corrsin (1966), figure 13)

(5) For many conventional flows $P \sim \varepsilon$ (eg homogeneous shear flow Chapter 6 Part 3 and discussed Chapter 4 Part 7 topics)

$$-\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} = \nu \overline{u_{i,j} u_{i,j}}$$

assume far from walls

$\overline{u_i u_j} \sim u^2 \quad U_{i,j} \sim u/l$

$\overline{u_i u_j} \sim u\, l\, u/l \sim u\, l\, U_{i,j}$

$-\overline{u_i u_j}\, U_{i,j} = C\, u\, l\, U_{i,j}^2$

$C = \text{const } O(1) \quad = \nu \overline{u_{i,j} u_{i,j}}$

so $\overline{u_{i,j} u_{i,j}} >> U_{i,j} U_{i,j}$

$t_d \sim (\overline{u_{i,j} u_{i,j}})^{-1/2}$

$t_\ell \sim (U_{i,j} U_{i,j})^{-1/2}$

$t_\ell >> t_d$ i.e. dissipative time scale much faster meanflow time scale, which implies small eddies not directly affected ensuring constant range

Example 4.3 Conversion between $E_{11}(k_1)$ and $E(k)$

The von Kármán spectrum (von Kármán, 1948)

$$E(k) = 0.97\frac{k^4}{(1+k^2)^{17/6}}$$

is commonly used to model the energy spectrum in isotropic turbulence. Using (4.30), the corresponding 1D spectrum is

$$E_{11}(k_1) = 0.97 \times \frac{18}{55}(k_1^2+1)^{-5/6}.$$

Note that $E_{11}(k_1)$ tends to a constant value as $k_1 \to 0$, whereas $E(k) = O(k^4)$ as $k \to 0$. Both functions are plotted in Figure 4.4.

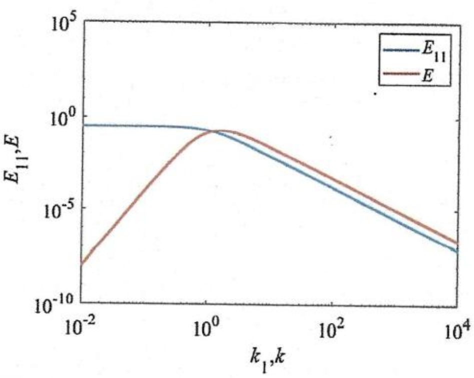


Fig. 4.4 Comparison of 1D and 3D energy spectra from Example 4.3.

Laws of Turbulence Decay

Self-similarity fixed-point/equilibrium solutions.

1. Final period/low Re$_T$: $R_{T_0} < 0.1, R_{T_\infty} = 0,$

$$k = \frac{5k_0}{2T_{t_0}} t^{-\frac{5}{2}}$$

$$\varepsilon = \frac{5\varepsilon_0}{2T_{t_0}} t^{-7/2}$$

2. High Re$_T$: $R^*_{T_0} = 10^3, R^*_{T_\infty} = 170$

$$k = \frac{k_0}{T_{t_0}} t^{-1}$$

$$\varepsilon = \frac{\varepsilon_0}{T_{t_0}} t^{-2}$$

As already mentioned, for High Re$_T$: Saffman theory, imposing the invariance of C, obtained

$$\overline{u^2} = KC^{2/5} t^{-6/5} \ (1.2)$$

Where K is a constant that depends upon the structure of the turbulence. To obtain this result, differently from Kolmogorov, Saffman only required self-similarity, not isotropy (Appendix A.6). Also obtained when discussing implications for turbulence modeling (Part 2) and Example 3.2 Moin and Chan (2025) Fundamentals of Turbulent Flows, Cambridge Press.

During the final decay predictions of the $-5/2$ (-2.5) law using different approaches agree such as Pope Ex. 6.10 and Hinze Section 3.3.

However, for high Re_T approaching or including the inertial sub range, there is no consensus many approaches, as vortex stretching, cannot be neglected and complex mathematical physics required leading to a large range of decay laws.

Kolmogorov, starting from the invariance of the Loitsyanskii integral, obtained that for isotropic turbulence $\overline{u^2} \propto t^{-10/7}$ (1.43) during decay (Appendix A.6).

Bachelor, Decay of turbulence in the initial period (1948) based on quasi-equilibrium/similarity, as also shown by Bernard: t^{-1}.

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Research

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Laws of turbulence decay from direct numerical simulations

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Inspection of available data on the decay exponent for the kinetic energy of homogeneous and isotropic turbulence (HIT) shows that it varies by as much as 100%. Measurements and simulations often show no correspondence with theoretical arguments, which are themselves varied. This situation is unsatisfactory given that HIT is a building block of turbulence theory and modelling. We take recourse to a large base of direct numerical simulations and study decaying HIT for a variety of initial conditions. We show that the Kolmogorov decay exponent and the Birkhoff–Saffman decay are both observed, albeit approximately, for long periods of time if the initial conditions are appropriately arranged. We also present, for both cases, other turbulent statistics such as the velocity derivative skewness, energy spectra and dissipation, and show that the decay and growth laws are approximately as expected theoretically, though the wavenumber spectrum near the origin begins to change relatively quickly, suggesting that the invariants do not strictly exist. We comment briefly on why the decay exponent has varied so widely in past experiments and simulations.

This article is part of the theme issue 'Scaling the turbulence edifice (part 1)'.

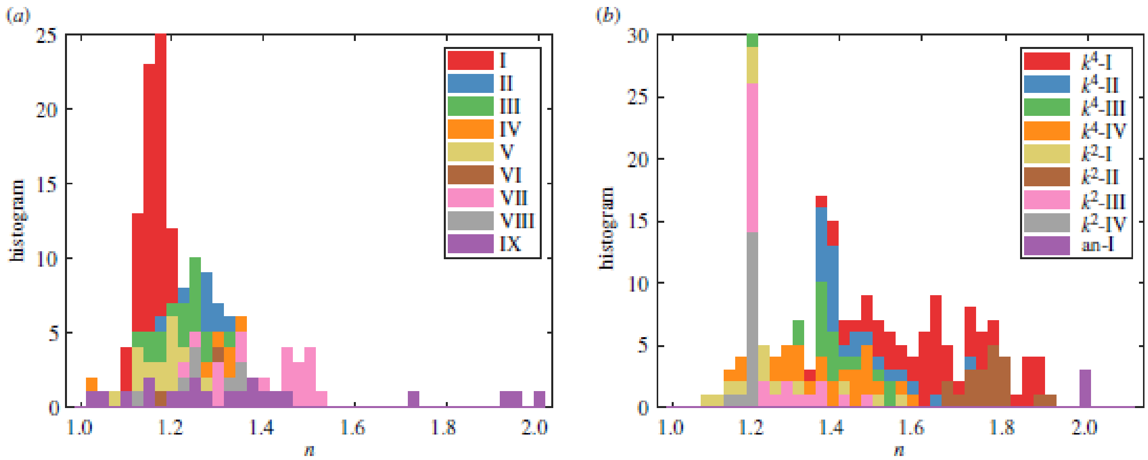

Figure 1. Histogram of decay exponents from the literature: (*a*) experiments, (*b*) simulations. Data in (*a*) are from I [5], II [6–9], III [10], IV [11–14], V [15–19], VI [20], VII [21], VIII [22–24], IX [10,14,22,23,25–28]. The data points IX correspond to fractal or active grids. In (*b*), the initial spectrum with $E(\kappa) \propto \kappa^4$ near the origin correspond to: κ^4-I [29–32], κ^4-II [33], κ^4-III [34–37], κ^4-IV [38]. The initial spectrum with $E(\kappa) \propto \kappa^2$ for small κ corresponds to cases: κ^2-I [39,40], κ^2-II [31], κ^2-III [33], κ^2-IV [35,36,41] an-I[42] corresponds to decay of an anisotropically forced turbulence. For the case [37], the simulations were compressible. Simulations here include DNS, LES using different numerical methods, and EDQNM. Some experiments and simulations are no doubt more thorough than others, but we cannot *a priori* discard any of them on the basis of available information. It should not be inferred that the 'correct' exponent is necessarily the most frequently observed one. (Online version in colour.)

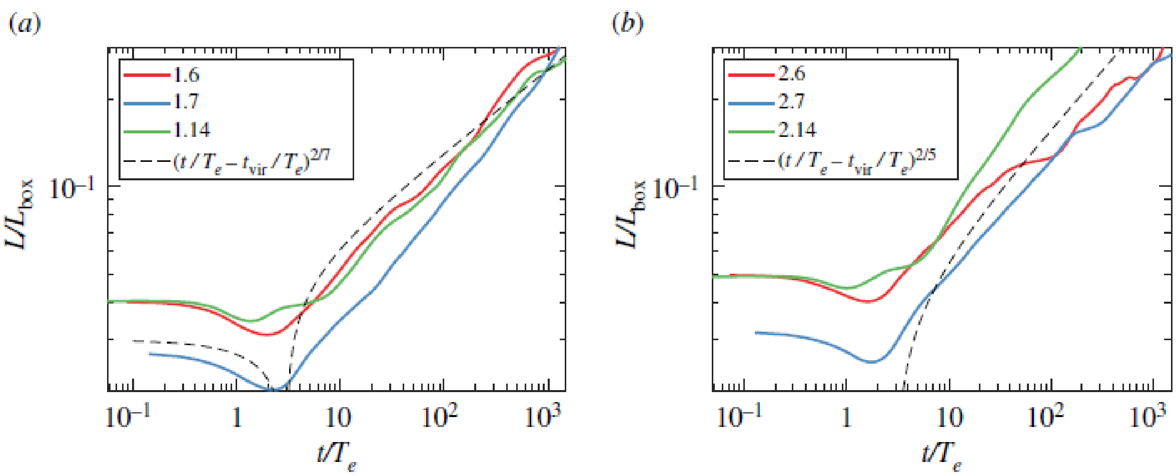

Figure 9. Scaling of integral length with time for (*a*) κ^4 and (*b*) κ^2; $t_{\rm vir}/T_e = 3$, but other nearby choices for it do not make a huge difference to our conclusions. (Online version in colour.)

Laws of turbulence decay from DNS

1. 0^{th} law of turbulence:

$$\varepsilon = C_\varepsilon u_0^3 / l_0 = L$$

$$\frac{dk}{dt} = -\varepsilon = \frac{3}{2}\frac{du_0^2}{dt} = -C_\varepsilon u_0^3 / L \qquad k = k_0 t^{-n}$$

$$k \sim t^{-2} \quad L \sim t^0 \qquad L = L_0 t^m$$

2. Self-Similarity: $k \sim t^{-1}$ $L \sim t^{1/2}$ 1 .5

I_{BS} 3. Birkhoff-Saffman: $n = 6/5$ $m = 2/5$ 1.2 .4

I_L 4. Loitsiansky: $n = 10/7$ $m = 2/7$ 1.43 .29

EFD: #3 DNS: #3 & #4 Both based on invariants of the K-H equation.

$$E(k,t) = \frac{2}{\pi} I_{BS}(t) k^2 + \frac{1}{3\pi} I_L(t) k^4 + \dots \quad \text{small } k$$

$$I_{BS} = u^2 \int_0^\infty r^2 \frac{1}{2r^2}\frac{\partial}{\partial r}\left[r^3 f(r,t)\right] dr \qquad I_L = u^2 \int_0^\infty r^4 f(r,t)\, dr$$

$I_{BS} = 0$ $E \propto k^4$: I_L = conservation angular momentum or depending I_L where $f(r,t)$ does not decay faster r^{-3} at large scales I_L diverges \& I_{BS} invariant such that $E \propto k^2$ as a result conservation linear momentum.

Appendix A

Note: To avoid confusion between $k(r,t)$ and TKE, a capital K will be used for the TKE, in this Appendix.

A.1

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = u_{rms}^3\left[\frac{\partial k}{\partial r} + \frac{4}{r}k\right] + 2\nu\overline{u^2}\left[\frac{4}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\right]$$

Multiply and divide by r^4

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\left[r^4\frac{\partial k}{\partial r} + 4r^3 k\right] + 2\nu\overline{u^2}\left[4r^3\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}\right]$$

$$= \frac{u_{rms}^3}{r^4}\left[\frac{\partial(r^4 k)}{\partial r} - \cancel{4r^3 k} + \cancel{4r^3 k}\right] + \frac{2\nu\overline{u^2}}{r^4}\left[\underbrace{4r^3\frac{\partial f}{\partial r} + r^4\frac{\partial^2 f}{\partial r^2}}_{\boxed{\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)}}\right]$$

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)$$

A.2

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (1A)$$

Taylor series for $f(r,t)$ and $k(r,t)$

$$f(r,t) = \underbrace{f(0,t)}_{\boxed{1}} + f''(0,t)\frac{r^2}{2!} + f^{IV}(0,t)\frac{r^4}{4!} + \ \ldots$$

$$k(r,t) = \underbrace{k'(0,t)}_{\boxed{=0}} r + k'''(0,t)\frac{r^3}{3!} + \cdots$$

Substitute into Eq. (1A)

$$\begin{aligned}\frac{\partial}{\partial t}&\left[\overline{u^2}\left(1 + f''(0,t)\frac{r^2}{2!} + f^{IV}(0,t)\frac{r^4}{4!}\right)\right] \\ &= \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}\left(r^4k'''(0,t)\frac{r^3}{3!}\right) \\ &+ \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\left(f''(0,t)r + f^{IV}(0,t)\frac{r^3}{3!}\right)\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial\overline{u^2}}{\partial t} &+ \frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2}f''(0,t)\right) + \frac{r^4}{4!}\frac{\partial}{\partial t}\left(f^{IV}(0,t)\right) \\ &= \frac{u_{rms}^3}{r^4}\left(\frac{7}{6}r^6k'''(0,t)\right) + \frac{2\nu\overline{u^2}}{r^4}\left(5r^4f''(0,t) + \frac{7}{6}r^6f^{IV}(0,t)\right)\end{aligned}$$

Simplify powers of r

$$\frac{\partial \overline{u^2}}{\partial t} + \frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2} f''(0,t)\right) + \frac{r^4}{4!}\frac{\partial}{\partial t}\left(f^{IV}(0,t)\right)$$
$$= \frac{7}{6} u_{rms}^3 r^2 k'''(0,t) + 10\nu\overline{u^2} f''(0,t) + \frac{7}{3}\nu\overline{u^2} r^2 f^{IV}(0,t)$$

Now, gather terms according to power of r

$$\boldsymbol{r^0}: \frac{\partial \overline{u^2}}{\partial t} = 10\nu\overline{u^2} f''(0,t) \quad (2A)$$

$$\boldsymbol{r^2}: \frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2} f''(0,t)\right) = \frac{7}{6} u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu\overline{u^2} r^2 f^{IV}(0,t) \quad (3A)$$

Only one term on the LHS for r^4, no need to consider it for this analysis.

Focus on Eq. (2A)

$$\frac{\partial \overline{u^2}}{\partial t} = 10\nu\overline{u^2} f''(0,t)$$

Using the definitions of turbulent kinetic energy

$$K = \frac{3}{2}\overline{u^2}$$

And Taylor microscale

$$\lambda_f^2 = -\frac{2}{f''(0,t)}$$

In Eq. (2A) yields

$$\frac{dK}{dt} = \frac{3}{2} 10\nu\overline{u^2}\left(-\frac{2}{\lambda_f^2}\right) = -30\frac{\nu\overline{u^2}}{\lambda_f^2}$$

And using the TKE equation for homogeneous isotropic turbulence

$$\frac{dK}{dt} = -\varepsilon$$

Gives

$$\varepsilon = 30\frac{\nu\overline{u^2}}{\lambda_f^2} = 15\frac{\nu\overline{u^2}}{\lambda_g^2}$$

Focus on Eq. (3A)

$$\frac{r^2}{2!}\frac{\partial}{\partial t}\left(\overline{u^2}f''(0,t)\right) = \frac{7}{6}u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu\overline{u^2}r^2 f^{IV}(0,t)$$

$$\frac{r^2}{2}\left(\frac{\partial\overline{u^2}}{\partial t}f''(0,t) + \overline{u^2}\frac{\partial f''(0,t)}{\partial t}\right) = \frac{7}{6}u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu\overline{u^2}r^2 f^{IV}(0,t) \quad (4A)$$

Now focus on term in parenthesis on LHS

$$\frac{\partial \overline{u^2}}{\partial t} f''(0,t) + \overline{u^2}\frac{\partial f''(0,t)}{\partial t} = \frac{2}{3}\frac{\partial K}{\partial t} f''(0,t) + \frac{2}{3}K\frac{\partial f''(0,t)}{\partial t} \quad (5A)$$

Substituting

$$\frac{dK}{dt} = -\varepsilon$$

Into Eq. (5A)

$$-\frac{2}{3}\varepsilon f''(0,t) + \frac{2}{3}K\frac{\partial f''(0,t)}{\partial t} \quad (6A)$$

And using

$$f''(0,t) = -\frac{\varepsilon}{15\nu\overline{u^2}} = -\frac{\varepsilon}{10\nu K}$$

Into Eq. (6A) gives

$$\frac{2}{3}\frac{\varepsilon^2}{10\nu K} - \frac{2}{3}K\frac{d}{dt}\left(\frac{\varepsilon}{10\nu K}\right) = \frac{\varepsilon^2}{15\nu K} - \frac{K}{15\nu}\left(\frac{1}{K}\frac{d\varepsilon}{dt} - \frac{\varepsilon}{K^2}\underbrace{\frac{dK}{dt}}_{\boxed{-\varepsilon}}\right)$$

$$= \frac{\varepsilon^2}{15\nu K} - \frac{K}{15\nu}\left(\frac{1}{K}\frac{d\varepsilon}{dt} + \frac{\varepsilon^2}{K^2}\right) = \cancel{\frac{\varepsilon^2}{15\nu K}} - \frac{1}{15\nu}\frac{d\varepsilon}{dt} - \cancel{\frac{\varepsilon^2}{15\nu K}}$$

Therefore, it was proved that,

$$\frac{\partial \overline{u^2}}{\partial t} f''(0,t) + \overline{u^2} \frac{\partial f''(0,t)}{\partial t} = -\frac{1}{15\nu}\frac{d\varepsilon}{dt} \qquad (7A)$$

Substituting Eq. (7A) into Eq. (4A) gives

$$-\frac{r^2}{2}\left(\frac{1}{15\nu}\frac{d\varepsilon}{dt}\right) = \frac{7}{6} u_{rms}^3 r^2 k'''(0,t) + \frac{7}{3}\nu \overline{u^2} r^2 f^{IV}(0,t)$$

And isolating $d\varepsilon/dt$

$$\frac{d\varepsilon}{dt} = -35\nu u_{rms}^3 k'''(0,t) - 70\nu^2 \overline{u^2} f^{IV}(0,t)$$

Which is equivalent to

$$\frac{d\varepsilon}{dt} = S_k^* R_T^{1/2} \frac{\varepsilon^2}{k} - G^* \frac{\varepsilon^2}{k}$$

As shown in Chapter 5 Part 1.

A.3

$$\frac{\partial}{\partial t}\left[\overline{u^2}f\right] = \frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \quad (8A)$$

Self-similarity

$$f(r,t) = \tilde{f}(r/\lambda(t)) = \tilde{f}(r/L(t)) = \tilde{f}(\eta)$$
$$k(r,t) = \tilde{k}(r/\lambda(t)) = \tilde{k}(\eta)$$

$$\eta = r/\lambda$$
$$\frac{\partial\eta}{\partial\lambda} = \frac{\partial}{\partial\lambda}\left(\frac{r}{\lambda}\right) = -\frac{r}{\lambda^2} = -\frac{\eta}{\lambda} \quad (9A)$$

Focus on LHS of Eq. (8A)

$$\frac{\partial}{\partial t}\left[\overline{u^2}\tilde{f}\right] = \frac{2}{3}\frac{dK}{dt}\tilde{f} + \frac{2}{3}K\frac{\partial\tilde{f}}{\partial t}$$
$$= \frac{2}{3}\frac{dK}{dt}\tilde{f} + \frac{2}{3}K\frac{\partial\tilde{f}}{\partial t} \quad (10A)$$

$$\frac{\partial\tilde{f}}{\partial t} = \frac{\partial\tilde{f}}{\partial\lambda}\underbrace{\frac{\partial\lambda}{\partial t}}_{\boxed{\dot{\lambda}}} = \frac{\partial\tilde{f}}{\partial\eta}\underbrace{\frac{\partial\eta}{\partial\lambda}}_{\boxed{-\frac{\eta}{\lambda}}}\dot{\lambda} = -\dot{\lambda}\frac{\eta}{\lambda}\frac{d\tilde{f}}{d\eta}$$

Therefore, Eq. (10A) becomes,

$$\frac{\partial}{\partial t}\left[\overline{u^2}\tilde{f}\right] = \frac{2}{3}\frac{dK}{dt}\tilde{f} - \frac{2}{3}K\dot{\lambda}\frac{\eta}{\lambda}\frac{d\tilde{f}}{d\eta}$$

Now, focus on RHS Eq. (8A)

$$\frac{u_{rms}^3}{r^4}\frac{\partial}{\partial r}(r^4 k) + \frac{2\nu\overline{u^2}}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right)$$

Assuming Self-similarity: $r = \eta\lambda_g$

$$\frac{u_{rms}^3}{(\eta\lambda_g)^4}\underbrace{\frac{\partial}{\partial r}}_{\boxed{\frac{1}{\lambda_g}\frac{d}{d\eta}}}\left((\eta\lambda_g)^4\tilde{k}\right) + \frac{2\nu\overline{u^2}}{(\eta\lambda_g)^4}\underbrace{\frac{\partial}{\partial r}}_{\boxed{\frac{1}{\lambda_g}\frac{d}{d\eta}}}\left((\eta\lambda_g)^4\frac{1}{\lambda_g}\frac{d\tilde{f}}{d\eta}\right)$$

$$\frac{u_{rms}^3}{\eta^4\lambda_g}\frac{d}{d\eta}(\eta^4\tilde{k}) + \frac{2\nu\overline{u^2}}{\eta^4\lambda_g^2}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

$$\sqrt{\frac{2}{3}}\frac{K^{3/2}}{\eta^4\lambda_g}\frac{d}{d\eta}(\eta^4\tilde{k}) + \frac{4\nu K}{3\eta^4\lambda_g^2}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

Therefore, Eq. (8A) becomes,

$$\frac{2}{3}\frac{dK}{dt}\tilde{f} - \frac{2}{3}K\dot{\lambda}_g\frac{\eta}{\lambda_g}\frac{d\tilde{f}}{d\eta} = \left(\frac{2}{3}\right)^{3/2}\frac{K^{3/2}}{\eta^4\lambda_g}\frac{d}{d\eta}(\eta^4\tilde{k}) + \frac{4\nu K}{3\eta^4\lambda_g^2}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

And multiplying by $3\lambda_g^2/2\nu K$

$$\underbrace{\frac{\lambda_g^2}{\nu K}\frac{dK}{dt}\tilde{f}}_{\boxed{1}} - \underbrace{\lambda_g\dot{\lambda}_g\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta}}_{\boxed{2}} = \underbrace{\left(\frac{2}{3}\right)^{\frac{1}{2}}\frac{\lambda_g K^{\frac{1}{2}}}{\nu\eta^4}\frac{d}{d\eta}(\eta^4\tilde{k})}_{\boxed{3}} + \frac{2}{\eta^4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right) \quad (11A)$$

Term 1:

$$\frac{\lambda_g^2}{\nu K}\frac{dK}{dt}\tilde{f} = -\frac{\lambda_g^2}{\nu K}\varepsilon\tilde{f} = -\frac{\lambda_g^2}{\nu K}\frac{10\nu K}{\lambda_g^2}\tilde{f} = -10\tilde{f}$$

Term 2:

$$-\lambda_g\dot{\lambda_g}\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = -\sqrt{\frac{10\nu K}{\varepsilon}}\frac{d}{dt}\left(\sqrt{\frac{10\nu K}{\varepsilon}}\right)\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = -\frac{1}{2}\frac{d}{dt}\left(\frac{10\nu K}{\varepsilon}\right)\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta}$$

$$= -5\eta\left(\underbrace{\frac{dK}{dt}}_{\boxed{-\varepsilon}}\frac{1}{\varepsilon} - \frac{K}{\varepsilon^2}\frac{d\varepsilon}{dt}\right)\frac{d\tilde{f}}{d\eta} = -5\eta\left(-1 - \frac{K}{\varepsilon^2}\frac{d\varepsilon}{dt}\right)\frac{d\tilde{f}}{d\eta}$$

Now substitute decay equation for ε

$$-\lambda_g\dot{\lambda_g}\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = -5\eta\left(-1 - \frac{K}{\varepsilon^2}\left(\frac{7}{3\sqrt{15}}S_{k_0}R_T^{\frac{1}{2}}\frac{\varepsilon^2}{k} - \frac{7}{15}G_0\frac{\varepsilon^2}{k}\right)\right)\frac{d\tilde{f}}{d\eta}$$

$$= 5\eta\frac{d\tilde{f}}{d\eta} + \frac{35}{3\sqrt{15}}\eta S_{k_0}R_T^{\frac{1}{2}}\frac{d\tilde{f}}{d\eta} - \frac{7}{3}\eta G_0\frac{d\tilde{f}}{d\eta} \quad (12A)$$

Recall relation between R_T and R_λ

$$\sqrt{R_T} = \sqrt{\frac{3}{20}}R_\lambda$$

And substitute into Eq. (12A)

$$-\lambda_g\dot{\lambda_g}\frac{\eta}{\nu}\frac{d\tilde{f}}{d\eta} = 5\eta\frac{d\tilde{f}}{d\eta} + \frac{7\cdot\cancel{5}}{3\sqrt{\cancel{5}\cdot\cancel{3}}}\eta S_{k_0}\sqrt{\frac{\cancel{3}}{\cancel{5}\cdot 2^2}}R_\lambda\frac{d\tilde{f}}{d\eta} - \frac{7}{3}\eta G_0\frac{d\tilde{f}}{d\eta}$$

$$= 5\eta\frac{d\tilde{f}}{d\eta} + \frac{7}{6}\eta S_{k_0}R_\lambda\frac{d\tilde{f}}{d\eta} - \frac{7}{3}\eta G_0\frac{d\tilde{f}}{d\eta}$$

Term 3:

$$\left(\frac{2}{3}\right)^{\frac{1}{2}}\frac{\lambda_g K^{\frac{1}{2}}}{\nu\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)=\frac{\lambda_g\sqrt{\overline{u^2}}}{\nu\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)=\frac{R_\lambda}{\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)$$

Therefore, Eq. (11A) becomes,

$$-10\tilde{f}+\frac{7}{6}\eta S_{k_0}R_\lambda\frac{d\tilde{f}}{d\eta}+\left(5-\frac{7}{3}G_0\right)\eta\frac{d\tilde{f}}{d\eta}=\frac{R_\lambda}{\eta^4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)+\frac{2}{\eta^4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)$$

Reordering the term yields

$$\frac{2}{\eta^4}\frac{d}{d\eta}\left(\eta^4\frac{d\tilde{f}}{d\eta}\right)+\eta\frac{d\tilde{f}}{d\eta}\left(\frac{7}{3}G_0-5\right)+10\tilde{f}=R_\lambda\left(\frac{7}{6}\eta S_{k_0}\frac{d\tilde{f}}{d\eta}-\eta^{-4}\frac{d}{d\eta}\left(\eta^4\tilde{k}\right)\right)$$

A.4

Loitsyanskii integral

$$B_2 = \int_0^\infty \overline{u^2} r^4 f(r,t) dr$$

Using (from Hinze)

$$\overline{u^2} = -4A_2 t^{-\frac{5}{2}} = c \times t^{-\frac{5}{2}}$$

$$f(r,t) = e^{-\frac{r^2}{8\nu t}}$$

And substituting into the expression for B_2

$$B_2 = c \times t^{-\frac{5}{2}} \int_0^\infty r^4 e^{-\frac{r^2}{8\nu t}} dr \qquad (13A)$$

Assuming full self-similarity, since the solution is in the final decay region, the variable η can be used to describe $f(r,t)$ such that,

$$f(r,t) = \tilde{f}\left(\frac{r}{\lambda(t)} = \eta\right)$$

Where $\lambda(t)$ represents the Taylor microscale, that varies with time:

$$\lambda(t) \propto \sqrt{t}$$

as shown in Chapter 5 Part 2.

Substituting $r = \eta\lambda$, $dr = \lambda d\eta$ in Eq. (13A) yields

$$B_2 = c \times t^{-\frac{5}{2}} \int_0^\infty \eta^4 \lambda^4 e^{-\frac{\eta^2\lambda^2}{8\nu t}} \lambda d\eta$$

$$B_2 = c \times \lambda^5 t^{-\frac{5}{2}} \int_0^\infty \eta^4 e^{-\frac{\eta^2\lambda^2}{8\nu t}} d\eta \qquad (14A)$$

Evaluating the integral in Eq. (14A) gives

$$\int_0^\infty \eta^4 e^{-a\eta^2} d\eta = \left[\frac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{a}\eta)}{8a^{\frac{5}{2}}} - \frac{(2a\eta^3 + 3\eta)e^{-a\eta^2}}{4a^2}\right]_0^\infty \qquad (15A)$$

Where:

$$a = \frac{\lambda^2}{8\nu t}$$

Evaluating Eq. (15A) at 0 and ∞ gives

$$\int_0^\infty \eta^4 e^{-a\eta^2} d\eta = \frac{3\sqrt{\pi}}{8a^{\frac{5}{2}}}$$

And substituting back into Eq. (13A)

$$B_2 = c \times \lambda^5 t^{-\frac{5}{2}} \frac{3\sqrt{\pi}}{8\left(\frac{\lambda^2}{8\nu t}\right)^{\frac{5}{2}}} = c \times \frac{\lambda^5 t^{-\frac{5}{2}}}{\lambda^5 t^{-\frac{5}{2}}} = c \neq f(t)$$

A.5

Handout 18: Decaying turbulence

In the absence of forcing, turbulence can only decay. The energy dissipation rate still plays an important role, but it is no longer constant, but it determines nor the decay of the mean squared velocity, specifically $\frac{1}{2}\langle \boldsymbol{u}^2 \rangle \equiv \mathcal{E}(t)$. We use here $\mathcal{E}(t)$ to denote the mean energy density and to distinguish it from the energy spectrum $E(k,t)$. The two are of course related via

$$\mathcal{E}(t) = \int_0^\infty E(k,t)\,\mathrm{d}k. \tag{1}$$

In decaying turbulence, both functions are decaying, and this is governed just by ϵ, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\epsilon(t). \tag{2}$$

Using dimensional arguments, we have $\epsilon \sim U^3/\xi$, where U is the typical velocity and ξ some typical length scale of the turbulence. Both are time-dependent; U can be related to $\mathcal{E}(t) \sim U^2$ and $\xi(t) \propto t^q$ is as yet undetermined, but in decaying turbulence, the small scales will die out, so only large eddies remain, so we expect $\xi(t)$ to grow, so $q > 0$. Once we know q, we can proceed and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\mathcal{E}^{3/2}t^{-q}. \tag{3}$$

This can be integrated

$$\int \mathcal{E}^{-3/2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\int t^{-q}\mathrm{d}t, \tag{4}$$

so $\mathcal{E}^{-1/2} \sim t^{1-q}$ or

$$\mathcal{E} \propto t^{-p}, \quad \text{with } p = 2(1-q). \tag{5}$$

1 Relation to conservation laws

Conflicting results about the decay can be found in the literature. The results depend on the governing physical processes involved and the initial conditions.

Kolmogorov made predictions under the assumption that the so-called Loitsianskii[1] integral is conserved. This integral is related to the angular momentum, $\boldsymbol{x} \times \boldsymbol{u}$. The actual Loitsianskii integral is defined in terms of a two-point correlation function as

$$\mathcal{L} = \int \boldsymbol{r}^2 \langle \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) \rangle \, d^3\boldsymbol{r}, \tag{6}$$

so it has dimensions

$$\mathcal{L} \propto \ell^5 u_\ell^2 \propto L^7 T^{-2}. \tag{7}$$

This suggests that $q = 2/7$. With such an assumption, one finds from Equation (5)

$$p = 2(1-2/7) = 2 \times 5/7 = 10/7 \approx 1.43. \tag{8}$$

This did not agree too well with the available experiments.

Later, Saffman (1967) proposed that the linear momentum might be "more conserved". The debate is ongoing (Davidson, 2000), but according to Saffman the relevant integral is

$$\mathcal{S} = \int \langle \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) \rangle \, d^3\boldsymbol{r}, \tag{9}$$

which has the dimension L^5T^{-2}, so $q = 2/5$ and therefore

$$p = 2(1-2/5) = 2 \times 3/5 = 6/5 \approx 1.2. \tag{10}$$

[1] Different spellings can be found, depending on who translated first into which language. Alternative spellings include: Loitsyanskii, Loitsiansky, and Loitsyansky

Table 1: Scaling exponents and relation to physical invariants and their dimensions.

α	p	q	inv.	dim.
4	$10/7 \approx 1.43$	$2/7 \approx 0.286$	$\mathcal{L}$	$[x]^7[t]^{-2}$
3	$8/6 \approx 1.33$	$2/6 \approx 0.333$		
2	$6/5 = 1.20$	$2/5 = 0.400$		
1	$4/4 = 1.00$	$2/4 = 0.500$	$\langle \boldsymbol{A}_{\rm 2D}^2 \rangle$	$[x]^4[t]^{-2}$
0	$2/3 \approx 0.67$	$2/3 \approx 0.667$	$\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$	$[x]^3[t]^{-2}$
-1	$0/2 = 0.00$	$2/1 = 1.000$		

2 Relation to initial conditions

A completely different idea is to assume a connetion with the initial spectrum, so let us write

$$E(k,t) \propto k^{\alpha}, \tag{11}$$

up to some cutoff scale, so we better write

$$E(k,t) \propto k^{\alpha} \psi k \xi(t), \tag{12}$$

Integrating over k yields

$$\mathcal{E}(t) = \int_0^\infty E(k,t)\,\mathrm{d}k = \xi^{-(\alpha+1)} \int_0^\infty (k\xi)^\alpha \psi(k\xi,t)\,\mathrm{d}k\xi \tag{13}$$

so

$$\mathcal{E}(t) \propto \xi^{-(\alpha+1)}. \tag{14}$$

The Navier-Stokes equations are invariant under rescaling, $x \to \tilde{x}\ell$ and $t \to \tilde{t}\ell^{1/q}$, which implies corresponding rescalings for velocity $u \to \tilde{t}\ell^{1-1/q}$ and viscosity $\nu \to \tilde{\nu}\ell^{2-1/q}$. However, ψ should be universal, but since it has dimensions,

$$[\psi] = [E][L]^\alpha = L^{3+\alpha}T^{-2}, \tag{15}$$

we can determine q; see Table 1 for a summary and Figure 1 for numerical results for α close to 4.

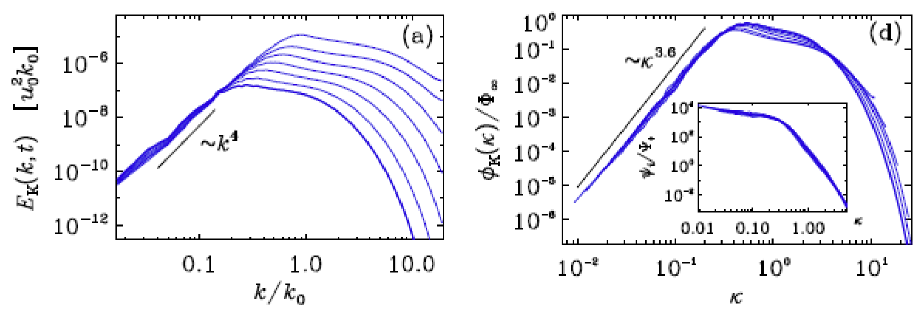

Figure 1: Spectra at different times (left) and collapsed spectra (right).

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A.6

Note on Decay of Homogeneous Turbulence

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The assumption of self-similarity and the existence of an exact invariant are combined to predict the decay rate of homogeneous turbulence.

On the assumptions that turbulence remains self-similar during decay and that the "Loitsianskii integral"

$$u^2 \int_0^\infty r^4 f(r)\, dr$$

remains invariant, Kolmogorov[1] predicted for isotropic turbulence that $u^2 \propto t^{-10/7}$ and $L \propto t^{2/7}$ during decay. Here, $u = \langle u_i^2 \rangle^{\frac{1}{2}}$ is the root-mean-square velocity, $f(r)$ is the longitudinal correlation function[2], and L is an integral length scale. An alternative deriviation of the Kolmogorov decay law based on equivalent assumptions has been given recently by Comte-Bellot and Corrsin[3], who have also discussed extensively the comparison with experiment.

Now, it has been known for some time[4,5] that the Loitsianskii integral is not invariant, so that the Kolmogorov decay law is of particularly doubtful significance unless it can be shown that the change in the Loitsianskii integral is slow compared with the energy decay. However, a more important objection is that recent work by the author[6] has confirmed the speculation by Birkhoff[7] that the Loitsianskii integral is in general divergent, and that it is only for a restricted type of isotropic turbulence that the Loitsianskii integral exists.[8]

On the other hand, for general homogeneous turbulence it was found that another invariant exists, namely

$$\int_0^\infty r^2 R(r)\, dr = C, \qquad (1)$$

where

$$R(r) = \frac{1}{8\pi r^2} \int_{|\mathbf{r}|=r} R_{ii}(\mathbf{r})\, dA(\mathbf{r});$$

$R_{ij}(\mathbf{r})$ is the velocity covariance tensor, and $dA(\mathbf{r})$ is the element of area on a sphere of radius r. The equivalent statement to (1) for the energy spectrum function is that $E(k) \sim (2C/\pi)\, k^2$ as $k \to 0$, where C is a constant which will not in general be zero. For isotropic turbulence, $R(r) = \frac{1}{6}\, u^2\, (3f + rf')$; and the condition for the Loitsianskii integral to exist is $C = 0$, since from (1) $f(r) \sim (6C/u^2) r^{-3}$ as $r \to \infty$.

If we now follow Kolmogorov[1], or Comte-Bellot and Corrsin[3], but replace the invariance of the Loitsianskii integral by the invariance of (1) or the equivalent condition on $E(k)$, we find for the decay rate

$$u^2 = KC^{2/5} t^{-6/5}, \qquad L = K' C^{1/5} t^{2/5}, \qquad (2)$$

where K, K' are constants that depend upon the structure of the turbulence. A simple way of deriving these results is to write $R(r) = u^2 \psi(r/L)$ from the assumption of self-similarity and $du^2/dt = -Au^3/L$ from the further assumption of Reynolds number independence (this is basically equivalent to the assumption[1,3] that an inertial subrange exists). The results (2) follow immediately with K, K' related to A and $\int_0^\infty \rho^2\, \psi(\rho) d\rho$.

Notice that there is no need to assume that the turbulence is isotropic, but the assumption of self-similarity is of course crucial. Comparison with the experimental data[3] shows that the results (2) fit the measurements for the initial period of decay at least as well and probably better than the Kolmogorov decay law. Indeed, the agreement is much closer than the nature of the assumptions would entitle one to expect.

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[8] Similar conclusions have been reached independently by Professor O. M. Phillips.

Turbulent Flow Measurements Utilizing the Doppler Shift of Scattered Laser Radiation

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The probability function for the turbulent velocity in a duct flow is determined from the frequency shift of laser illumination scattered by small particles contained in the flow. From this, the mean turbulent velocity and the intensity of turbulence are obtained.

The feasibility of measuring steady fluid velocities from the Doppler shift of scattered laser radiation was first demonstrated by Yeh and Cummins.[1]

A.7

We start with the given general solution:

$$R_{ii}(r,t) = \varphi(t)\psi(\chi) \tag{1}$$

$$= \frac{\sqrt{8\nu}}{r}\exp(-r^2/8\nu t)\sum_{p=1}^{\infty}\frac{A_p}{t^p}H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) \tag{2}$$

where H_{2p-1} are the Hermite polynomials.

We impose the second moment condition:

$$\int_0^{\infty} dr\ r^2 R_{ii}(r,t) = 0. \tag{3}$$

Substituting the general solution,

$$\int_0^{\infty} dr\ r^2\frac{\sqrt{8\nu}}{r}\exp(-r^2/8\nu t)\sum_{p=1}^{\infty}\frac{A_p}{t^p}H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) = 0. \tag{4}$$

Rewriting:

$$\sum_{p=1}^{\infty}\frac{A_p}{t^p}\int_0^{\infty} dr\ r\exp(-r^2/8\nu t)H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) = 0. \tag{5}$$

Due to the orthogonality of Hermite polynomials, the integral vanishes for all $p \neq 1$, but the term for $p = 1$ does not satisfy the condition, forcing $A_1 = 0$.

A.8

The fourth moment condition states:

$$I_4 = \int_0^\infty dr\, r^4 R_{ii}(r,t) = \text{constant} \tag{1}$$

where I_4 is known as the **Loitsyanskii integral**, which is conserved in certain turbulence models.

From the general solution:

$$R_{ii}(r,t) = \frac{\sqrt{8\nu}}{r} \exp(-r^2/8\nu t) \sum_{p=1}^{\infty} \frac{A_p}{t^p} H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) \tag{2}$$

Substituting this into the fourth moment integral:

$$I_4 = \int_0^\infty dr\, r^4 \frac{\sqrt{8\nu}}{r} \exp(-r^2/8\nu t) \sum_{p=1}^{\infty} \frac{A_p}{t^p} H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right) \tag{3}$$

$$= \sqrt{8\nu} \sum_{p=1}^{\infty} \frac{A_p}{t^p} \int_0^\infty dr\, r^3 \exp(-r^2/8\nu t) H_{2p-1}\left(\frac{r}{\sqrt{8\nu t}}\right). \tag{4}$$

Let:

$$\chi = \frac{r}{\sqrt{8\nu t}} \Rightarrow dr = \sqrt{8\nu t}\, d\chi. \tag{5}$$

Rewriting the integral:

$$I_4 = \sqrt{8\nu} \sum_{p=1}^{\infty} \frac{A_p}{t^p} \int_0^\infty d\chi\, (8\nu t)^{5/2} \chi^3 \exp(-\chi^2) H_{2p-1}(\chi). \tag{6}$$

Factoring out the time dependence:

$$I_4 = (8\nu)^{3/2} t^{(5/2)-p} \sum_{p=1}^{\infty} A_p \int_0^\infty d\chi\, \chi^3 \exp(-\chi^2) H_{2p-1}(\chi). \tag{7}$$

Using the orthogonality properties of Hermite polynomials and known integral results:

$$\int_0^\infty d\chi\, \chi^3 \exp(-\chi^2) H_{2p-1}(\chi) \neq 0 \tag{8}$$

only if $2p - 1 = 3$, which implies:

$$p = 2. \tag{9}$$

For I_4 to remain constant, the exponent of t in the prefactor must be zero:

$$\frac{5}{2} - p = 0 \Rightarrow p = 2. \tag{10}$$

Thus, only the $p = 2$ term contributes to the fourth moment integral, while all other terms vanish. Therefore, the condition implies that only A_2 can be nonzero, ensuring consistency with the conservation of I_4.