# Chapter 4: Turbulence at Small Scales
## Part 8: Structure functions

### 6.1.4 *Restatement of the Kolmogorov hypotheses*

In order to deduce precise consequences from them, it is worthwhile to provide here more precise statements of the Kolmogorov (1941) hypotheses. Kolmogorov presented these in terms of an $N$-point distribution in the four-dimensional $x$–$t$ space. Here, however, we consider the $N$-point distribution in physical space ($\boldsymbol{x}$) at a fixed time $t$ – which is sufficiently general for most purposes.

Consider a simple domain $\mathcal{G}$ within the turbulent flow, and let $\boldsymbol{x}^{(0)}$, $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N)}$ be a specified set of points within $\mathcal{G}$. New coordinates and velocity differences are defined by

$$\boldsymbol{y} \equiv \boldsymbol{x} - \boldsymbol{x}^{(0)}, \tag{6.20}$$

$$\boldsymbol{v}(\boldsymbol{y}) \equiv \boldsymbol{U}(\boldsymbol{x}, t) - \boldsymbol{U}(\boldsymbol{x}^{(0)}, t), \tag{6.21}$$

and the joint PDF of $\boldsymbol{v}$ at the $N$ points $\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \ldots, \boldsymbol{y}^{(N)}$ is denoted by $f_N$.

> **The definition of local homogeneity**. The turbulence is locally homogeneous in the domain $\mathcal{G}$, if for every fixed $N$ and $\boldsymbol{y}^{(n)} (n = 1, 2, \ldots, N)$, the $N$-point PDF $f_N$ is independent of $\boldsymbol{x}^{(0)}$ and $\boldsymbol{U}(\boldsymbol{x}^{(0)}, t)$.
>
> **The definition of local isotropy**. The turbulence is locally isotropic in the domain $\mathcal{G}$ if it is locally homogeneous and if in addition the PDF $f_N$ is invariant with respect to rotations and reflections of the coordinate axes.
>
> **The hypothesis of local isotropy**. In any turbulent flow with a sufficiently large Reynolds number ($\mathrm{Re} = \mathcal{U}\mathcal{L}/\nu$), the turbulence is, to a good approximation, locally isotropic if the domain $\mathcal{G}$ is sufficiently small (i.e., $|\boldsymbol{y}^{(n)}| \ll \mathcal{L}$, for all $n$) and is not near the boundary of the flow or its other singularities.
>
> **The first similarity hypothesis**. For locally isotropic turbulence, the $N$-point PDF $f_N$ is uniquely determined by the viscosity $\nu$ and the dissipation rate $\varepsilon$.
>
> **The second similarity hypothesis**. If the moduli of the vectors $\boldsymbol{y}^{(m)}$ and of their differences $\boldsymbol{y}^{(m)} - \boldsymbol{y}^{(n)}$ $(m \neq n)$ are large compared with the Kolmogorov scale $\eta$, then the $N$-point PDF $f_N$ is uniquely determined by $\varepsilon$ and does not depend on $\nu$.

It is important to observe that the hypotheses apply specifically to velocity differences. The use of the $N$-point PDF $f_N$ allows the hypotheses to be applied to any turbulent flow, whereas statements in terms of wavenumber spectra apply only to flows that are statistically homogeneous (in at least one direction).

For inhomogeneous flows, local isotropy is possible only 'to a good approximation' (as stated in the hypothesis). For example, taking $\boldsymbol{y}^{(1)} = \boldsymbol{e}\ell$ and $\boldsymbol{y}^{(2)} = -\boldsymbol{e}\ell$ (where $\ell$ is a specified length and $\boldsymbol{e}$ a specified unit vector), we have

$$\begin{aligned}\langle \boldsymbol{v}(\boldsymbol{y}^{(1)}) - \boldsymbol{v}(\boldsymbol{y}^{(2)}) \rangle &= \langle \boldsymbol{U}(\boldsymbol{y}^{(1)}) \rangle - \langle \boldsymbol{U}(\boldsymbol{y}^{(2)}) \rangle \\ &\approx 2\frac{\ell}{\mathcal{L}} \boldsymbol{e} \cdot \mathcal{L} \nabla \langle \boldsymbol{U} \rangle. \end{aligned} \tag{6.22}$$

Evidently this simple statistic is not exactly isotropic, but instead has a small anisotropic component – of order $\ell/\mathcal{L}$ – arising from large-scale inhomogeneities.

## 6.2 Structure functions

To illustrate the correct application of the Kolmogorov hypotheses, we consider – as did Kolmogorov (1941b) – the second-order velocity structure functions. The predictions of the hypotheses are deduced, and then compared with experimental data.

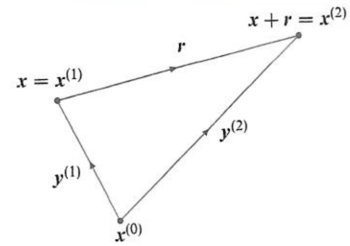


Fig. 6.3. A sketch showing the points $x$ and $x + r$ in terms of $x^{(n)}$ and $y^{(n)}$. All points are within the domain $\mathcal{G}$.

By definition, the second-order velocity structure function is the covariance of the difference in velocity between two points $x + r$ and $x$:

$$D_{ij}(r, x, t) \equiv \langle [U_i(x + r, t) - U_i(x, t)][U_j(x + r, t) - U_j(x, t)] \rangle. \qquad (6.23)$$

## 6.3 Two-point correlation

The Kolmogorov hypotheses, and deductions drawn from them, have no direct connection to the Navier–Stokes equations (although, as in the previous section, the continuity equation is usually invoked). Although, in the description of the energy cascade, the transfer of energy to successively smaller scales has been identified as a phenomenon of prime importance, the precise mechanism by which this transfer takes place has not been identified or quantified. It is natural, therefore, to try to extract from the Navier–Stokes equations useful information about the energy cascade. The earliest attempts

(outlined in this section) are those of Taylor (1935a) and of von Kármán and Howarth (1938), which are based on the two-point correlation. The next two sections give the view from wavenumber space in terms of the energy spectrum – the Fourier transform of the two-point correlation.

### *Autocorrelation functions*

Consider homogeneous isotropic turbulence, with zero mean velocity, r.m.s. velocity $u'(t)$, and dissipation rate $\varepsilon(t)$. Because of homogeneity, the two-point correlation

$$R_{ij}(r, t) \equiv \langle u_i(x + r, t) u_j(x, t) \rangle, \qquad (6.41)$$

is independent of $x$. At the origin it is

$$R_{ij}(0, t) = \langle u_i u_j \rangle = u'^2 \delta_{ij}. \qquad (6.42)$$

Kolmogorov spectra can be obtained via two paths:

1. Use Fourier transforms of structure functions (physical space)
2. Apply Kolmogorov hypothesis directly to the spectra (wave number space) using Fourier transform of $\mathcal{R}_{ij}(\underline{r},t) = \overline{u_i(\underline{x},t)u_j(\underline{x}+\underline{r},t)}$ .

The second approach is less rigorous but simpler as we have done. The first approach was originally used by Kolmogorov. Also, the second approach can be connected to the Navier-Stokes equations to extract further information on the energy cascade.

Second order velocity structure function is co-variance of the difference in velocity between two points $\underline{x}+\underline{r}$ and $\underline{x}$: 2nd order tensor which is determined by all eddies with size less than or comparable with $|\underline{r}|$.

$$D_{ij}(\underline{x},\underline{r},\tau) = \overline{[u_i(\underline{x}+\underline{r},t) - u_i(\underline{x},t)][u_j(\underline{x}+\underline{r},t) - u_j(\underline{x},t)]} \quad (1)$$

Which can be expressed in terms of the two-point velocity correlation tensor (for convenience dropping the time dependence):

$$D_{ij}(\underline{x},\underline{r}) = \mathcal{R}_{ij}(\underline{x},0) + \mathcal{R}_{ij}(\underline{x}+\underline{r},0) - \mathcal{R}_{ij}(\underline{x},\underline{r}) - \mathcal{R}_{ij}(\underline{x}+\underline{r},-\underline{r})$$

To within scalar multiples, the only second-order tensors that can be formed from the vector $\underline{r}$ are $\delta_{ij}$ and $r_i r_j$. Consequently $D_{ij}$ can be written as

$$D_{ij}(\underline{r},t) = D_{NN}(r,t)\delta_{ij} + [D_{LL}(r,t) - D_{NN}(r,t)]\frac{r_i r_j}{r^2} \quad (2)$$

Where the scalar functions $D_{LL}$ and $D_{NN}$ are called, respectively, the longitudinal and transverse structure functions. If the coordinate system is chosen so that $\underline{r} = r\widehat{e_1}$

$$D_{11} = D_{LL} \quad D_{22} = D_{33} = D_{NN} \quad D_{ij} = 0 \;\; i \neq j$$

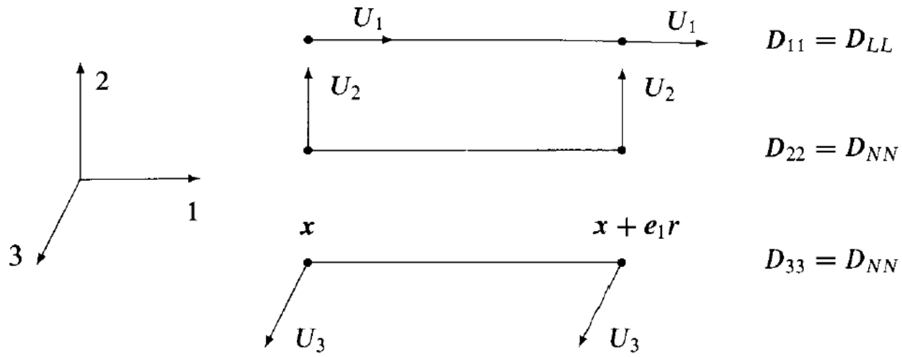


Fig. 6.4. A sketch of the velocity components involved in the longitudinal and transverse structure functions for $\boldsymbol{r} = \boldsymbol{e}_1 r$.

As for $\mathcal{R}_{ij}$,

$$\frac{\partial}{\partial r_j} D_{ij} = 0 \qquad \text{Due to incompressibility}$$

Combining isotropic theory with the incompressibility condition

$$D_{NN}(r,t) = D_{LL}(r,t) + \tfrac{1}{2} r \tfrac{\partial}{\partial r} D_{LL}(r,t) \quad (3)$$

$D_{ij}(\underline{r},t) = f\big(D_{LL}(r,t)\big)$ , i.e., determined by single scalar function $D_{LL}(r,t)$.

According to the 1st similarity hypothesis, given $\underline{r}$ $\big(|\underline{r}| \ll L\big)$, $D_{ij}$ is uniquely determined by $\varepsilon$ and v.

$(\varepsilon r)^{2/3}$ has dimensions of velocity squared, and so can be used to make $D_{ij}$ non-dimensional. There is only one independent non-dimensional group that can be formed from $r, \varepsilon, \nu$ which can be taken to be $r\varepsilon^{1/4}/\nu^{3/4} = r/\eta$, where $\eta = \varepsilon^{-1/4}\nu^{3/4}$, as per Chapter 5 Part 0.

Thus,

$$D_{LL}(r,t) = (\varepsilon r)^{2/3}\widehat{D}_{LL}(r/\eta) \quad (4)$$

where $\widehat{D}_{LL}(r/\eta)$ is a universal, non-dimensional function.

According to the 2nd similarity hypothesis, for large $r/\eta$ $(L \gg r \gg \eta)$, $D_{LL}$ is independent of $\nu$ and, in this case, there is no non-dimensional group that can be formed from $\varepsilon$ and $r$, so $D_{LL}$ is given by:

$$D_{LL}(r,t) = C_2(\varepsilon r)^{2/3} \quad (5)$$

Where $C_2$ is a universal constant → For large $r/\eta$, $\widehat{D}_{LL}(r/\eta)$ asymptotically goes to a constant value $C_2$. Using Eq. (3 and (5)):

$$D_{NN}(r,t) = \tfrac{4}{3}D_{LL}(r,t) = \tfrac{4}{3}C_2(\varepsilon r)^{2/3} \quad (6)$$ in inertial subrange

Therefore using Eq. (2)

$$D_{ij}(\underline{r},t) = C_2(\varepsilon r)^{2/3}\left(\tfrac{4}{3}\delta_{ij} - \tfrac{1}{3}\tfrac{r_i r_j}{r^2}\right) = f(C_2,\varepsilon,r) \quad (7)$$

For $\underline{r}$ = $\hat{e}_1$r and $L \gg r \gg \eta$:

$$\frac{D_{11}}{(\varepsilon r)^{2/3}} = C_2$$

$$\frac{D_{22}}{(\varepsilon r)^{2/3}} = \frac{D_{33}}{(\varepsilon r)^{2/3}} = \frac{4}{3}C_2$$

$$D_{ij} = 0 \quad i \neq j$$

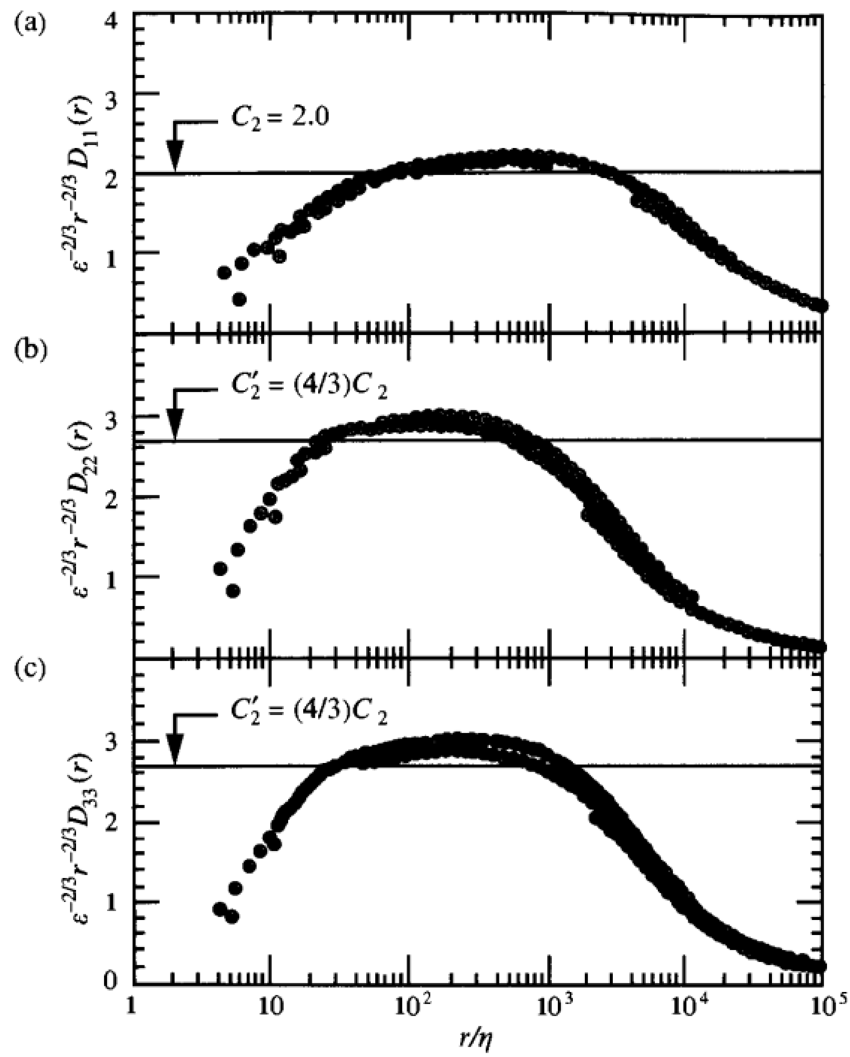


Fig. 6.5. Second-order velocity structure functions measured in a high-Reynolds-number turbulent boundary layer. The horizontal lines show the predictions of the Kolmogorov hypotheses in the inertial subrange, Eqs. (6.33) and (6.34). (From Sad-doughi and Veeravalli (1994).)

the above predictions can readily be examined. Taking the value $C_2 = 2.0$ suggested by these and other data, we draw the following conclusions.

(i) For $7{,}000\ \eta \approx \frac{1}{2}\mathcal{L} > r > 20\eta$, $D_{11}/(\varepsilon r)^{2/3}$ is within $\pm 15\%$ of $C_2$.

(ii) There is no perceptible difference between $D_{22}$ and $D_{33}$.

(iii) For $1{,}200\ \eta \approx \frac{1}{10}\mathcal{L} > r > 12\eta$, $D_{22}/(\varepsilon r)^{2/3}$ is within $\pm 15\%$ of $\frac{4}{3}C_2$.

Over the ranges of $r$ given above, $D_{11}$ and $D_{22}$ change by factors of 50 and 20, respectively, and so $\pm 15\%$ variations can be considered small in comparison.

Structure Functions: Bernard

Kolmogorov put forth his hypotheses in physical space using structure functions. Bernard vs. Pope only considers

$$D_{ii} = S_n(\underline{x}, \underline{r}, t) = \langle | u_i(\underline{x}+\underline{r}, t) - u_i(\underline{x}, t)|^2 \rangle$$

$$S_2 = \langle \underline{u}(\underline{x}+\underline{r}, t) \cdot \underline{u}(\underline{x}+\underline{r}, t) \rangle + \langle \underline{u}(\underline{x}, t) \cdot \underline{u}(\underline{x}, t) \rangle - 2 \langle \underline{u}(\underline{x}, t)\, \underline{u}(\underline{x}+\underline{r}, t) \rangle$$

For homogeneous turbulence

$$= R_{ii}(0) + R_{ii}(0) - 2R_{ii}(\underline{r})$$

$$= 2k + 2k - 4 \int_0^\infty \frac{\sin kr}{kr} E(k,t)\, dk \quad \text{using Bernard Eq. (4.180)}$$

$$= 4 \int_0^\infty E(k,t) \left(1 - \frac{\sin kr}{kr}\right) dk \quad (1) \quad \text{(needs derivation)}$$

$$S_n^L(r) = \langle | u(x+r) - u(x)|^n \rangle \qquad n = 2 = D_{11}$$

$$S_n^T(r) = \langle | v(x+r) - v(x)|^n \rangle \qquad n = 2 = D_{22}$$

For small r: dissipation range

For intermediate r between small and large (energy containing range): inertial subrange

For high Re use $E(k) = C_K k^{-5/3} \varepsilon^{2/3}$; since, integrand in (1) is small for both (a) small and (b) large k. For (a): $\sin kr / kr \approx 1$ so term in ( ) small. For (b): $E(k,t)$ in dissipation range so small. Integration (1) yields:

$$S_2 = 4.82\, C_K\, (\varepsilon r)^{2/3} \quad (2)$$

Dimensional analysis yields: $(\varepsilon_r r)^{1/3} = \frac{m}{s}$

$\therefore \tau$ dimensionless

$|u(x+r) - u(x)| \sim \tau (\varepsilon_r r)^{1/3}$ $\tau$ = random variable $\neq f(\varepsilon, r)$

Substituting into $S_n^L(r)$:

$$S_n^L(r) = C_n r^{n/3} \overline{\varepsilon_r^{n/3}}$$

$\varepsilon_r = \varepsilon$ ave over sphere radius r

$\overline{\varepsilon_r^{n/3}}$ obeys power law $r^{\alpha}$ where $\alpha = \alpha(n/3)$ & $S_n^L(r)$ obeys power law $r^{n/3+\alpha(n/3)}$ such that $\overline{\varepsilon_r^{n/3}}$ (fluctuating dissipation) modifies $r^{n/3}$ power law. That is for (2) to be true: $\alpha(2/3) = 0$. Similarly Kolmogorov obtained $S_3^L(r) = -\frac{4}{5} r \bar{\varepsilon}_r$ which implies $\alpha(1) = 0$.

$S_2^L / (\bar{\varepsilon}_r r)^{2/3}$ & $-S_3^L(\bar{\varepsilon}_r r)$ should be constant when power law given by n/3. Compensated spectrums show nearly constant values for $r/\eta > 100/\eta$ = mean of sub range. Tilt in spectrums suggest $\alpha(2/3)$ & $\alpha(1) \neq 0$. Explanation, similar that of "bottleneck" effect of Kolmogorov compensated spectrum, as per Chapter 4 Part 6 pg 18.

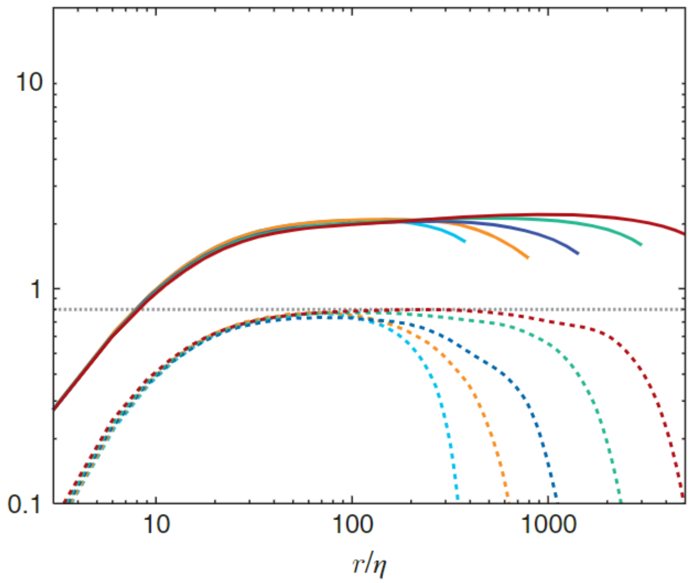


**Figure 4.10** Compensated longitudinal structure functions computed in isotropic turbulence [24]. $S_2^L/(\epsilon_r r)^{2/3}$, top curves; $-S_3^L/(\epsilon_r r)$, bottom curves. Constant, dotted line is 0.8. The sequence of curves in each group covers an increasing $r/\eta$ domain correspond to $R_\lambda = 167, 257, 471, 732, 1131$.

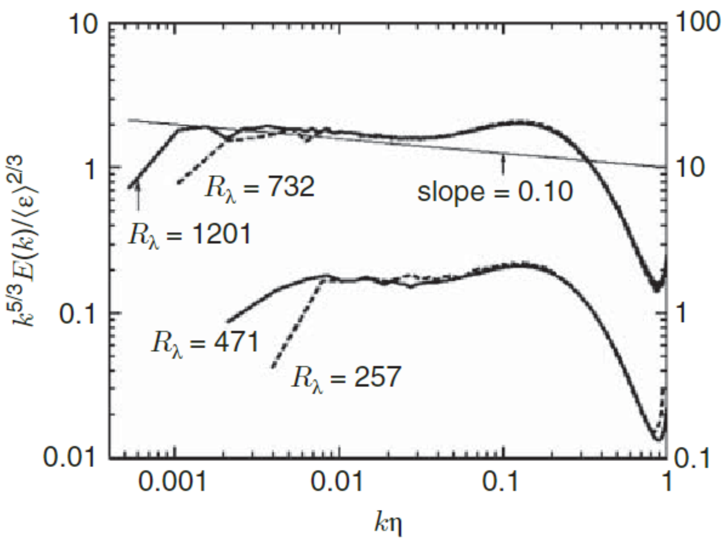


**Figure 4.8** Compensated energy spectrum as given in [21]. With increasing $R_\lambda$ the simulations used $512^3$, $1024^3$, $2048^3$, and $4096^3$ meshes. Scales on the left and right are for the upper and lower curves, respectively. Reproduced from *Physics of Fluids*, Vol. 15, pp. L21–L24, 2003, with the permission of AIP Publishing.