Chapter 5: Energy Decay in Isotropic Turbulence

Decay process of TKE by viscous dissipation is ideally studied for homogeneous isotropic turbulence since it contains all essential physics while yielding equations in their simplest forms. The results are used in many turbulence models, which are applied to general flows.

Part 1: Energy Decay

Idealized problem: large region far from boundaries or box with periodic boundary conditions. In either case region/box large enough such that BCs do not influence core flow and $f(r)$ and $k(r)$ decay to zero with r well within the domain.

Concept is that a complete statistical description of a homogenous isotropic turbulence is specified in the region/box of interest; and the GDE are solved for the calculation of the energy decay via the solution of the k, ε equations.

Note that all variables are f(t) such that ensemble or spatial averaging is required.

Turbulent Kinetic Energy Equation

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i{}^2}\right) + \cancel{\overline{U_j}\left(\frac{1}{2}\overline{u_i{}^2}\right)_{,j}}}_{\boxed{\frac{Dk}{Dt}}} + \cancel{\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\overline{u_j p'}\right) + \frac{1}{2}\overline{u_i{}^2 u_j}\right]} = \cancel{\nu\nabla^2 k} + \cancel{P} - \tilde{\varepsilon}$$

Where:

$$P = -\overline{u_i u_j}\,\overline{U_{i,j}}$$

$$\tilde{\varepsilon} = \nu\overline{u_{i,j}{}^2}$$

ε Equation

$$\frac{\partial \varepsilon}{\partial t} + \cancel{\overline{U_j}\frac{\partial \varepsilon}{\partial x_j}} = \frac{D\epsilon}{Dt} = \cancel{P_\epsilon^1} + \cancel{P_\epsilon^2} + \cancel{P_\epsilon^3} + P_\epsilon^4 + \cancel{\Pi_\epsilon} + \cancel{T_\epsilon} + \cancel{D_\epsilon} - \Upsilon_\epsilon,$$

$$P_\epsilon^1 = -\epsilon_{ij}^c \frac{\partial \overline{U}_i}{\partial x_j} \tag{3.43}$$

$$P_\epsilon^2 = -\epsilon_{ij} \frac{\partial \overline{U}_i}{\partial x_j} \tag{3.44}$$

$$P_\epsilon^3 = -2\nu \overline{u_k \frac{\partial u_i}{\partial x_j}} \frac{\partial^2 \overline{U}_i}{\partial x_k \partial x_j} \tag{3.45}$$

$$P_\epsilon^4 = -2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_j}} \tag{3.46}$$

$$\Pi_\epsilon = -\frac{2\nu}{\rho}\frac{\partial}{\partial x_i}\left(\overline{\frac{\partial p}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\right) \tag{3.47}$$

$$T_\epsilon = -\nu\frac{\partial}{\partial x_k}\left(\overline{u_k \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\right) \tag{3.48}$$

$$D_\epsilon = \nu\,\nabla^2 \epsilon \tag{3.49}$$

$$\Upsilon_\epsilon = 2\nu^2 \overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)^2}. \tag{3.50}$$

In Eqs (3.43) and (3.44),

$$\epsilon_{ij}^c = 2\nu \overline{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}} \tag{3.51}$$

$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{3.52}$$

Enstrophy equation

$\zeta = \overline{\underline{\omega} \cdot \underline{\omega}} = \overline{\omega_i^2} = \frac{\varepsilon}{\nu} - \overline{u_{i,j} u_{j,i}} = \frac{\tilde{\varepsilon}}{\nu}$

$$\frac{D\zeta}{Dt} = \cancel{P_\zeta^1} + \cancel{P_\zeta^2} + \cancel{P_\zeta^3} + P_\zeta^4 + \cancel{T_\zeta} + \cancel{D_\zeta} - \Upsilon_\zeta, \quad (3.82)$$

where

$$P_\zeta^1 = 2\overline{\omega_i \omega_k} \frac{\partial \overline{U}_i}{\partial x_k} \quad (3.83)$$

$$P_\zeta^2 = 2\overline{\omega_i \frac{\partial u_i}{\partial x_k}} \overline{\Omega}_k \quad (3.84)$$

$$P_\zeta^3 = -2\overline{u_k \omega_i} \frac{\partial \overline{\Omega}_i}{\partial x_k} \quad (3.85)$$

$$P_\zeta^4 = 2\overline{\omega_i \omega_k \frac{\partial u_i}{\partial x_k}} \quad (3.86)$$

$$T_\zeta = -\frac{\partial}{\partial x_k} \overline{(u_k \omega_i \omega_i)} \quad (3.87)$$

$$D_\zeta = \nu \nabla^2 \zeta \quad (3.88)$$

$$\Upsilon_\zeta = 2\nu \overline{\frac{\partial \omega_i}{\partial x_k} \frac{\partial \omega_i}{\partial x_k}}. \quad (3.89)$$

Similar TKE, but in this case dissipation of vorticity

ζ equation derived from $\underline{\Omega}$ equation by writing Ω_i form $\times\ \omega_i$ and average

Note absence p, but many difficulties in modeling as per ε

P_ζ terms = production; P_ζ^2 and P_ζ^3 include transport and stretching

T_ζ term = transport

D_ζ term = diffusion

Υ_ζ term = dissipation

For homogenous isotropic turbulence:

$$\frac{dk}{dt} = -\varepsilon \quad (1)$$

where $\varepsilon = \tilde{\varepsilon} = \nu\overline{u_{i,j}u_{i,j}}$. Clearly, the other terms in the TKE equation are zero for isotropic turbulence. The governing equation for ε simplifies as

$$\frac{d\varepsilon}{dt} = P_\varepsilon^4 - \Upsilon_\varepsilon = -2\nu\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}} - 2\nu^2\overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2} \quad (2a)$$

Since clearly the other terms in the ε equation are zero for isotropic turbulence and it will be shown that both terms on RHS of Eq. (2a) are equal to a constant and therefore cannot be reduced to the gradient of fluctuating terms, which would be equal to zero.

Since $\tilde{\varepsilon} = \nu\zeta$

$$\zeta = \overline{\underline{\omega} \cdot \underline{\omega}} = \overline{\omega_i^2} = \frac{\varepsilon}{\nu} - \frac{\overline{u_{i,j}u_{j,i}}}{\nu} = \frac{\tilde{\varepsilon}}{\nu}$$

$$\frac{d\tilde{\varepsilon}}{dt} = \nu P_\zeta^4 - \nu\Upsilon_\zeta = +2\nu\overline{\omega_i\omega_k\frac{\partial u_i}{\partial x_k}} - 2\nu^2\overline{\frac{\partial \omega_i}{\partial x_k}\frac{\partial \omega_i}{\partial x_k}} \quad (2b)$$

The physics of isotropic decay is governed by the decay of k at rate ε, and the evolution of ε due to the balance of the RHS of Eq. (2a) or (2b). Note that:

$$-\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}} = \overline{\omega_i\omega_k\frac{\partial u_i}{\partial x_k}}$$

P_ε^4 and νP_ζ^4: effects of vortex stretching > 0 $\therefore$ represents production of ε. Similar but more complex than $\omega_k \frac{\partial u_i}{\partial x_k}$ term in fluctuating vorticity equation (Chapter 2 Part 4 pp. 11 -12).

$-\Upsilon_\varepsilon$ and $-\nu\Upsilon_\zeta$: effects of dissipation of dissipation

Vorticity Definition:

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

Product of Vorticity Components:

$$\omega_i \omega_k = \left(\epsilon_{ipq} \frac{\partial u_q}{\partial x_p}\right)\left(\epsilon_{krs} \frac{\partial u_s}{\partial x_r}\right)$$

Levi-Civita Identity:

$$\epsilon_{ipq}\epsilon_{krs} = \delta_{ik}\left(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr}\right) + \delta_{ir}\left(\delta_{ps}\delta_{qk} - \delta_{pk}\delta_{qs}\right) + \delta_{is}\left(\delta_{pk}\delta_{qr} - \delta_{pr}\delta_{qk}\right)$$

Contracted Form:

$$\omega_i \omega_k \frac{\partial u_i}{\partial x_k} = \epsilon_{ipq}\epsilon_{krs} \frac{\partial u_q}{\partial x_p}\frac{\partial u_s}{\partial x_r}\frac{\partial u_i}{\partial x_k}$$

$$\omega_i \omega_k \frac{\partial u_i}{\partial x_k} = \left[\delta_{ik}\left(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr}\right) + \delta_{ir}\left(\delta_{ps}\delta_{qk} - \delta_{pk}\delta_{qs}\right) + \delta_{is}\left(\delta_{pk}\delta_{qr} - \delta_{pr}\delta_{qk}\right)\right] \frac{\partial u_q}{\partial x_p}\frac{\partial u_s}{\partial x_r}\frac{\partial u_i}{\partial x_k}$$

1st term of RHS:

$$\delta_{ik}\left(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr}\right)\frac{\partial u_q}{\partial x_p}\frac{\partial u_s}{\partial x_r}\frac{\partial u_i}{\partial x_k} = \left(\frac{\partial u_q}{\partial x_r}\frac{\partial u_q}{\partial x_r} - \frac{\partial u_r}{\partial x_p}\frac{\partial u_p}{\partial x_r}\right)\underbrace{\frac{\partial u_i}{\partial x_i}}_{=0} = 0$$

2nd term of RHS:

$$\delta_{\mathrm{ir}}\left(\delta_{\mathrm{ps}}\delta_{\mathrm{qk}} - \delta_{\mathrm{pk}}\delta_{\mathrm{qs}}\right)\frac{\partial \mathrm{u_q}}{\partial \mathrm{x_p}}\frac{\partial \mathrm{u_s}}{\partial \mathrm{x_r}}\frac{\partial \mathrm{u_i}}{\partial \mathrm{x_k}} = \underbrace{\frac{\partial \mathrm{u_q}}{\partial \mathrm{x_p}}\frac{\partial \mathrm{u_p}}{\partial \mathrm{x_r}}\frac{\partial \mathrm{u_r}}{\partial \mathrm{x_q}}}_{=0} - \frac{\partial \mathrm{u_q}}{\partial \mathrm{x_p}}\frac{\partial \mathrm{u_q}}{\partial \mathrm{x_r}}\frac{\partial \mathrm{u_r}}{\partial \mathrm{x_p}} = -\frac{\partial \mathrm{u_q}}{\partial \mathrm{x_p}}\frac{\partial \mathrm{u_q}}{\partial \mathrm{x_r}}\frac{\partial \mathrm{u_r}}{\partial \mathrm{x_p}}$$

$$\frac{\partial u_q}{\partial x_p}\frac{\partial u_p}{\partial x_r}\frac{\partial u_r}{\partial x_q} = H_{qprprq} = H_{ijkjki} = 0 \quad \text{(see Pope Exe. 6.11; A4 pg. 32)}$$

3rdterm of RHS:

$$\delta_{is}\left(\delta_{pk}\delta_{qr} - \delta_{pr}\delta_{qk}\right)\frac{\partial u_q}{\partial x_p}\frac{\partial u_s}{\partial x_r}\frac{\partial u_i}{\partial x_k} = \frac{\partial u_q}{\partial x_p}\frac{\partial u_i}{\partial x_q}\frac{\partial u_i}{\partial x_p} - \frac{\partial u_q}{\partial x_p}\frac{\partial u_i}{\partial x_p}\frac{\partial u_i}{\partial x_q} = 0$$

Therefore,

$$\omega_i \omega_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial u_q}{\partial x_p}\frac{\partial u_q}{\partial x_r}\frac{\partial u_r}{\partial x_p} = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_l}\frac{\partial u_l}{\partial x_j} = -\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}$$

Derivation of Eq. (5-75) in Bernard's book:

$$\overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2} = \overline{\left(\frac{\partial \omega_i}{\partial x_j}\right)^2}$$

The vorticity ω_i is defined as: $\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$

The derivative of ω_i with respect to x_j is:

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\epsilon_{ikl}\frac{\partial u_l}{\partial x_k}\right) = \epsilon_{ikl}\frac{\partial^2 u_l}{\partial x_k \partial x_j}$$

The square of this derivative is

$$\left(\frac{\partial \omega_i}{\partial x_j}\right)^2 = \left(\epsilon_{ikl}\frac{\partial^2 u_l}{\partial x_k \partial x_j}\right)^2$$

Expanding this, we get:

$$\left(\frac{\partial \omega_i}{\partial x_j}\right)^2 = \left(\epsilon_{ikl}\frac{\partial^2 u_l}{\partial x_k \partial x_j}\right)\left(\epsilon_{imn}\frac{\partial^2 u_n}{\partial x_m \partial x_j}\right)$$

The product of Levi-Civita symbols is simplified using:

$$\epsilon_{ikl}\epsilon_{imn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$$

Substituting this into the expression,

$$\left(\frac{\partial \omega_i}{\partial x_j}\right)^2 = (\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm})\frac{\partial^2 u_l}{\partial x_k \partial x_j}\frac{\partial^2 u_n}{\partial x_m \partial x_j}$$

Expanding the expression using the Kronecker delta properties,

$$\left(\frac{\partial \omega_i}{\partial x_j}\right)^2 = \frac{\partial^2 u_l}{\partial x_k \partial x_j}\frac{\partial^2 u_l}{\partial x_k \partial x_j} - \frac{\partial^2 u_l}{\partial x_k \partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_j}$$

The incompressibility condition $\nabla \cdot u = 0$ implies,

$$\frac{\partial u_k}{\partial x_k} = 0$$

Note that

$$\frac{\partial^2 u_l}{\partial x_k \partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_j} = \underbrace{\frac{\partial}{\partial x_k}\left(\frac{\partial u_l}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_j}\right)}_{=0} - \underbrace{\frac{\partial u_l}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{\partial^2 u_k}{\partial x_l \partial x_j}\right)}_{=0} = 0$$

This means $\frac{\partial^2 u_l}{\partial x_k \partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_j}$ vanishes,

$$\left(\frac{\partial \omega_i}{\partial x_j}\right)^2 = \frac{\partial^2 u_l}{\partial x_k \partial x_j}\frac{\partial^2 u_l}{\partial x_k \partial x_j}$$

Exercise 2.10 in Pope's book (see derivation Chapter 3 part 1)

$$\frac{D\omega^2}{Dt} = \nu\nabla^2\omega^2 + 2\omega_i\omega_j\frac{\partial U_i}{\partial x_j} - 2\nu\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} \quad (1)$$

Equation 3.36 in Davidson's Book (see derivation Chapter 3 part 1)

$$\frac{D}{Dt}\left(\frac{\vec{\omega}^2}{2}\right) = \omega_1\omega_j S_{ij} - \nu(\nabla\times\vec{\omega})^2 + \nu\nabla\cdot[\vec{\omega}\times(\nabla\times\vec{\omega})] \quad (2)$$

The sums of the two highlighted terms in both above equations are equal: Eq. (1) /2= Eq. (2)

$$\nabla^2\frac{\omega^2}{2} - \nabla\omega\cdot\nabla\omega = -(\nabla\times\omega)^2 + \nabla\cdot[\omega\times(\nabla\times\omega)] = \underbrace{-\omega\cdot\nabla\times(\nabla\times\omega)}_{Palinstrophy} = \omega\cdot\nabla^2\omega$$

For homogeneous turbulence,

$\nabla^2\frac{\omega^2}{2} = \nabla\cdot(\omega\cdot\nabla\omega) = 0$, and $\nabla\cdot[\omega\times(\nabla\times\omega)] = 0$

which implies that $\nabla\omega\cdot\nabla\omega = (\nabla\times\omega)^2$.

The following derivation shows the terms are equal for homogeneous turbulence.

$$\begin{aligned}(\nabla\times A)_k(\nabla\times A)_k &= (\nabla_i A_j\varepsilon_{ijk})(\varepsilon_{kpq}\nabla_p A_q)\\ &= (\nabla_i A_j)(\varepsilon_{ijk}\varepsilon_{kpq})(\nabla_p A_q)\\ &= (\nabla_i A_j)(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})(\nabla_p A_q)\\ &= (\nabla_i A_j)(\nabla_i A_j) - (\nabla_i A_j)(\nabla_j A_i)\\ &= \frac{\partial A_j}{\partial x_i}\frac{\partial A_j}{\partial x_i} - \frac{\partial A_j}{\partial x_i}\frac{\partial A_i}{\partial x_j}\end{aligned}$$

$$\frac{\partial A_j}{\partial x_i}\frac{\partial A_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(A_j\frac{\partial A_i}{\partial x_j}\right) - A_j\frac{\partial^2 A_i}{\partial x_i\partial x_j}$$

A is divergence free, $A_j\frac{\partial^2 A_i}{\partial x_i\partial x_j} = 0$,

$$\frac{\partial A_j}{\partial x_i}\frac{\partial A_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(A_j\frac{\partial A_i}{\partial x_j}\right)$$

For homogeneous turbulence,

$$\frac{\partial A_j}{\partial x_i}\frac{\partial A_i}{\partial x_j} = 0$$

Therefore, for homogeneous turbulence,

$$(\nabla\times A)_k(\nabla\times A)_k = \frac{\partial A_j}{\partial x_i}\frac{\partial A_j}{\partial x_i}$$

In summary, for homogeneous turbulence: $-\nabla\omega\cdot\nabla\omega = -(\nabla\times\omega)^2 = \underbrace{-\omega\cdot\nabla\times(\nabla\times\omega)}_{Palinstrophy}$

RHS of Eq. (2a) can be expressed in terms of $\mathcal{R}_{ii}$ and $S_{il,i}$ via extension of identity (see derivation Eq. (6) Chapter 4 Part 1):

$$\overline{\frac{\partial u_i(\underline{x})}{\partial x_k} u_j\left(\underline{y}\right)} = -\frac{\partial \mathcal{R}_{ij}\left(\underline{y}-\underline{x}\right)}{\partial r_k} \quad (3)$$

If $k = j$ and $j = i$

$$\overline{\frac{\partial u_i(\underline{x})}{\partial x_j} u_i\left(\underline{y}\right)} = -\frac{\partial \mathcal{R}_{ii}\left(\overbrace{\underline{y}-\underline{x}}^{\underline{r}}\right)}{\partial r_j} \quad (4)$$

Taking a derivative with respect to x_l in Eq. (4) yields

$$\overline{\frac{\partial^2 u_i(\underline{x})}{\partial x_j \partial x_l} u_i\left(\underline{y}\right)} = -\frac{\partial}{\partial r_l}\left(\frac{\partial \mathcal{R}_{ii}(\underline{r})}{\partial r_j}\right)\underbrace{\frac{\partial r_l}{\partial x_l}}_{\boxed{-1}} = \frac{\partial^2 \mathcal{R}_{ii}(\underline{r})}{\partial r_j \partial r_l} \quad (5)$$

$$\boxed{\frac{\partial (y_l - x_l)}{\partial x_l} = -\frac{\partial x_l}{\partial x_l} = -\delta_{ll}}$$

Similarly, taking two derivatives with respect to y_j and y_l in Eq. (5)

$$\overline{\frac{\partial^2 u_i(\underline{x})}{\partial x_j \partial x_l} \frac{\partial^2 u_i\left(\underline{y}\right)}{\partial y_j \partial y_l}} = \frac{\partial^4 \mathcal{R}_{ii}(\underline{r})}{\partial {r_j}^2 \partial {r_l}^2}$$

And taking the limit for $\underline{r} \to 0$, $\underline{y} \to \underline{x}$

$$\overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2} = \frac{\partial^4 \mathcal{R}_{ii}(0)}{\partial {r_j}^2 \partial {r_l}^2} \quad (6)$$

A similar simplification of the triple velocity correlation can be obtained starting from

$$S_{il,i}(\underline{r}) = \overline{u_i(\underline{x}) u_l(\underline{x}) u_i(\underline{y})} \quad (7)$$

$$\boxed{\underline{r} = \underline{y} - \underline{x}}$$

Note that il, i notation emphasizes fact that second u_i component is at a different location $\underline{y}$ than $u_i\ u_l$ at location $\underline{x}$.

Taking a derivative with respect to x_j in Eq. (7)

$$\frac{\partial \overline{u_i(\underline{x}) u_l(\underline{x}) u_i(\underline{y})}}{\partial x_j} = \frac{\partial S_{il,i}}{\partial x_j}$$

$$\overline{\frac{\partial u_i(\underline{x})}{\partial x_j} u_l(\underline{x}) u_i(\underline{y})} + \overline{u_i(\underline{x}) \frac{\partial u_l(\underline{x})}{\partial x_j} u_i(\underline{y})} = \frac{\partial S_{il,i}}{\partial r_j} \underbrace{\frac{\partial r_j}{\partial x_j}}_{\boxed{-1}} = -\frac{\partial S_{il,i}}{\partial r_j} \quad (8)$$

Taking an additional derivative with respect to x_l in Eq. (8) yields

$$\overline{\frac{\partial u_i(\underline{x})}{\partial x_l}\frac{\partial u_l(\underline{x})}{\partial x_j}u_i(\underline{y})} + \cancel{\overline{u_i(\underline{x})\frac{\partial^2 u_l(\underline{x})}{\partial x_j \partial x_l}u_i(\underline{y})}} + \cancel{\overline{\frac{\partial u_i(\underline{x})}{\partial x_j}\frac{\partial u_l(\underline{x})}{\partial x_l}u_i(\underline{y})}} + \overline{\frac{\partial^2 u_i(\underline{x})}{\partial x_j \partial x_l}u_l(\underline{x})u_i(\underline{y})} = -\frac{\partial}{\partial r_l}\left(\frac{\partial S_{il,i}}{\partial r_j}\right)\underbrace{\frac{\partial r_l}{\partial x_l}}_{\boxed{-1}} = \frac{\partial^2 S_{il,i}}{\partial r_j \partial r_l} \quad (9)$$

Where the two crossed terms are zero due to continuity.

Finally, taking a derivative with respect to y_j in Eq. (9) yields

$$\overline{\frac{\partial u_i(\underline{x})}{\partial x_l}\frac{\partial u_l(\underline{x})}{\partial x_j}\frac{\partial u_i(\underline{y})}{\partial y_j}} + \overline{\frac{\partial^2 u_i(\underline{x})}{\partial x_j \partial x_l}u_l(\underline{x})\frac{\partial u_i(\underline{y})}{\partial y_j}} = \frac{\partial}{\partial r_j}\left(\frac{\partial^2 S_{il,i}}{\partial r_j \partial r_l}\right)\underbrace{\frac{\partial r_j}{\partial y_j}}_{\boxed{1}}$$

Or equivalently

$$\frac{\partial^3 S_{il,i}}{\partial {r_j}^2 \partial r_l}(\underline{r}) = \overline{\frac{\partial u_i(\underline{x})}{\partial x_l}\frac{\partial u_l(\underline{x})}{\partial x_j}\frac{\partial u_i(\underline{y})}{\partial y_j}} + \overline{\frac{\partial^2 u_i(\underline{x})}{\partial x_j \partial x_l}u_l(\underline{x})\frac{\partial u_i(\underline{y})}{\partial y_j}} \quad (10)$$

Taking the limit for $\underline{r} \to 0$, $\underline{y} \to \underline{x}$

$$\frac{\partial^3 S_{il,i}(0)}{\partial {r_j}^2 \partial r_l} = \overline{\frac{\partial u_i(\underline{x})}{\partial x_l} \frac{\partial u_l(\underline{x})}{\partial x_j} \frac{\partial u_i(\underline{x})}{\partial x_j}} \qquad (11)$$

Since after using the incompressibility condition again

$$\overline{\frac{\partial^2 u_i(\underline{x})}{\partial x_j \partial x_l} u_l(\underline{x}) \frac{\partial u_i(\underline{x})}{\partial x_j}} = \overline{\frac{\partial}{\partial x_l}\left(\frac{\partial u_i(\underline{x})}{\partial x_j}\right) \frac{\partial u_i(\underline{x})}{\partial x_j} u_l(\underline{x})} = \frac{1}{2}\overline{\frac{\partial}{\partial x_l}\left(\frac{\partial u_i(\underline{x})}{\partial x_j}\right)^2 u_l(\underline{x})}$$

$$= \frac{1}{2}\left\{\overline{\frac{\partial}{\partial x_l}\left[\left(\frac{\partial u_i(\underline{x})}{\partial x_j}\right)^2 u_l(\underline{x})\right]} - \overline{\left(\frac{\partial u_i(\underline{x})}{\partial x_j}\right)^2 \frac{\partial u_l(\underline{x})}{\partial x_l}}\right\}$$

$$= \frac{1}{2}\overline{\frac{\partial}{\partial x_l}\left[\left(\frac{\partial u_i(\underline{x})}{\partial x_j}\right)^2 u_l(\underline{x})\right]} = 0$$

in homogeneous turbulence.

Substituting Eq. (6) and (11) into Eq. (2a)

$$\frac{d\varepsilon}{dt} = -2\nu \frac{\partial^3 S_{il,i}(0)}{\partial {r_j}^2 \partial r_l} - 2\nu^2 \frac{\partial^4 \mathcal{R}_{ii}(0)}{\partial {r_j}^2 \partial {r_l}^2}$$

To simplify the relation even more, recall from Chapter 4: Part 2, Eq. (9) and (11):

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}\frac{df}{dr}\right]$$

$$S_{ijl}(\underline{r}) = u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_j r_l}{2r^3} - \frac{k}{2}\delta_{ij}\frac{r_l}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

Using $\mathcal{R}_{ij}(\underline{r})$ and specifying for $i = j$ and taking four derivatives of $\mathcal{R}_{ii}(\underline{r})$, two with respect to r_j and two with respect to r_l yields

$$\frac{\partial \mathcal{R}_{ii}}{\partial r_j}(\underline{r}) = \overline{u^2}\left(4f'\frac{r_j}{r} + r_j f''\right)$$ Chapter 4: Part 3, Eq. (6)

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial {r_j}^2}(\underline{r}) = \overline{u^2}\left(7f'' + \frac{8f'}{r} + rf'''\right)$$ Chapter 4: Part 3, Eq. (7)

$$\frac{\partial^3 \mathcal{R}_{ii}}{\partial {r_j}^2 \partial r_l}(\underline{r}) = \overline{u^2}\left(8f''\frac{r_l}{r^2} + 8f'''\frac{r_l}{r} + r_l f^{IV} - 8f'\frac{r_l}{r^3}\right)$$

$$\frac{\partial^4 \mathcal{R}_{ii}}{\partial {r_j}^2 \partial {r_l}^2}(\underline{r}) = \overline{u^2}\left[\frac{24}{r}f'''(r) + 11f^{IV}(r) + rf^{V}(r)\right]$$ Proof in Appendix A.1

To evaluate this expression at $r = 0$, use Taylor series of $f'''(r)$

$$f'''(r) = rf^{IV}(0) + \frac{r^3}{3}f^{VI}(0) + \cdots$$

Since f is an even function of r. Consequently,

$$\lim_{r\to 0}\frac{f'''(r)}{r} = f^{IV}(0)$$

Such that

$$\frac{\partial^4 \mathcal{R}_{ii}}{\partial {r_j}^2 \partial {r_l}^2}(0) = \overline{u^2}[35f^{IV}(0) + rf^{V}(0)]$$

Similarly, it can be shown that,

$$\frac{\partial^3 S_{il,i}}{\partial {r_j}^2 \partial r_l}(0) = \frac{35}{2} u_{rms}^3 k'''(0)$$

Proof in Appendix A.2

Note that $k'''(0) < 0$ such that $P_\varepsilon^4 > 0$ and represents production of ε via vortex stretching. Thus,

$$\frac{d\varepsilon}{dt} = -35 u_{rms}^3 k'''(0) - 70\overline{u^2} f^{IV}(0) \quad (12)$$

i.e., only depends on two time-dependent scalars, along with k ($u_{rms} = [\frac{2}{3}k]^{1/2}$) and ε.

Using $\mathcal{R}_{11}(\underline{r}) = \overline{u^2(x)} f(r) = \overline{u(x)u(x+r)}$ and $S_{111}(\underline{r}) = u_{rms}^3 k(r) = \overline{u(x)u(x)u(x+r)}$:

$$\overline{(u_{xx})^2} = u_{rms}^2 f^{IV}(0) \quad (13)$$

Proof in Appendix A.3

$$\overline{(u_x)^3} = u_{rms}^3 k'''(0) \quad (14)$$

Proof in Appendix A.4

The proof for Eq. (13) is done using both scalar and vector approaches, as per Chapter 4 Part 3 for the derivation of $\overline{u^2} f''(0) = -\overline{{u_x}^2}$; and the proof for Eq. (14) is done using both scalar and tensor approaches.

The skewness of u_x is defined as

See Appendix A.5

$$S_k = -\frac{\overline{(u_x)^3}}{\overline{(u_x)^2}^{3/2}} \quad (15)$$

And found to be positive due minus sign on RHS.

$$S_k \overline{(u_x)^2}^{3/2} = -\overline{(u_x)^3} = -u_{rms}^3 k'''(0)$$

$$\overline{(u_x)^2} = -\overline{u^2} f''(0)$$

where

Chapter 4: Part 3

$$f''(0) = \frac{\varepsilon}{-15\nu\overline{u^2}}$$

Therefore

$$\overline{(u_x)^2} = \frac{\varepsilon}{15\nu} = \frac{15\nu\overline{u^2}}{\lambda_g^2}\frac{1}{15\nu} = \frac{\overline{u^2}}{\lambda_g^2} \quad (16)$$

$$\overline{u^2} = u_{rms}^2$$

Using Eq. (14) and (16), it follows that Eq. (15) becomes,

$$k'''(0) = -\frac{S_k}{\lambda_g^3} = -S_k\left(\frac{\varepsilon}{15u_{rms}^2\nu}\right)^{\frac{3}{2}} \quad (17)$$

Palenstrophy coefficient of u_x can be defined as

See Appendix A.5 and Eq. (2a) and (2b)

$$G = \frac{\overline{u^2}\ \overline{(u_{xx})^2}}{\overline{(u_x)^2}^2}$$

Eqs. (13) and (16) show that

$$f^{IV}(0) = \frac{G}{\lambda_g^4} = G\left(\frac{\varepsilon}{15u_{rms}^2 \nu}\right)^2 \quad (18)$$

Substituting Eqs. (17) and (18) into Eq. (12), gives the ε equation for homogeneous isotropic turbulence in the standard form

$$\frac{d\varepsilon}{dt} = S_k^* R_T^{1/2} \frac{\varepsilon^2}{k} - G^* \frac{\varepsilon^2}{k} \quad (19)$$

Where:

$$S_K^* = \frac{7}{3\sqrt{15}} S_k$$

$$G^* = \frac{7}{15} G$$

$$R_T = \frac{k^2}{\nu\varepsilon}$$

$$\frac{d\varepsilon}{dt} = P_\varepsilon^4 - \Upsilon_\varepsilon = \quad (2a)$$

$$P_\varepsilon^4 = S_k^* R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k}$$

$$\Upsilon_\varepsilon = G^* \frac{\varepsilon^2}{k}$$

This equation, along with Eq. (1) represent two equations in the four unknowns k, ε, S_K^* and G^* , all of which are $f(t)$, i.e., not closed. RHS term 1 = gain and term 2 = loss.

Initial state needs to be specified, i.e., at $t = 0$, $k_0, \varepsilon_0, S_{K_0}^*$, and G_0^*. Alternatively, using Eqs. (17) and (18), initial forms for $f(r)$ and $k(r)$ can be specified, from which S_{k_0} and G_0 can be obtained.

$u_{rms} = (\overline{u^2})^{1/2}$

$u^3_{rms} = \overline{u^2}^{3/2}$

$$\varepsilon_t = -35\nu\, u^3_{rms}\left[-S\left(\frac{\varepsilon}{15\nu\overline{u^2}}\right)^{3/2}\right] - 70\nu^2\,\overline{u^2}\left[G\left(\frac{\varepsilon}{15\nu\overline{u^2}}\right)^2\right]$$

$K = \frac{3}{2}\overline{u^2}$

$\frac{2}{3}K = \overline{u^2}$

$$-35\nu\,\overline{u^2}^{3/2}\left(-S\left(\frac{\varepsilon}{15\nu\overline{u^2}}\right)^{3/2}\right) - \frac{70\nu^2}{\overline{u^2}}\,\frac{G\,\varepsilon^2}{(15\nu)^2}$$

$$+\frac{35\nu S\,\varepsilon^{3/2}}{(15\nu)^{3/2}}$$

$$-\frac{70}{\frac{2}{3}K}\,\frac{G\,\varepsilon^2}{15^2} = -\frac{7}{15}G\frac{\varepsilon^2}{K}$$

$$\frac{5\times 7}{15\;15^{1/2}} = \frac{7}{3\sqrt{15}} \qquad \frac{35 S\,\varepsilon^{3/2}}{15^{3/2}\,\nu^{1/2}}$$

$$\frac{7}{3\sqrt{15}}\,\frac{K}{\nu^{1/2}\varepsilon^{1/2}}\,\frac{\varepsilon^2}{K} = \frac{7}{3\sqrt{15}}\cdot\frac{1}{\nu^{1/2}}\,\varepsilon^{3/2}$$

$$\varepsilon_t = S_K^* R_T^{1/2}\,\varepsilon^2/K - G^*\,\varepsilon^2/K \quad \wedge \quad K_t = -\varepsilon$$

$S_K^* = \frac{7}{3\sqrt{15}} S_K$

$G^* = \frac{7}{15} G$

$R_T = \frac{K^2}{\nu\varepsilon}$

2 equations

4 unknowns $= K, \varepsilon$

S_K^*, G^*

IC: $t=0$ $K_0, \varepsilon_0, S_{K0}^*, G_0^*$

or $f(\nu)$

$\wedge\ R(\nu)$

at $t=0$

For (17): From (16) $\overline{(u_x)^2}^{3/2} = u^3_{rms}/\lambda_g^3$

$S_K u^3_{rms}/\lambda_g^3 = -u^3_{rms} f_K'''$

For (18): Using (13) $\overline{(u_{xx})^2} = u^2_{rms} f^{IV}$

$G\,\overline{(u_x)^2}^2 = \overline{u^2}\,\overline{(u_{xx})^2}$

and from (16) $G\,u^4_{rms}/\lambda_g^4 = u^4_{rms} f^{IV}$

Turbulent Reynolds Number (also see Chapter 4 Part 3 pg. 14 where $Re_L = R_T$)

$$R_T = \frac{k^2}{\nu\varepsilon} = \text{ turbulent Re } = \frac{\sqrt{k}k^{3/2}/\varepsilon}{\nu}$$

Velocity scale $u = \sqrt{k}$ and length scale $l = k^{3/2}/\varepsilon$, where l is related to the eddy turnover time:

$$T_t = \frac{k}{\varepsilon}$$

Which shows that,

$$\frac{1}{T_t} = \frac{\varepsilon}{k} = -\frac{1}{k}\frac{dk}{dt}$$

And can be interpreted as the fractional rate of energy dissipation and $T_t =$ time scale of TKE dissipation.

Alternatively,

$R_T = (k/\varepsilon)/(\nu/k) =$ ratio of turbulent and viscous time scales $= T_t/T_\mu$, where

$T_\mu = \nu/k =$ time scale of viscous dissipation

Large $R_T =$ very energetic turbulence and far from being dissipated, i.e., $T_\mu \ll T_t$.

Small $R_T =$ energy in dissipation range since rate flow energy drops = rate at which energy is dissipated = weak turbulence, i.e., $T_\mu \gg T_t$.

Therefore, $R_T \to 0$ during decay of isotropic turbulence

Since R_T appears in the stretching term of $d\varepsilon/dt$, the equation indicates that stretching is important for energetic turbulence vs. dissipative range.

Another useful turbulence Reynolds number is,

$$R_\lambda = \frac{\lambda u_{rms}}{\nu}$$

Where $\lambda = \lambda_g$ or λ_f.

Using

$$\varepsilon = \frac{30\nu\overline{u^2}}{{\lambda_f}^2} = \frac{15\nu\overline{u^2}}{{\lambda_g}^2}$$

it is possible to obtain the relationship between R_T and R_λ

$$R_T = \frac{k^2}{\nu\varepsilon} = \frac{k^2{\lambda_g}^2}{15\nu^2\overline{u^2}} = \frac{9\overline{u^2}^2{\lambda_g}^2}{60\nu^2\overline{u^2}} = \frac{3}{20}{R_\lambda}^2 \qquad \boxed{k = \frac{3}{2}\overline{u^2}}$$

R_T or R_λ can be used to characterize degree of turbulence for homogeneous flow:

- $R_\lambda > 100$ turbulence not weak
- $R_\lambda > 1000$ strong turbulence
- $R_\lambda < 1$ very weak turbulence, final period decay before it relaminarizes

Interest is in decay process from initial state $R_T \gg 1$ to $R_T < 1$.

Eqs. (1) and (19) can be combined into a single equation for R_T. Starting from

$$R_T = \frac{k^2}{\nu\varepsilon}$$

$$\frac{dR_T}{dt} = \frac{2k}{\nu\varepsilon}\frac{dk}{dt} - \frac{k^2}{\nu\varepsilon^2}\frac{d\varepsilon}{dt}$$

Substituting Eqs. (1) and (19)

$$\frac{dR_T}{dt} = -\frac{2k}{\nu} - S_k^*\sqrt{R_T}\frac{k}{\nu} + G^*\frac{k}{\nu} \quad (20)$$

Since k and ε are always positive, a dimensionless time can be defined as ($t' = \frac{\varepsilon}{k}\tau$)

$$\tau(t) = \int_0^t \frac{\varepsilon(t')}{k(t')}dt' \quad (21)$$

Where it is assumed that $\tau(0) = 0$. Note that $\tau \to \infty$ as $t \to \infty$. This can be integrated exactly using Eq. (1), to obtain,

$$\tau(t) = \ln\big(k(0)/k(t)\big)$$

It is also possible to obtain the inverse mapping of τ to t.

Defining

$$R_T^*(\tau) = R_T(t(\tau))$$

Or equivalently

$$R_T^*(\tau(t)) = R_T(t)$$

Such that,

$$\frac{dR_T}{dt} = \frac{dR_T^*}{dt}\frac{d\tau}{dt} = \frac{\varepsilon}{k}\frac{dR_T^*}{d\tau} \quad (22)$$

using Eq. (21).

Substituting Eq. (22) into (20) yields

$$\frac{dR_T^*}{d\tau} = R_T^*\left(G^* - 2 - S_k^*\sqrt{R_T^*}\right) \quad (23)$$

See Appendix A.6

Thus, an alternative to solving the decay problem via Eqs. (1) and (19) is the option of solving Eq. (23).

G^* and S_k^* are $f(t)$ such that represents one equation in three unknowns, i.e., additional assumptions are required.

No matter which way the decay problem is approached, solving for k and ε requires additional assumptions so that a closed system of equations can be deduced.

Appendix A

A.1

P73.

Eq 5.10

Started from Eqs 4.42 and 4.43

Eq. 4.43

$$\frac{\partial R_{ii}}{\partial r_j} = \overline{u^2}\left[4\frac{r_j}{r}f' + r_j f^2\right] \Rightarrow \qquad f^2 = f'' \quad f^3 = f'''$$

$$\frac{\partial^2 R_{ii}}{\partial^2 r_j} = \overline{u^2}\frac{\partial}{\partial r_j}\left[4\frac{r_j}{r}f' + r_j f^2\right] \Rightarrow \frac{\partial}{\partial r_j}\left[4\frac{r_j}{r}f' + r_j f^2\right] = 4\frac{r_j}{r}\frac{\partial f'}{\partial r}\frac{\partial r}{\partial r_j} +$$

$$4f'\frac{\partial\left(\frac{r_j}{r}\right)}{\partial r_j} + r_j\frac{\partial f''}{\partial r}\frac{\partial r}{\partial r_j} + f''\frac{\partial r_j}{\partial r_j} =$$

$$4\frac{r_j}{r}\frac{r_j}{r}f'' + 4f'\frac{1}{r}\frac{\partial(r_j)}{\partial r_j} + 4f' r_j\left(-\frac{1}{r^2}\right)\frac{r_j}{r} + r_j f'''\frac{r_j}{r} + 3f'' \Rightarrow$$

$$\frac{\partial^2 R_{ii}}{\partial^2 r_j} = \left[4f'' + 12f'\frac{1}{r} + rf''' + 3f'' - 4f'\frac{1}{r}\right]\overline{u^2} \Rightarrow$$

$$\boxed{\frac{\partial^2 R_{ii}}{\partial r_j^2} = \left[7f'' + rf''' + 8f'\frac{1}{r}\right]\overline{u^2} = \left[8f'\frac{1}{r} + 7f'' + rf'''\right]\overline{u^2}}$$

$$\frac{\partial^3 R_{ii}}{\partial r_j^2 \partial r_l} = \overline{u^2}\frac{\partial}{\partial r_l}\left[8f'\frac{1}{r} + 7f'' + rf'''\right] \Rightarrow$$

$$\frac{\partial}{\partial r_l}[\quad] = 8\frac{1}{r}\frac{\partial f'}{\partial r}\frac{\partial r}{\partial r_l} + 8f'\frac{\partial}{\partial r_l}\left(\frac{1}{r}\right) + 7\frac{\partial f''}{\partial r}\frac{\partial r}{\partial r_l} + f'''\frac{\partial r}{\partial r_l} + r\frac{\partial f'''}{\partial r}\frac{\partial r}{\partial r_l}$$

$$= 8f''\frac{r_l}{r^2} - 8f'\frac{1}{r^2}\frac{r_l}{r} + 7f'''\frac{r_l}{r} + f'''\frac{r_l}{r} + r_l f^4$$

$$= 8f''\frac{r_l}{r^2} + 8f'''\frac{r_l}{r} + r_l f^4 - 8f'\frac{r_l}{r^3} \Rightarrow$$

$$\frac{\partial^3 R_{ii}}{\partial r_j^2 \partial r_l} = \overline{U^2}\left[8f''\frac{r_l}{r^2} + 8f'''\frac{r_l}{r} + r_l f^4 - 8f'\frac{r_l}{r^3}\right]$$

$$f^4 = f^{iv}$$

$$\frac{\partial^4 R_{ii}}{\partial r_j^2 \partial r_i^2} = \bar{u}^2 \frac{\partial}{\partial r_i}\left[-8f'\frac{r_i}{r^3} + 8f''\frac{r_i}{r^2} + 8f'''\frac{r_i}{r} + f^4 r_i\right] \Rightarrow$$

$$\frac{\partial}{\partial r_i}[\ ] = 8\frac{r_i}{r}\frac{\partial f'''}{\partial r}\frac{\partial r}{\partial r_i} + 8f'''\frac{\partial}{\partial r_i}\left(\frac{r_i}{r}\right) + 8\frac{r_i}{r^2}\frac{\partial f''}{\partial r_i} + 8f''\frac{\partial \frac{r_i}{r^2}}{\partial r_i}$$

$$+ f^4\frac{\partial r_i}{\partial r_i} + r_i\frac{\partial f^4}{\partial r}\frac{\partial r}{\partial r_i} - 8f'\frac{\partial}{\partial r_i}\left(\frac{r_i}{r^3}\right) - 8\frac{r_i}{r^3}\frac{\partial f'}{\partial r}\frac{\partial r}{\partial r_i} =$$

$$8\frac{r_i r_i}{r^2}f^4 + 8f'''r_i\frac{\partial}{\partial r_i}\left(\frac{1}{r}\right) + 8f'''\frac{1}{r}\frac{\partial r_i}{\partial r_i} + 8\frac{r_i r_i}{r^3}f''' + 8f''r_i\frac{\partial}{\partial r_i}\left(\frac{1}{r^2}\right) +$$

$$\frac{1}{r^2}8f''\frac{\partial r_i}{\partial r_i} + 3f^4 + \frac{r_i r_i}{r}f^5 - \frac{1}{r^3}8f'\frac{\partial r_i}{\partial r_i} - 8f'r_i\frac{\partial}{\partial r_i}\left(\frac{1}{r^3}\right) - 8f''\frac{1}{r^2}$$

$$= 8f^4 - 8f'''\frac{r_i}{r^2}\frac{r_i}{r} + 8f'''\frac{3}{r} + 8f'''\frac{1}{r} + 8f''r_i(-2)\frac{1}{r^3}\frac{r_i}{r} +$$

$$\frac{8}{r^2}f''(3) + 3f^4 + rf^5 - \frac{8f'(3)}{r^3} - 8f'r_i\frac{-3}{r^4}\frac{r_i}{r} - 8f''\frac{1}{r^2}$$

$$= 8f^4 - 8f'''\frac{1}{r} + 24f'''\frac{1}{r} + 8f'''\frac{1}{r} - 16f''\frac{1}{r^2} +$$

$$24f''\frac{1}{r^2} + 3f^4 + rf^5 - 24f'\frac{1}{r^3} + 24f'\frac{1}{r^3} - 8f''\frac{1}{r^2}$$

$$= 8f^4 + 3f^4 + rf^5 + 24f'''\frac{1}{r} = 11f^4 + rf^5 + 24f'''\frac{1}{r} \Rightarrow$$

$$\frac{\partial^4 R_{ii}}{\partial r_j^2 \partial r_i^2} = \bar{u}^2\left[\frac{24}{r}f''' + 11f^4 + rf^5\right] \quad \text{Eq5.10}$$

A.2

$j \neq l.\ l = i$

P73. Eq. 5.14

$$S_{il,i} = u_{rms}^3 \left[(k - rk') r_i r_l r_i / 2r^3 - \frac{k}{2}\delta_{il}\frac{r_i}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{ii}\frac{r_l}{r} + \delta_{li}\frac{r_i}{r}\right)\right]$$

∂S

$$S_{il,i} = u_{rms}^3 \left[(k - rk')\frac{r_l}{2r} - \frac{k r_l}{2r} + \left(\frac{r}{4}k' + \frac{1}{2}k\right)\left(\frac{3r_l}{r} + \frac{r_l}{r}\right)\right]$$

$$= u_{rms}^3 \left[\frac{r_l}{2r}(k - rk' - k) + \left(\frac{r}{4}k' + \frac{1}{2}k\right)\left(\frac{4r_l}{r}\right)\right]$$

$$= u_{rms}^3 \left[\frac{-r_l}{2r} rk' + r_l k' + \frac{2kr_l}{r}\right]$$

$$S_{il,i} = u_{rms}^3 \left[\frac{1}{2} r_l k' + 2r_l \frac{k}{r}\right]$$

$$\frac{\partial S_{il,i}}{\partial r_l} = u^3 \frac{\partial}{\partial r_l}\left[\frac{1}{2} r_l k' + 2r_l\frac{k}{r}\right]$$

$$\frac{\partial}{\partial r_l}[\ ] = \frac{1}{2} r_l \frac{\partial k'}{\partial r}\frac{\partial r}{\partial r_l} + \frac{1}{2}k'\frac{\partial r_l}{\partial r_l} + 2r_l\frac{\partial\left(\frac{k}{r}\right)}{\partial r_l} + \frac{2k}{r}\frac{\partial r_l}{\partial r_l}$$

$$= \frac{1}{2} r_l k'' \frac{r_l}{r} + \frac{3}{2}k' + \frac{2r_l}{r}\frac{\partial k}{\partial r}\frac{\partial r}{\partial r_l} + 2r_l k\frac{\partial\left(\frac{1}{r}\right)}{\partial r_l} + \frac{6k}{r}$$

$$= \frac{1}{2} rk'' + \frac{3}{2}k' + 2\frac{r_l r_l}{r^2}k' + \frac{2r_l k(-1)}{r^2}\frac{\partial r}{\partial r_l} + \frac{6k}{r}$$

$$= \frac{1}{2} rk'' + \frac{3}{2}k' + 2k' - \frac{2r_l k}{r^2}\frac{r_l}{r} + \frac{6k}{r}$$

$$\frac{\partial}{\partial r_l}[\ ] = \frac{1}{2} rk'' + \frac{7}{2}k' - \frac{2k}{r} + \frac{6k}{r} = \frac{1}{2} rk'' + \frac{7}{2}k' + \frac{4k}{r} \Rightarrow$$

$$\boxed{\frac{\partial S_{il,i}}{\partial r_l} = u_{rms}^3\left[\frac{1}{2} rk'' + \frac{7}{2}k' + \frac{4k}{r}\right]}$$

$$\frac{\partial^2 S_{il,i}}{\partial r_l \partial r_j} = u_{rms}^3 \frac{\partial}{\partial r_j}[\quad]$$

eq. 5-14 continued ⓔ

$$\frac{\partial^2 S_{i1,i}}{\partial r_k \partial r_j} = u_{rms}^3 \frac{\partial}{\partial r_j}[\ \ ] \Rightarrow$$

$$\frac{\partial}{\partial r_j}[\ ] = \frac{\partial}{\partial r_j}\left[\frac{1}{2} r k''' + \frac{7}{2}k' + \frac{4k}{r}\right]$$

$$= \frac{1}{2} r \frac{\partial k''}{\partial r_j} + \frac{1}{2} k'' \frac{\partial r}{\partial r_j} + \frac{7}{2}\frac{\partial k'}{\partial r_j} + \frac{4}{r}\frac{\partial k}{\partial r_j} + 4k \frac{\partial(\frac{1}{r})}{\partial r_j}$$

$$= \frac{1}{2} r \frac{\partial k''}{\partial r}\frac{\partial r}{\partial r_j} + \frac{1}{2} k'' \frac{r_j}{r} + \frac{7}{2}\frac{\partial k'}{\partial r}\frac{\partial r}{\partial r_j} + \frac{4}{r}\frac{\partial k}{\partial r}\frac{\partial r}{\partial r_j} + 4k(-1)\frac{1}{r^2}\frac{\partial r}{\partial r_j}$$

$$= \frac{1}{2} r k''' \frac{r_j}{r} + \frac{1}{2} k'' \frac{r_j}{r} + \frac{7}{2} k'' \frac{r_j}{r} + \frac{4}{r} k' \frac{r_j}{r} - 4k \frac{1}{r^2}\frac{r_j}{r}$$

$$= \frac{1}{2} k''' r_j + \frac{1}{2} k'' \frac{r_j}{r} + \frac{7}{2} k'' \frac{r_j}{r} + 4k' \frac{r_j}{r^2} - 4k \frac{r_j}{r^3} \Rightarrow$$

$$\boxed{\frac{\partial^2 S_{i1,i}}{\partial r_k \partial r_j} = \left[\frac{1}{2} k''' r_j + 4 k'' r_j \frac{1}{r} + 4k' \frac{r_j}{r^2} - 4k \frac{r_j}{r^3}\right] u_{rms}^3}$$

$$\frac{\partial^3 S_{i1,i}}{\partial r_k \partial r_j^2} = u_{rms}^3 \frac{\partial}{\partial r_j}[\ \ \ ]$$

$$\frac{\partial}{\partial r_j}[\ \ ] = \frac{1}{2} k''' \frac{\partial r_j}{\partial r_j} + \frac{1}{2} r_j \frac{\partial k'''}{\partial r}\frac{\partial r}{\partial r_j} + 4\frac{r_j}{r}\frac{\partial k''}{\partial r}\frac{\partial r}{\partial r_j} + 4k'' \frac{\partial(\frac{r_j}{r})}{\partial r_j} +$$

$$\frac{4r_j}{r^2}\frac{\partial k'}{\partial r}\frac{\partial r}{\partial r_j} + 4k' \frac{\partial}{\partial r_j}\left(\frac{r_j}{r^2}\right) - \frac{4r_j}{r^3}\frac{\partial k}{\partial r}\frac{\partial r}{\partial r_j} - 4k\frac{\partial}{\partial r_j}\left(\frac{r_j}{r^3}\right)$$

$$= \frac{1}{2} k'''(3) + \frac{1}{2} k^4 r_j \frac{r_j}{r} + 4\frac{r_j}{r} k''' \frac{r_j}{r} + 4k'' r_j \frac{\partial(\frac{1}{r})}{\partial r_j} + \frac{4k''}{r}\frac{\partial r_j}{\partial r_j} +$$

$$\frac{4r_j}{r^2} k'' \frac{r_j}{r} + \frac{4k'}{r^2}\frac{\partial r_j}{\partial r_j} + 4k' r_j \frac{\partial}{\partial r_j}\left(\frac{1}{r^2}\right) - \frac{4r_j}{r^3} k' \frac{r_j}{r} - 4k r_j \frac{\partial}{\partial r_j}\left(\frac{1}{r^3}\right)$$

$$- 4k \frac{1}{r^3}\frac{\partial}{\partial r_j}(r_j)$$

$$= \frac{3}{2} k''' + \frac{1}{2} k^4 r + 4k''' + \left(-\frac{4k'' r_j}{r^2}\frac{r_j}{r}\right) + \frac{12k''}{r} +$$

$$4k'' \frac{1}{r} + \frac{12k'}{r^2} + \left(-8k' r_j \frac{1}{r^3}\frac{r_j}{r}\right) - \frac{4k'}{r^2} - 4k r_j(-3)\frac{1}{r^4}\frac{r_j}{r} - \frac{12k}{r^3}$$

Eq. 5.14 continued. (3)

$$\frac{\partial}{\partial r_j}[\ ] = \frac{3}{2}k''' + \frac{1}{2}rk^4 + 4k''' - 4\frac{k''}{r} + \frac{12k''}{r} + 4k''\frac{1}{r} + \frac{12k'}{r^2}$$
$$- 8k'\frac{1}{r^2} - 4k'\frac{1}{r^2} + 12k\frac{1}{r^3} - \frac{12k}{r^3}$$

$$\frac{\partial}{\partial r_j}[\ ] = \frac{1}{2}rk^4 + \frac{11}{2}k''' + 12k''\frac{1}{r} \Rightarrow$$

$$\frac{\partial^3 S_{il,i}}{\partial r_l \partial r_j^2} = u_{rms}^3\left[\frac{1}{2}rk^4 + \frac{11}{2}k''' + 12k''\frac{1}{r}\right]$$

$$k''(r) = rk'''(0) + \frac{r^3}{3!}k^5(0) + \cdots$$

$$\lim_{r \to 0}\frac{k''(r)}{r} = k'''(0).$$

$$\frac{\partial^3 S_{il,i}}{\partial r_l \partial r_j^2}(0) = \left[\frac{11}{2}k'''(0) + 12k'''(0)\right]u_{rms}^3 = \frac{35}{2}k'''(0)\,u_{rms}^3$$

$$\frac{\partial^3 S_{il,i}}{\partial r_l \partial r_j^2}(0) = \frac{35}{2}u_{rms}^3\,k'''(0) \qquad (5.14)$$

A.3

Scalar approach

Define

$$\mathcal{R}_{11}(\underline{r}) = \overline{u^2(x)}f(r) = \overline{u(x)u(x')} = \overline{u(x)u(x+r)}$$

$$x + r = x'$$

Where x, r, x' represent scalar quantities and x, r are two independent variables.

$$\overline{u^2(x)}f(r) = \overline{u(x)u(x+r)}$$

Taking two derivatives with respect to r, we have shown that (Chapter 4 Part 3)

$$\overline{u^2(x)}f''(r) = \overline{u(x)\frac{\partial^2 u(x')}{\partial x'^2}}$$

Where the following rules were used

$$\frac{\partial f}{\partial r} = f'$$

$$\frac{\partial x'}{\partial r} = 1$$

Taking two additional derivatives with respect to r yields

$$\overline{u^2(x)}f'''(r) = \overline{u(x)\frac{\partial}{\partial r}\left(\frac{\partial^2 u(x')}{\partial x'^2}\right)}$$

$$= \overline{u(x)\frac{\partial}{\partial x'}\left(\frac{\partial^2 u(x')}{\partial x'^2}\right)\underbrace{\frac{\partial x'}{\partial r}}_{\boxed{1}}} = \overline{u(x)\frac{\partial^3 u(x')}{\partial x'^3}}$$

$$\overline{u^2(x)}f^{IV}(r) = \overline{u(x)\frac{\partial}{\partial r}\left(\frac{\partial^3 u(x')}{\partial x'^3}\right)}$$

$$= \overline{u(x)\frac{\partial}{\partial x'}\left(\frac{\partial^3 u(x')}{\partial x'^3}\right)\underbrace{\frac{\partial x'}{\partial r}}_{\boxed{1}}} = \overline{u(x)\frac{\partial^4 u(x')}{\partial x'^4}}$$

Therefore

$$\overline{u^2(x)}f^{IV}(r) = \overline{u(x)\frac{\partial^4 u(x')}{\partial x'^4}}$$

Taking the limit for $r \to 0, x' \to x$

$$\overline{u^2(x)}f^{IV}(0) = \overline{u(x)\frac{\partial^4 u(x)}{\partial x^4}}$$

Focus on the RHS

$$\overline{u(x)\frac{\partial^4 u(x)}{\partial x^4}} = \frac{\partial}{\partial x}\left[\overline{u(x)\frac{\partial^3 u(x)}{\partial x^3}}\right] - \overline{\frac{\partial u(x)}{\partial x}\frac{\partial^3 u(x)}{\partial x^3}}$$

Where the first term on the RHS is zero due to hypothesis of homogeneous turbulence (gradient of fluctuating quantity).

Apply same step one more time

$$-\overline{\frac{\partial u(x)}{\partial x}\frac{\partial^3 u(x)}{\partial x^3}} = -\frac{\partial}{\partial x}\left[\overline{\frac{\partial u(x)}{\partial x}\frac{\partial^2 u(x)}{\partial x^2}}\right] + \overline{\frac{\partial^2 u(x)}{\partial x^2}\frac{\partial^2 u(x)}{\partial x^2}}$$

Therefore

$$\overline{u^2(x)}f^{IV}(0) = \overline{u_1(x)\frac{\partial^4 u(x)}{\partial x^4}} = \overline{\frac{\partial^2 u(x)}{\partial x^2}\frac{\partial^2 u(x)}{\partial x^2}}$$

$$\overline{u^2(x)}f^{IV}(0) = \overline{\frac{\partial^2 u(x)}{\partial x^2}\frac{\partial^2 u(x)}{\partial x^2}} = \overline{u_{,xx}^2}$$

Vector approach

$$\mathcal{R}_{11}(\underline{r}) = \overline{u^2} f(r\widehat{e_1}) = \overline{u_1(\underline{x})u_1(\underline{x}')} = \overline{u_1(\underline{x})u_1(\underline{x} + r\widehat{e_1})}$$

$$\underline{y} = \underline{x} + \underline{r}$$

where

$$\underline{r} = r\widehat{e_1}$$

And

$$y_l = x_l + r_l$$

Taking a first derivative with respect to r

$$\overline{u^2} f'(r) = \overline{u(\underline{x}) \frac{\partial u(\underline{x} + r\widehat{e_1})}{\partial r}} + \overline{\frac{\partial u(\underline{x})}{\partial r} u(\underline{x} + r\widehat{e_1})}$$

$$= \overline{u(\underline{x}) \frac{\partial u(\underline{x} + r\widehat{e_1})}{\partial y_l} \frac{\partial y_l}{\partial r}} + \overline{\frac{\partial u(\underline{x})}{\partial x_l} \underbrace{\frac{\partial x_l}{\partial r}}_{\boxed{0}} u(\underline{x} + r\widehat{e_1})}$$

$$= \overline{u(\underline{x}) \frac{\partial u(\underline{x} + r\widehat{e_1})}{\partial y_l} \frac{\partial r_l}{\partial r}} = \overline{u(x) \frac{\partial u(\underline{x} + r\widehat{e_1})}{\partial y_l} \frac{r}{r_l}}$$

$$\overline{u^2} f'(r) = \overline{u(\underline{x}) \frac{\partial u(\underline{x} + r\widehat{e_1})}{\partial y_l} \frac{r}{r_l}}$$

Taking a second derivative with respect to r

$$\overline{u^2} f''(r) = \frac{\partial}{\partial r}\left[\overline{u(\underline{x})\frac{\partial u(\underline{x}+r\widehat{e_1})}{\partial y_l}\frac{r}{r_l}}\right]$$

$$= \overline{\underbrace{\frac{\partial u(\underline{x})}{\partial r}}_{\boxed{0}}\frac{\partial u(\underline{x}+r\widehat{e_1})}{\partial y_l}\frac{r}{r_l}} + \overline{u(\underline{x})\frac{\partial}{\partial r}\left(\frac{\partial u(\underline{x}+r\widehat{e_1})}{\partial y_l}\right)\frac{r}{r_l}}\ \overline{u(\underline{x})\frac{\partial u(\underline{x}+r\widehat{e_1})}{\partial y_l}\underbrace{\frac{\partial}{\partial r}\left(\frac{r}{r_l}\right)}_{\boxed{0}}}$$

$$\frac{\partial}{\partial r}\left(\frac{r}{r_l}\right) = \frac{1}{r_l} - \frac{1}{r}\frac{\partial r_l}{\partial r} = \frac{1}{r_l} - \frac{1}{r}\frac{r}{r_l} = 0$$

$$\overline{u^2} f''(r) = \overline{u(\underline{x})\frac{\partial^2 u(\underline{x}+r\widehat{e_1})}{\partial y_l^2}}$$

Taking a third derivative with respect to r

$$\overline{u^2} f'''(r) = \frac{\partial}{\partial r}\left[\overline{u(\underline{x})\frac{\partial^2 u(\underline{x}+r\widehat{e_1})}{\partial y_l^2}}\right]$$

$$= \overline{\underbrace{\frac{\partial u(\underline{x})}{\partial r}}_{\boxed{0}}\frac{\partial^2 u(\underline{x}+r\widehat{e_1})}{\partial y_l^2}} + \frac{\partial}{\partial y_l}\left[\overline{u(\underline{x})\frac{\partial^2 u(\underline{x}+r\widehat{e_1})}{\partial y_l^2}}\right]\frac{\partial y_l}{\partial r}$$

$$\overline{u^2} f'''(r) = \overline{u(\underline{x})\frac{\partial^3 u(\underline{x}+r\widehat{e_1})}{\partial y_l^3}\frac{r}{r_l}}$$

Taking a fourth derivative with respect to r (like the second derivative) yields

$$\overline{u^2} f^{IV}(r) = \overline{u(\underline{x})\frac{\partial^4 u(\underline{x}+r\widehat{e_1})}{\partial y_l^4}}$$

Taking the limit for $r \to 0,\ y_l \to x_l$

$$\overline{u^2} f^{IV}(0) = \overline{u(\underline{x})\frac{\partial^4 u(\underline{x})}{\partial x_l^4}}$$

And applying homogeneity follows same steps as scalar proof.

A.4

Scalar approach

$$S_{111}(r\hat{e_1} = \underline{r}) = u_{rms}^3 k(r)$$

$$S_{111}(\underline{r}) = \overline{u(x)u(x)u(x+r)}$$

$$x + r = x'$$

Where x, r, x' represent scalar quantities and x, r are two independent variables.

$$u_{rms}^3 k(r) = \overline{u(x)u(x)u(x+r)}$$

Taking three derivatives with respect to r yields (same procedure as $f'''(r)$)

$$u_{rms}^3 k'''(r) = \overline{u(x)u(x)\frac{\partial^3 u(x')}{\partial x'^3}}$$

Taking the limit for $r \to 0, x' \to x$

$$u_{rms}^3 k'''(0) = \overline{u(x)u(x)\frac{\partial^3 u(x)}{\partial x^3}}$$

Focus on the RHS

$$\overline{u(x)u(x)\frac{\partial^3 u(x)}{\partial x^3}} = \cancel{\frac{\partial}{\partial x}\left[\overline{u(x)u(x)\frac{\partial^2 u(x)}{\partial x^2}}\right]} - 2\overline{\frac{\partial u(x)}{\partial x}u(x)\frac{\partial^2 u(x)}{\partial x^2}}$$

Where the first term on the RHS is zero due to hypothesis of homogeneous turbulence (gradient of fluctuating quantity).

$$\frac{\partial}{\partial x}\left[\overline{\frac{\partial u(x)}{\partial x}u(x)\frac{\partial u(x)}{\partial x}}\right] = \overline{\frac{\partial^2 u(x)}{\partial x^2}u(x)\frac{\partial u(x)}{\partial x}} + \overline{\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}} + \overline{\frac{\partial u(x)}{\partial x}u(x)\frac{\partial^2 u(x)}{\partial x^2}} = 2\overline{\frac{\partial u(x)}{\partial x}u(x)\frac{\partial^2 u(x)}{\partial x^2}} + \overline{\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}}$$

Therefore, multiplying the last relation by -1 and isolating the first term on the RHS

$$-2\overline{\frac{\partial u(x)}{\partial x}u(x)\frac{\partial^2 u(x)}{\partial x^2}} = -\frac{\partial}{\partial x}\left[\overline{\frac{\partial u(x)}{\partial x}u(x)\frac{\partial u(x)}{\partial x}}\right] + \overline{\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}}$$

Or equivalently

$$\overline{u(x)u(x)\frac{\partial^3 u(x)}{\partial x^3}} = \overline{\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}\frac{\partial u(x)}{\partial x}} = u_{rms}^3 k'''(0)$$

$$u_{rms}^3 k'''(0) = \overline{u_x^3}$$

6^{th} order tensor approach

P205 pope Exe 6.11 ①

$H_{ijkiqr} = 0$

$i = p$

$$i = p \Rightarrow H_{ijkiqr} = \left\langle \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_q} \frac{\partial u_k}{\partial x_r} \right\rangle = 0.$$

$$= a_1 \delta_{ii} \delta_{jq} \delta_{kr} + a_2 (\delta_{ii} \delta_{jk} \delta_{qr} + \delta_{jq} \delta_{ik} \delta_{ir} + \delta_{kr} \delta_{ij} \delta_{iq}) +$$

$$a_3 (\delta_{ii} \delta_{jr} \delta_{qk} + \delta_{jq} \delta_{ir} \delta_{ik} + \delta_{kr} \delta_{iq} \delta_{ij}) + a_4 (\delta_{iq} [illegible] \delta_{ik} \delta_{jr} + \delta_{ir} \delta_{ij} \delta_{qk}) +$$

$$a_5 (\delta_{ij} \delta_{ik} \delta_{qr} + \delta_{ij} \delta_{qk} \delta_{ir} + \delta_{ik} \delta_{ij} \delta_{qr} + \delta_{ik} \delta_{rj} \delta_{iq} + \delta_{jk} \delta_{qi} \delta_{ir} + \delta_{jk} \delta_{ri} \delta_{iq})$$

$$= 3a_1 \delta_{jq} \delta_{kr} + a_2 (3 \delta_{jk} \delta_{qr} + \delta_{jq} \delta_{kr} + \delta_{kr} \delta_{jq}) +$$

$$a_3 (3 \delta_{jr} \delta_{qk} + \delta_{jq} \delta_{kr} + \delta_{kr} \delta_{jq}) + a_4 (\delta_{kq} \delta_{jr} + \delta_{jr} \delta_{kq}) +$$

$$a_5 (\delta_{jk} \delta_{rq} + \delta_{kq} \delta_{jr} + \delta_{jk} \delta_{rq} + [illegible] \delta_{kq} \delta_{jr} + \delta_{jk} \delta_{rq} + \delta_{jk} \delta_{rq})$$

$$= 3a_1 \delta_{jq} \delta_{kr} + a_2 3 \delta_{jk} \delta_{qr} + 2a_2 \delta_{jq} \delta_{kr} +$$

$$3a_3 \delta_{jr} \delta_{kq} + 2a_3 \delta_{jq} \delta_{kr} + a_4 \delta_{kq} \delta_{jr} + a_4 \delta_{jr} \delta_{kq} +$$

$$4a_5 \delta_{jk} \delta_{qr} + 2a_5 \delta_{kq} \delta_{jr} = 0 \Rightarrow$$

$$\delta_{jk} \delta_{qr} (3a_2 + 4a_5) + \delta_{jq} \delta_{kr} (3a_1 + 2a_2 + 2a_3) +$$

$$\delta_{jr} \delta_{kq} (3a_3 + 2a_4 + 2a_5) = 0 \Rightarrow$$

$$\begin{cases} 3a_1 + 2a_2 + 2a_3 = 0 \\ 3a_2 + 4a_5 = 0 \\ 3a_3 + 2a_4 + 2a_5 = 0 \end{cases} \quad \text{eq. 6.96}$$

Bob Pope Exe 6.11 ②

① $\frac{\partial}{\partial x_j}\langle u_k \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_k}\rangle = 0 \Rightarrow$

$\langle u_k \frac{\partial^2 u_j}{\partial x_i \partial x_j}\frac{\partial u_i}{\partial x_k} + u_k \frac{\partial u_j}{\partial x_i}\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_k}\rangle = 0 \Rightarrow$

$\langle \underbrace{u_k \frac{\partial u_j}{\partial x_i}\frac{\partial^2 u_i}{\partial x_j \partial x_k}}_{A} + \underbrace{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{C = H_{ijkkij}}\rangle = 0$

② $\frac{\partial}{\partial x_i}\langle u_k \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k}\rangle = 0 \Rightarrow$

$\langle u_k \frac{\partial^2 u_i}{\partial x_i \partial x_j}\frac{\partial u_j}{\partial x_k} + u_k \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_j}{\partial x_i \partial x_k} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k}\rangle = 0 \Rightarrow$

$\langle \underbrace{u_k \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_j}{\partial x_i \partial x_k}}_{B} + \underbrace{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_i}}_{D}\rangle = 0$

③ $\frac{\partial}{\partial x_k}\langle u_k \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\rangle = 0 \Rightarrow$

$\langle \frac{\partial u_k}{\partial x_k}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + u_k \frac{\partial^2 u_i}{\partial x_j \partial x_k}\frac{\partial u_j}{\partial x_i} + u_k \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_j}{\partial x_k \partial x_i}\rangle = 0 \Rightarrow$

$\langle \underbrace{u_k \frac{\partial^2 u_i}{\partial x_j \partial x_k}\frac{\partial u_j}{\partial x_i}}_{A} + \underbrace{u_k \frac{\partial^2 u_j}{\partial x_k \partial x_i}\frac{\partial u_i}{\partial x_j}}_{B}\rangle = 0$

$\begin{cases} \langle A + C\rangle = 0 \\ \langle B + D\rangle = 0 \\ \langle A + B\rangle = 0 \end{cases}$

$C = H_{jkiijk}$. $D = H_{ijkjki}$ For symmetries:

$C = H_{jkiijk} = H_{ijkjki} = D \Rightarrow C = D$

$\begin{cases} \langle A + C\rangle = 0 \\ \langle B + C\rangle = 0 \\ \langle A + B\rangle = 0 \end{cases} \Rightarrow \begin{cases} \langle A - B\rangle = 0 \\ \langle A + B\rangle = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases} \Rightarrow$

$C = D = 0$. i.e. $\boxed{H_{ijkjki} = 0}$

P206 pope Exe 6.11 ③.

$p=\delta, q=k, r=i$

$$H_{i\delta k\delta ki}=0=a_1\delta_{ij}\delta_{jk}\delta_{ki}+a_2(\delta_{ij}\delta_{jk}\delta_{ki}+\delta_{jk}\delta_{ik}\delta_{ji}+\delta_{ki}\delta_{ij}\delta_{jk})$$
$$+a_3(\delta_{ij}\delta_{ji}\delta_{kk}+\delta_{jk}\delta_{ii}\delta_{jk}+\delta_{ki}\delta_{ik}\delta_{jj})+a_4(\delta_{ik}\delta_{jk}\delta_{ji}+$$
$$\delta_{ii}\delta_{jj}\delta_{kk})+a_5(\delta_{ij}\delta_{jk}\delta_{ki}+\delta_{ij}\delta_{kk}\delta_{ji}+\delta_{ik}\delta_{jj}\delta_{ki}+$$
$$\delta_{ik}\delta_{ij}\delta_{jk}+\delta_{jk}\delta_{ki}\delta_{ji}+\delta_{jk}\delta_{ii}\delta_{jk})=0 \Rightarrow$$

$$a_1\delta_{ii}+a_2(\delta_{ii}+\delta_{ii}+\delta_{kk})+a_3(\delta_{ii}\delta_{kk}+\delta_{jj}\delta_{ii}+\delta_{ii}\delta_{jj})+a_4(\delta_{ii}+27)+$$
$$a_5(\overset{3}{\delta_{ii}}+\overset{9}{\delta_{ii}\delta_{kk}}+\overset{9}{\delta_{ii}\delta_{jj}}+\overset{3}{\delta_{jj}}+\overset{3}{\delta_{ii}}+\overset{9}{\delta_{jj}\delta_{ii}})=0 \Rightarrow$$

$$3a_1+9a_2+27a_3+30a_4+36a_5=0 \Rightarrow$$

$$a_1+3a_2+9a_3+10a_4+12a_5=0 \qquad \text{eq. 6.97}$$

From eqs. 6.96~6.97: $a_2=-\frac{4}{3}a_1,\ a_3=-\frac{1}{6}a_1,\ a_4=-\frac{3}{4}a_1,\ a_5=a_1$

eq. 6.98

$$\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^3\right\rangle=H_{111111}=a_1+3a_2+3a_3+2a_4+6a_5$$
$$=a_1-4a_1-\frac{1}{2}a_1-\frac{3}{2}a_1+6a_1=a_1$$
$$=S\left(\frac{\varepsilon}{15\nu}\right)^{3/2} \qquad \text{eq. 6.99}$$

From 6.84. $\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^3\right\rangle=-\frac{2}{35}\left\langle\omega_i\omega_j\frac{\partial u_i}{\partial x_j}\right\rangle=a_1 \Rightarrow$

$$\left\langle\omega_i\omega_j\frac{\partial u_i}{\partial x_j}\right\rangle=-\frac{35}{2}a_1 \qquad \text{eq. 6.100}$$

$$H_{iikkqq}=3a_1+21a_2+15a_3+12a_4+54a_5$$
$$=3a_1-28a_1-\frac{5}{2}a_1-9a_1+54a_1=\frac{35}{2}a_1 \qquad \text{eq. 6.101}$$

$\mathcal{H}_{ijkpqr}$ is a sixth order tensor,

$$\mathcal{H}_{ijkpqr} \equiv \overline{\frac{\partial u_i}{\partial x_p}\frac{\partial u_j}{\partial x_q}\frac{\partial u_k}{\partial x_r}}$$

And we have shown that,

$$\mathcal{H}_{111111} = \overline{\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1}} = \overline{(u_x)^3} = a_1$$

And

$$\mathcal{H}_{iikkqq} = \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_q}\frac{\partial u_k}{\partial x_q}} = \frac{35}{2}a_1$$

If we make the following change of indices: $k = l$ and $q = j$, we obtain,

$$\mathcal{H}_{iilljj} = \overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}} = \frac{35}{2}a_1$$

And comparison with Eq. (11), results in,

$$\mathcal{H}_{iilljj} = \frac{\partial^3 S_{il,i}}{\partial {r_j}^2 \partial r_l}(0)$$

Similarly,

$$\mathcal{H}_{111111} = \frac{\partial^3 S_{11,1}}{\partial {r_1}^3}(0) = \overline{\frac{\partial u_1(\underline{x})}{\partial x_1}\frac{\partial u_1(\underline{x})}{\partial x_1}\frac{\partial u_1(\underline{x})}{\partial x_1}} = \overline{u_x^3}$$

A.5

Definition Skewness and Palinstrophy (related to Palenstrophy?)

The **skewness** is the third moment of v', normalized by the variance:

$$\text{skewness} = \frac{< v'^3 >}{< v'^2 >^{3/2}} \tag{3.5}$$

A PDF which is symmetric about the mean $< v >$ will have zero skewness. All higher odd moments of such a symmetric PDF will also be identically zero. The skewness reveals information about the asymmetry of the PDF. Positive skewness indicates that the PDF has a longer tail for $v - \langle v \rangle > 0$ than for $v - \langle v \rangle < 0$. Hence a positive skewness means that variable v' is more likely to take on large positive values than large negative values. A time series with long stretches of small negative values and a few instances of large positive values, with zero time mean, has positive skewness (Fig. 3.1).

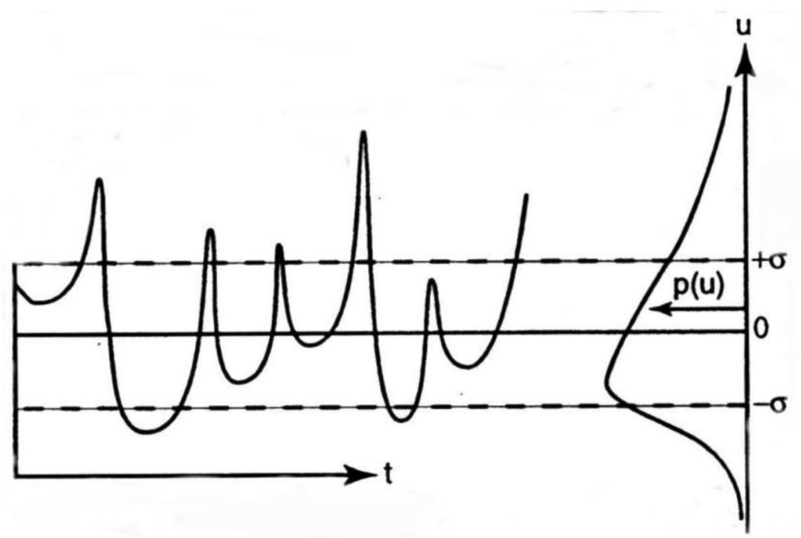

Figure 3.1: Signal with a positive skewness.

Davidson, Turbulence, Chapter 10, Two-Dimensional Turbulence, 2004.

[2] Palinstrophy is defined as $\frac{1}{2}(\nabla \times \boldsymbol{\omega})^2$, which in two-dimensions is $\frac{1}{2}(\nabla \omega)^2$. The etymology of the word is given in Lesieur (1990). It was introduced by Pouquet et al. (1975) and is constructed from *palin* and *strophy*, which are the Greek for *again* and *rotation* respectively. Thus Palinstrophy is 'again rotation' or 'curl curl'.

A.5

$$R_T = k^2/\nu\varepsilon$$

$$\frac{dR_T}{dt} = \frac{2k}{\nu\varepsilon}\frac{dk}{dt} - \frac{k^2}{\nu\varepsilon^2}\frac{d\varepsilon}{dt}$$

$$= -\frac{2k}{\nu} - \frac{k^2}{\nu\varepsilon^2}\left[S_K^* R_T^{1/2}\frac{\varepsilon^2}{k} - G^*\frac{\varepsilon^2}{k}\right]$$

$$= -\frac{2k}{\nu} - S_K^*\sqrt{R_T}\,\frac{k}{\nu} + G^*\frac{k}{\nu}$$

$$\tau(t) = \int_0^t \frac{\varepsilon}{k}\,dt \quad \text{using} \quad \frac{dk}{dt} = -\varepsilon$$

$$= -\int_{k(0)}^{k(t)} \frac{dk}{k} = -\ln k\,\Big|_{k(0)}^{k(t)} = \ln k_0/k(t)$$

$$R_T^*(\tau) = R_T(t) \Rightarrow \frac{dR_T}{dt} = \frac{dR_T^*}{d\tau}\frac{d\tau}{dt} = \frac{dR_T^*}{d\tau}\frac{\varepsilon}{k}$$

$$\frac{dR_T^*}{d\tau} = \frac{k}{\varepsilon}\left[-\frac{2k}{\nu} - S_K^*\sqrt{R_T}\,\frac{k}{\nu} + G^*\frac{k}{\nu}\right]$$

$$= \frac{k^2}{\varepsilon\nu}\left[G^* - 2 - S_K^*\sqrt{R_T}\right]$$