Chapter 5: Energy Decay in Isotropic Turbulence

Part 1: Energy Decay Highlights

Decay process of TKE by viscous dissipation is ideally studied for homogeneous isotropic turbulence since it contains all essential physics while yielding equations in their simplest forms. The results are used in many turbulence models, which are applied to general flows.

$$\frac{dk}{dt} = -\varepsilon \quad (1)$$

$$\frac{d\varepsilon}{dt} = P_\varepsilon^4 - \Upsilon_\varepsilon = -2\nu\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_j}} - 2\nu^2\overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2} \quad (2a)$$

$$\frac{d\tilde{\varepsilon}}{dt} = \nu P_\zeta^4 - \nu\Upsilon_\zeta = +2\nu\overline{\omega_i\omega_k\frac{\partial u_i}{\partial x_k}} - 2\nu^2\overline{\frac{\partial \omega_i}{\partial x_k}\frac{\partial \omega_i}{\partial x_k}} \quad (2b)$$

P_ε^4 and νP_ζ^4: effects of vortex stretching $> 0 \therefore$ represents production of ε.

$-\Upsilon_\varepsilon$ and $-\nu\Upsilon_\zeta$: effects of dissipation of dissipation.

$$\overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2} = \overline{\frac{\partial \omega_i}{\partial x_k}\frac{\partial \omega_i}{\partial x_k}} = \underbrace{-\omega \cdot \nabla \times (\nabla \times \omega)}_{Palinstrophy} = \frac{\partial^4 \mathcal{R}_{ii}(0)}{\partial {r_j}^2 \partial {r_l}^2} \quad (6)$$

Dissipation/Palenstrophy.

$$-\overline{\frac{\partial u_i(\underline{x})}{\partial x_l}\frac{\partial u_l(\underline{x})}{\partial x_j}\frac{\partial u_i(\underline{x})}{\partial x_j}} = 2\nu\overline{\omega_i\omega_k\frac{\partial u_i}{\partial x_k}} = \frac{\partial^3 S_{il,i}(0)}{\partial {r_j}^2 \partial r_l} \quad (11) \quad \text{Vortex stretching}$$

Using definitions $\mathcal{R}_{ij}(\underline{r})$ and $Sil,i(r)$:

$$\frac{d\varepsilon}{dt} = -2\nu\frac{\partial^3 S_{il,i}(0)}{\partial {r_j}^2 \partial r_l} - 2\nu^2\frac{\partial^4 \mathcal{R}_{ii}(0)}{\partial {r_j}^2 \partial {r_l}^2}$$

Recall Chapter 4 Part 2:

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}\frac{df}{dr}\right]$$

$$S_{ijl}(\underline{r}) = u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_j r_l}{2r^3} - \frac{k}{2}\delta_{ij}\frac{r_l}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

$$\frac{\partial^4 \mathcal{R}_{ii}}{\partial {r_j}^2 \partial {r_l}^2}(0) = \overline{u^2}[35 f^{IV}(0) + r f^{V}(0)]$$

$$\frac{\partial^3 S_{il,i}}{\partial {r_j}^2 \partial r_l}(0) = \frac{35}{2} u_{rms}^3 k'''(0)$$

$$\frac{d\varepsilon}{dt} = -35 u_{rms}^3 k'''(0) - 70\overline{u^2} f^{IV}(0) \quad (12)$$

i.e., only depends on two time-dependent scalars, along with k ($u_{rms} = [\frac{2}{3}k]^{1/2}$) and ε.

$$\overline{(u_{xx})^2} = u_{rms}^2 f^{IV}(0) \quad (13)$$

$$\overline{(u_x)^3} = u_{rms}^3 k'''(0) \quad (14)$$

Using:

$$\mathcal{R}_{11}(\underline{r}) = \overline{u(x)u(x+r)}$$

$$S_{111}(\underline{r}) = \overline{u(x)u(x)u(x+r)}$$

The skewness of u_x is defined (positive due minus sign on RHS):

$$S_k = -\frac{\overline{(u_x)^3}}{\overline{(u_x)^2}^{3/2}} \quad (15)$$

$$k'''(0) = -\frac{S_k}{\lambda_g^3} = -S_k\left(\frac{\varepsilon}{15 u_{rms}^2 \nu}\right)^{\frac{3}{2}} \quad (17) \quad \text{used in Part 2}$$

Palenstrophy coefficient of u_x can be defined as

$$G = \frac{\overline{u^2}\,\overline{(u_{xx})^2}}{\overline{(u_x)^2}^2}$$

$$f^{IV}(0) = \frac{G}{\lambda_g^4} = G\left(\frac{\varepsilon}{15 u_{rms}^2 \nu}\right)^2 \quad (18) \quad \text{used in Part 2}$$

$$\frac{d\varepsilon}{dt} = S_k^* R_T^{1/2} \frac{\varepsilon^2}{k} - G^* \frac{\varepsilon^2}{k} \quad (19)$$

$$S_K^* = \frac{7}{3\sqrt{15}} S_k$$

$$G^* = \frac{7}{15} G$$

$$R_T = \frac{k^2}{\nu\varepsilon}$$

This equation, along with Eq. (1) represent two equations in the four unknowns k, ε, S_K^* and G^* , all of which are $f(t)$, i.e., not closed. RHS term 1 = gain and term 2 = loss. Initial state needs to be specified, i.e., at $t = 0$, $k_0, \varepsilon_0, S_{K_0}^*$, and G_0^*. Alternatively, using Eqs. (17) and (18), initial forms for $f(r)$ and $k(r)$ can be specified, from which S_{k_0} and G_0 can be obtained.

Turbulent Reynolds Number ($Re_L = R_T$)

$$R_T = \frac{k^2}{\nu\varepsilon} = \text{ turbulent Re} = \frac{\sqrt{k}k^{3/2}/\varepsilon}{\nu}$$

$$\frac{dR_T}{dt} = -\frac{2k}{\nu} - S_k^* \sqrt{R_T} \frac{k}{\nu} + G^* \frac{k}{\nu} \quad (20)$$

$$\tau(t) = \int_0^t \frac{\varepsilon(t')}{k(t')} dt' \quad (21)$$

$$\tau(t) = \ln\big(k(0)/k(t)\big)$$

$$\frac{dR_T^*}{d\tau} = R_T^*\left(G^* - 2 - S_k^* \sqrt{R_T^*}\right) \quad (23)$$

Thus, an alternative to solving the decay problem via Eqs. (1) and (19) is the option of solving Eq. (23). G^* and S_k^* are $f(t)$ such that represents one equation in three unknowns, i.e., additional assumptions are required.

No matter which way the decay problem is approached, solving for k and ε requires additional assumptions so that a closed system of equations can be deduced.