Anisotropy

Definitions

Two-point velocity correlation tensor:

$$\mathcal{R}_{ij}\left(\underline{x},\underline{y},t\right)=\overline{u_i(\underline{x},t)u_j\left(\underline{y},t\right)}$$

$U_i=\overline{U_i}+u_i$

Where u_i represents the fluctuating component of the velocity in i −direction.

When $\underline{x}=\underline{y}$,

$$\mathcal{R}_{ij}(\underline{x},\underline{x},t)=\overline{u_i(\underline{x},t)u_j(\underline{x},t)}=\mathcal{R}_{ij}(\underline{x},t)$$

Which represents the Reynolds stress tensor (assuming constant density):

$$\mathcal{R}_{ij}(\underline{x},t)=\begin{pmatrix}\overline{u_1{}^2} & \overline{u_1u_2} & \overline{u_1u_3}\\ \overline{u_2u_1} & \overline{u_2{}^2} & \overline{u_2u_3}\\ \overline{u_3u_1} & \overline{u_3u_2} & \overline{u_3{}^2}\end{pmatrix}\quad(1)$$

Schuman (1977) investigated the realizability of Reynolds-stress turbulence models and in doing so provided the following realizability conditions for the Reynolds stress tensor based on physical considerations:

$$\mathcal{R}_{ij}\geq 0\quad for\ i=j\quad(2)$$

$$\mathcal{R}_{ij}^2\leq\mathcal{R}_{ii}\mathcal{R}_{jj}\quad for\ i\neq j\quad(3)$$

$$\det\left(\mathcal{R}_{ij}\right)\geq 0\quad(4)$$

Eq. (2) is a consequence of real velocities, and it requires nonnegative energy. Eq. (3) is a consequence of the Schwarz' inequality and it states that the cross-correlation between the velocity components u_i and u_j is bounded by the magnitude of the autocorrelations. Eq. (4) states that the three cross-correlations cannot take on arbitrary values. Eqs. (2), (3), and (4) represent five checks.

However, it also shown that if a stronger condition is satisfied, i.e., that $\mathcal{R}_{ij}$ is a positive semi-definite matrix

$$Q=x_i\mathcal{R}_{ij}x_j\geq 0\quad(5.1)$$

for arbitrary real nonvanishing vectors x_i that not only are Eq. (2) - (4) satisfied but additionally nonnegative eigenvalues and principal invariants are implied. The necessary and sufficient conditions for the positive semi-definiteness of $\mathcal{R}_{ij}$ are as follows:

$$\overline{u_i u_i} \geq 0, \ \ \overline{u_i u_i} + \overline{u_j u_j} \geq \ 2\left|\overline{u_i u_j}\right|, \ \ \det(\mathcal{R}_{ij}) \geq 0, \qquad i,j = \{1,2,3\}. \quad (5.2)$$

Eq. (5) has the advantage of only requiring three checks vs. five.

Dissipation tensor:

$$\varepsilon_{ij} = \nu \begin{pmatrix} 2\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} + \left[\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2 + \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}}\right] + \left[\overline{\left(\frac{\partial u_1}{\partial x_3}\right)^2 + \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}}\right] \\ + \left[\overline{\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2}}\right] + 2\overline{\left(\frac{\partial u_2}{\partial x_2}\right)^2} + \left[\overline{\left(\frac{\partial u_2}{\partial x_3}\right)^2 + \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}}\right] \\ + \left[\overline{\left(\frac{\partial u_3}{\partial x_1}\right)^2 + \frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3}}\right] + \left[\overline{\left(\frac{\partial u_3}{\partial x_2}\right)^2 + \frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3}}\right] + 2\overline{\left(\frac{\partial u_3}{\partial x_3}\right)^2} \end{pmatrix}$$

represents the definition of the dissipation $\left(\varepsilon = \nu\overline{u_{i,j}u_{i,j}} + \upsilon\frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j}\right)$ prior to contraction of the indices, where $\tilde{\varepsilon} = \nu\overline{u_{i,j}u_{i,j}}$ is the pseudo-dissipation; and the second term is zero for homogeneous/isotropic turbulence.

Isotropic and anisotropic tensors

It is always possible to express a second order tensor as the sum of an isotropic tensor (with elements only on the diagonal) and an anisotropic tensor. For a generic tensor A_{ij}:

$$A_{ij} = A_{ij}^I + A_{ij}^A$$

Where the isotropic tensor can be expressed as:

$$A_{ij}^I = \frac{1}{3} A_{kk}\delta_{ij}$$

and A_{ij}^A represents the anisotropic part of A_{ij}. The trace of A_{ij}^I is equivalent to the trace of A_{ij}, such that A_{ij}^A is a traceless tensor, i.e., $A_{kk}^A = 0$. Applying this to the Reynolds stress tensor gives:

$$\mathcal{R}_{ij} = \mathcal{R}_{ij}^I + \mathcal{R}_{ij}^A = \frac{2}{3}k\delta_{ij} + a_{ij} \quad (6)$$

Where a_{ij} is the anisotropic stress tensor and k represents the turbulent kinetic energy of the flow and it is equal to half the trace of $\mathcal{R}_{ij}$:

$$k = \frac{1}{2}\left(\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}\right)$$

Using Eq. (6), the following expression for the anisotropic stress tensor is obtained:

$$a_{ij} = \mathcal{R}_{ij} - \frac{2}{3}k\delta_{ij} \quad (7)$$

a_{ij} can be normalized using a factor equal to two times the turbulent kinetic energy to obtain the normalized anisotropic stress tensor b_{ij}:

$$b_{ij} = \frac{a_{ij}}{2k} \quad (8)$$

Therefore, the corresponding matrix representations of a_{ij} and b_{ij} are:

$$a_{ij} = \begin{pmatrix} \overline{u_1{}^2} - \frac{2}{3}k & \overline{u_1 u_2} & \overline{u_1 u_3} \\ \overline{u_2 u_1} & \overline{u_2{}^2} - \frac{2}{3}k & \overline{u_2 u_3} \\ \overline{u_3 u_1} & \overline{u_3 u_2} & \overline{u_3{}^2} - \frac{2}{3}k \end{pmatrix}$$

$$b_{ij} = \begin{pmatrix} \frac{\overline{u_1{}^2}}{2k} - \frac{1}{3} & \frac{\overline{u_1 u_2}}{2k} & \frac{\overline{u_1 u_3}}{2k} \\ \frac{\overline{u_2 u_1}}{2k} & \frac{\overline{u_2{}^2}}{2k} - \frac{1}{3} & \frac{\overline{u_2 u_3}}{2k} \\ \frac{\overline{u_3 u_1}}{2k} & \frac{\overline{u_3 u_2}}{2k} & \frac{\overline{u_3{}^2}}{2k} - \frac{1}{3} \end{pmatrix}$$

As already mentioned, a_{ij} and b_{ij} are traceless tensors, i.e., $a_{ii} = b_{ii} = 0$.

The required conditions for the positive semi-definiteness of $\mathcal{R}_{ij}$ as per Eq. (5) can be expressed as constraints on the elements of b_{ij}. Banerjee et al. (2007) shows that the diagonal component b_{ii} of the anisotropy tensor takes its minimal value of (-1/3) if $\mathcal{R}_{ii} = 0$ and its maximal value (2/3) when $\mathcal{R}_{ii} = 2k$.

Minimum value:

$$\mathcal{R}_{ii} = 0$$

$$b_{ii} = \frac{a_{ii}}{2k} = \cancel{\frac{\mathcal{R}_{ii}}{2k}} - \frac{2k\delta_{ii}}{2k \cdot 3} = -\frac{1}{3}$$

Maximum value:

$$\mathcal{R}_{ii} = 2k$$

$$b_{ii} = \frac{a_{ii}}{2k} = \frac{\mathcal{R}_{ii}}{2k} - \frac{2k\delta_{ii}}{2k \cdot 3} = 1 - \frac{1}{3} = \frac{2}{3}$$

Therefore, the diagonal terms b_{ii} are bounded as follows:

$$-\frac{1}{3} \leq b_{ii} \leq \frac{2}{3}$$

Because of the positive semi-definiteness of $\mathcal{R}_{ij}$, the off-diagonal elements of b_{ij} reach their minimal values (-1/2) in the cases when the corresponding element $\mathcal{R}_{ij} = -k$ and their maximum values (1/2) if $\mathcal{R}_{ij} = k$.

Minimum value:

$$\mathcal{R}_{ij} = -k$$

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\mathcal{R}_{ij}}{2k} - \cancel{\frac{2k\delta_{ij}}{2k \cdot 3}} = -\frac{k}{2k} = -\frac{1}{2}$$

Maximum value:

$$\mathcal{R}_{ij} = k$$

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\mathcal{R}_{ij}}{2k} - \cancel{\frac{2k\delta_{ij}}{2k \cdot 3}} = \frac{k}{2k} = \frac{1}{2}$$

Therefore, the off-diagonal terms b_{ij} are bounded as follows:

$$-\frac{1}{2} \leq b_{ij} \leq \frac{1}{2}, i \neq j$$

A similar process can be applied to the dissipation tensor, such that:

$$\varepsilon_{ij} = \varepsilon_{ij}^{I} + \varepsilon_{ij}^{A}$$

Where:

$$\varepsilon_{ij}^{I} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{2}{3}\nu\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}\delta_{ij}$$

$$\varepsilon_{ij}^{I} = \nu\begin{pmatrix} \frac{2}{3}\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} + 0 + 0 \\ +0 + \frac{2}{3}\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} + 0 \\ +0 + 0 + \frac{2}{3}\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} \end{pmatrix}$$

The anisotropic dissipation tensor can be expressed as:

$$\begin{aligned}\varepsilon_{ij}^{A} &= \varepsilon_{ij} - \varepsilon_{ij}^{I} \\ &= \nu\begin{pmatrix} 2\overline{\left(\frac{\partial u}{\partial x}\right)^2} - \frac{2}{3}\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} + \left[\overline{\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}\right] + \left[\overline{\left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x}}\right] \\ + \left[\overline{\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}\right] + 2\overline{\left(\frac{\partial v}{\partial y}\right)^2} - \frac{2}{3}\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} + \left[\overline{\left(\frac{\partial v}{\partial z}\right)^2 + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}}\right] \\ + \left[\overline{\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z}}\right] + \left[\overline{\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z}}\right] + 2\overline{\left(\frac{\partial w}{\partial z}\right)^2} - \frac{2}{3}\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} \end{pmatrix}\end{aligned}$$

And normalized using a factor equal to $2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}$ to obtain the normalized anisotropic dissipation tensor $\widetilde{\varepsilon_{ij}^{A}}$, in a form like b_{ij}:

$$\widetilde{\varepsilon_{ij}^{A}} = \nu \begin{pmatrix} \frac{\overline{\left(\frac{\partial u}{\partial x}\right)^2}}{\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3} & + \frac{\left[\overline{\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}\right]}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} & + \frac{\left[\overline{\left(\frac{\partial u}{\partial z}\right)^2 + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x}}\right]}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} \\ + \frac{\left[\overline{\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}\right]}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} & + \frac{\overline{\left(\frac{\partial v}{\partial y}\right)^2}}{\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3} & + \frac{\left[\overline{\left(\frac{\partial v}{\partial z}\right)^2 + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}}\right]}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} \\ + \frac{\left[\overline{\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z}}\right]}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} & + \frac{\left[\overline{\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z}}\right]}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} & + \frac{\overline{\left(\frac{\partial w}{\partial z}\right)^2}}{\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3} \end{pmatrix} \quad (9)$$

Note that ε_{ij}^{I} is not equivalent to $\tilde{\varepsilon} = \nu\overline{u_{i,j}u_{i,j}}$. ε_{ij}^{I} represents the isotropic part of the dissipation tensor prior to contraction, whereas $\tilde{\varepsilon}$ represents the non-zero terms of ε_{ij} in isotropic turbulence. The usual decomposition for the dissipation tensor is:

$$\varepsilon = \nu\overline{u_{i,j}u_{i,j}} + \upsilon\frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j}$$

$$= \nu \begin{pmatrix} \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} & + \left[\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2}\right] & + \left[\overline{\left(\frac{\partial u_1}{\partial x_3}\right)^2}\right] \\ + \left[\overline{\left(\frac{\partial u_2}{\partial x_1}\right)^2}\right] & + \overline{\left(\frac{\partial u_2}{\partial x_2}\right)^2} & + \left[\overline{\left(\frac{\partial u_2}{\partial x_3}\right)^2}\right] \\ + \left[\overline{\left(\frac{\partial u_3}{\partial x_1}\right)^2}\right] & + \left[\overline{\left(\frac{\partial u_3}{\partial x_2}\right)^2 +}\right] & + \overline{\left(\frac{\partial u_3}{\partial x_3}\right)^2} \end{pmatrix} + \nu \begin{pmatrix} \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} & + \left[\overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}}\right] & + \left[\overline{\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}}\right] \\ + \left[\overline{\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2}}\right] & + \overline{\left(\frac{\partial u_2}{\partial x_2}\right)^2} & + \left[\overline{\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}}\right] \\ + \left[\overline{\frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3}}\right] & + \left[\overline{\frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3}}\right] & + \overline{\left(\frac{\partial u_3}{\partial x_3}\right)^2} \end{pmatrix}$$

In homogeneous turbulence, the fluctuating velocity $\underline{u}(\underline{x}, t)$ is statistically homogeneous, and the time-averaged properties of the flow are uniform and independent of position, i.e., $\frac{\partial}{\partial x_j}\overline{fluctuating\ terms} = 0$. The second term in this decomposition is equal to $\frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} = \overline{u_{i,j} u_{j,i}} = 0$ for homogeneous/isotropic turbulence.

Invariant Equation, eigenvalues, and eigenvectors

For a general second order tensor A_{ij}, the eigenvalue problem involves determining eigenvalues $\lambda \in C$ and $x_i \in \Re^n, x_i \neq 0$ such that:

$$A_{ij} x_k = \lambda x_k$$

C =set of complex numbers

$\Re^n$ = real coordinate n-space

Where x_k represent the eigenvectors of A_{ij}

Or equivalently:

$$(\lambda \delta_{ij} - A_{ij}) x_k = 0$$

Which is equivalent to solving the linear system:

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} = 0 \qquad (10)$$

Where a_{ij} are the elements of the second order tensor A_{ij}.
When A_{ij} is represented in the basis formed by its eigenvectors, it assumes a diagonal form:

$$\hat{A}_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Every second order tensor possesses three principal invariants, which are independent of the reference system the tensor is represented in. These invariants can be related to the elements of A_{ij} and the eigenvalues of A_{ij}:

$$\mathrm{I_A} = trace(A_{ij}) = a_{11} + a_{22} + a_{33} = trace(\hat{A}_{ij}) = \lambda_1 + \lambda_2 + \lambda_3 \quad (11)$$

$$\mathrm{II_A} = \frac{1}{2}\left\{\left[trace(A_{ij})\right]^2 - trace(A_{ij}^{\mathbf{2}})\right\}$$
$$= a_{11}a_{33} + a_{22}a_{33} + a_{11}a_{22} - a_{23}a_{32} - a_{12}a_{21} - a_{13}a_{31}$$
$$= a_{11}a_{33} + a_{22}a_{33} + a_{11}a_{22} - {a_{23}}^2 - {a_{12}}^2 - {a_{13}}^2$$
$$= \frac{1}{2}\left\{\left[trace(\hat{A}_{ij})\right]^2 - trace\left({\hat{A}_{ij}}^{\mathbf{2}}\right)\right\} = \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_2 \quad (12)$$

$$\mathrm{III_A} = \det(A_{ij})$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} = \det(\hat{A}_{ij}) = \lambda_1\lambda_2\lambda_3 \quad (13)$$

The resulting characteristic polynomial is obtained by carrying out the determinant in Eq. (10) and equating it to zero:

$$\lambda^3 - \mathrm{I_A}\lambda^2 + \mathrm{II_A}\lambda - \mathrm{III_A} = 0 \quad (14)$$

Eqs. (11), (12), and (13) form a non-linear system of three equations in three unknowns (λ_1, λ_2, λ_3) with 9 parameters given by the elements of A_{ij} (6 if the tensor is symmetric).

The general solution of the cubic equation shown in Eq. (14) can be obtained by substituting:

$$\lambda = x + \frac{\mathrm{I_A}}{3}$$

Such that the cubic equation gets reduced to the normal form:

$$x^3 + hx + k = 0$$

Where:

$$h = \frac{1}{3}\left(3\mathrm{II_A} - {\mathrm{I_A}}^2\right)$$

$$k = \frac{1}{27}\left(-2{\mathrm{I_A}}^3 + 9\mathrm{I_A}\mathrm{II_A} - 27\mathrm{III_A}\right)$$

Which has the solutions:

$$x_1 = A + B$$

$$x_2, x_3 = -\frac{1}{2}(A+B) \pm \frac{i\sqrt{3}}{2}(A-B)$$

Where:

$$i^2 = -1$$

$$A = \sqrt[3]{-\frac{k}{2} + \sqrt{\frac{k^2}{4} + \frac{h^3}{27}}}$$

$$B = \sqrt[3]{-\frac{k}{2} - \sqrt{\frac{k^2}{4} + \frac{h^3}{27}}}$$

If $\mathrm{I_A}, \mathrm{II_A}, \mathrm{III_A}$ are real (and hence if h and k are real)

- If $\frac{k^2}{4} + \frac{h^3}{27} > 0$ there are one real root and two conjugate imaginary roots
- If $\frac{k^2}{4} + \frac{h^3}{27} = 0$ there are three real roots of which at least two are equal.
- If $\frac{k^2}{4} + \frac{h^3}{27} < 0$ there are three real and unequal roots

Solving this system analytically, i.e., obtaining $\lambda_k = f(a_{ij})$, may not be possible in the general case due to the nonlinearity of the equations. However, ML and AI techniques could be applied to understand which elements a_{ij} have larger influence on λ_k. When numerical data are given, numerical methods can be used to obtain an approximate solution for the inverted system.

For turbulence data, MATLAB is used to solve numerically the eigenvalue problem, instead of the general solution to the cubic equation. Once the three eigenvalues of A_{ij} are determined, it is possible to obtain the corresponding eigenvectors from the solution of the equations:

$$\left(\lambda_k \delta_{ij} - A_{ij}\right) x_k = 0 \quad (15)$$

Where $\lambda_k, k = \{1,2,3\}$ represent the three eigenvalues of A_{ij}. For λ_1 the resulting system of equation is:

$$\left(\lambda_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = 0$$

And solving for x_{11}, x_{12}, x_{13} provides the three components of the first eigenvector x_1. A similar system of equations can be solved for x_2 and x_3.

The corresponding eigenvectors can be used to form a 3x3 matrix P_{ij}:

$$P_{ij} = (x_1 \quad x_2 \quad x_3)$$

where x_i represents the i^{th} column eigenvector.

The matrix P_{ij} diagonalizes A_{ij}:

$$\hat{A}_{ij} = P_{ij}^{-1} A_{ij} \, P_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

The eigenvectors x_i represent the principal axes for A_{ij}. The diagonal tensor $\hat{\mathcal{R}}_{ij}$ is used in the following sections as a starting point for the anisotropy analysis.

Relationship between the eigenvalues and eigenvectors of $\mathcal{R}_{ij}$ and b_{ij}

Beginning with the general expression for the Reynolds stress tensor shown in Eq. (1), it is necessary to solve Eq. (14) to derive an expression for the eigenvalues λ_{u_i} as a function of the Reynolds stress tensor components $\mathcal{R}_{ij.}$ As shown before, this can only be done when numerical data is provided for $\mathcal{R}_{ij.}$ It is assumed that the eigenvalues are obtained numerically solving Eq. (14). They are ordered from largest to smallest ($\lambda_{u_1} \geq \lambda_{u_2} \geq \lambda_{u_3}$). Then, using the corresponding eigenvectors, $\mathcal{R}_{ij}$ can be represented in its principal axes, where the only non-zero components belong to the main diagonal and are equal to the eigenvalues λ_{u_i}:

$$\hat{\mathcal{R}}_{ij} = \begin{pmatrix} \lambda_{u_1} & 0 & 0 \\ 0 & \lambda_{u_2} & 0 \\ 0 & 0 & \lambda_{u_3} \end{pmatrix}$$

Starting from $\hat{\mathcal{R}}_{ij}$ and combining it with Eqs. (7) and (8), an expression for $\hat{b}_{ij}$ is obtained:

$$\hat{a}_{ij} = \hat{\mathcal{R}}_{ij} - \frac{2}{3}k\delta_{ij} = \begin{pmatrix} \lambda_{u_1} & 0 & 0 \\ 0 & \lambda_{u_2} & 0 \\ 0 & 0 & \lambda_{u_3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3}k & 0 & 0 \\ 0 & \frac{2}{3}k & 0 \\ 0 & 0 & \frac{2}{3}k \end{pmatrix}$$

$$\hat{b}_{ij} = \frac{\hat{a}_{ij}}{2k} = \begin{pmatrix} \frac{\lambda_{u_1}}{2k} - \frac{1}{3} & 0 & 0 \\ 0 & \frac{\lambda_{u_2}}{2k} - \frac{1}{3} & 0 \\ 0 & 0 & \frac{\lambda_{u_3}}{2k} - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \lambda_{b_1} & 0 & 0 \\ 0 & \lambda_{b_2} & 0 \\ 0 & 0 & \lambda_{b_3} \end{pmatrix} \quad (15)$$

Now, it is possible to calculate the principal invariants of $\hat{b}_{ij}$, using Eqs. (11), (12), and (13).

$$I_b = trace(\hat{b}_{ij}) = \frac{\lambda_{u_1}}{2k} + \frac{\lambda_{u_2}}{2k} + \frac{\lambda_{u_3}}{2k} - 1$$

$$II_b = \frac{1}{2}\left\{\left[trace(\hat{b}_{ij})\right]^2 - trace\left(\hat{b}_{ij}{}^{\mathbf{2}}\right)\right\}$$
$$= \left(\frac{\lambda_{u_1}}{2k} - \frac{1}{3}\right)\left(\frac{\lambda_{u_3}}{2k} - \frac{1}{3}\right) + \left(\frac{\lambda_{u_2}}{2k} - \frac{1}{3}\right)\left(\frac{\lambda_{u_3}}{2k} - \frac{1}{3}\right) + \left(\frac{\lambda_{u_1}}{2k} - \frac{1}{3}\right)\left(\frac{\lambda_{u_2}}{2k} - \frac{1}{3}\right)$$
$$= \frac{\lambda_{u_1}\lambda_{u_3} + \lambda_{u_2}\lambda_{u_3} + \lambda_{u_1}\lambda_{u_2}}{4k^2} - \frac{1}{3}\frac{\lambda_{u_1} + \lambda_{u_2} + \lambda_{u_3}}{k} + \frac{1}{3}$$

$$III_b = \det\left(\hat{b}_{ij}\right)$$
$$= \left(\frac{\lambda_{u_1}\lambda_{u_2}\lambda_{u_3}}{8k^3} - \frac{\lambda_{u_1}\lambda_{u_3} + \lambda_{u_1}\lambda_{u_2} + \lambda_{u_2}\lambda_{u_3}}{12k^2} + \frac{\lambda_{u_1} + \lambda_{u_2} + \lambda_{u_3}}{18k} - \frac{1}{27}\right)$$

The resulting characteristic polynomial, in analogy with Eq. (14) is:

$$\lambda^3 - I_b\lambda^2 + II_b\lambda - III_b = 0$$

The characteristic polynomial is solved using MATLAB Symbolic Math Toolbox, to keep the dependency for the elements of $\mathcal{R}_{ij}$ and its eigenvalues. Therefore, the invariants I_b, II_b, and III_b are expressed as $f\left(\mathcal{R}_{ij}, \lambda_{u_k}\right)$ and the cubic equation is solved analytically using symbolic values for $\mathcal{R}_{ij}$ and λ_{u_k}.

Solving for the eigenvalues gives:

$$\lambda_{b_1} = -\frac{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2} - 3\lambda_{u_1}}{3\overline{u_1{}^2} + 3\overline{u_2{}^2} + 3\overline{u_3{}^2}} = -\frac{1}{3}\frac{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}} + \frac{3}{3}\frac{\lambda_{u_1}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}$$

$$\lambda_{b_2} = -\frac{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2} - 3\lambda_{u_2}}{3\overline{u_1{}^2} + 3\overline{u_2{}^2} + 3\overline{u_3{}^2}} = -\frac{1}{3}\frac{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}} + \frac{3}{3}\frac{\lambda_{u_2}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}$$

$$\lambda_{b_3} = -\frac{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2} - 3\lambda_{u_3}}{3\overline{u_1{}^2} + 3\overline{u_2{}^2} + 3\overline{u_3{}^2}} = -\frac{1}{3}\frac{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}} + \frac{3}{3}\frac{\lambda_{u_3}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}$$

Which can be simplified as:

$$\lambda_{b_1} = -\frac{1}{3} + \frac{\lambda_{u_1}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}$$

$$\lambda_{b_2} = -\frac{1}{3} + \frac{\lambda_{u_2}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}$$

$$\lambda_{b_3} = -\frac{1}{3} + \frac{\lambda_{u_3}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}$$

Moreover, the trace of $\mathcal{R}_{ij}$ is an invariant, per Eq. (11), i.e., it has the same value in any reference system.

$$\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2} = \lambda_{u_1} + \lambda_{u_2} + \lambda_{u_3} \quad (16)$$

Finally, an expression relating λ_{b_i} to λ_{u_i} can be obtained, i.e., relating the eigenvalues of the normalized anisotropic tensor and the Reynolds stress tensor:

$$\lambda_{b_i} = -\frac{1}{3} + \frac{\lambda_{u_i}}{\overline{u_1{}^2} + \overline{u_2{}^2} + \overline{u_3{}^2}}$$

Using Eq. (16):

$$\lambda_{b_i} = -\frac{1}{3} + \frac{\lambda_{u_i}}{\lambda_{u_1} + \lambda_{u_2} + \lambda_{u_3}} \qquad (17)$$

As stated in Pope Eq. (7) and (8) in **Problem 11.4**.

$b_{ij}$is a traceless tensor; therefore, the sum of the eigenvalues of b_{ij} is zero, i.e., only two eigenvalues are linearly independent, while the third one can be expressed as a combination of the first two:

$$\lambda_{b_3} = -\lambda_{b_1} - \lambda_{b_2}$$

The elements on the diagonal of $\hat{b}_{ij}$ are equal to the eigenvalues λ_{b_i}, since the diagonal representation of the normalized anisotropic tensor is used, as shown in Eq. (15).

In general, the eigenvectors of $\mathcal{R}_{ij}$ and b_{ij} are the same. This can be justified by the fact that the eigenvectors are invariant with respect to addition or multiplication by a constant.

The eigenvectors for $\mathcal{R}_{ij}$ ($x_{\mathcal{R}_k}$) and b_{ij} (x_{b_k})satisfy the property:

$$\mathcal{R}_{ij}x_{\mathcal{R}_k} = \lambda_{u_k}x_{\mathcal{R}_k} \quad (18b)$$

$$b_{ij}x_{b_k} = \lambda_{b_k}x_{b_k} \quad (18b)$$

Subtracting $\frac{2}{3}k\delta_{ij}$ on both sides of Eq. (18a) and dividing by $2k$ gives:

$$\underbrace{\frac{\left(\mathcal{R}_{ij} - \frac{2}{3}k\delta_{ij}\right)}{2k}}_{\boxed{b_{ij}}} x_{\mathcal{R}_k} = \frac{\left(\lambda_{u_k} - \frac{2}{3}k\delta_{ij}\right)}{2k} x_{\mathcal{R}_k}$$

$$a_{ij} = \mathcal{R}_{ij} - \frac{2}{3}k\delta_{ij} \quad (7)$$

$$b_{ij} = \frac{a_{ij}}{2k} \quad (8)$$

$$b_{ij}x_{\mathcal{R}_k} = \underbrace{\frac{\left(\lambda_{u_k} - \frac{2}{3}k\delta_{ij}\right)}{2k}}_{\boxed{\lambda_{b_k}}} x_{\mathcal{R}_k}$$

$$b_{ij}x_{\mathcal{R}_k} = \lambda_{b_k}x_{\mathcal{R}_k} \quad (19)$$

Comparing Eqs. (18b) and (19) reveals that the eigenvectors of $\mathcal{R}_{ij}$ and b_{ij} are identical, whereas their eigenvalues are related, as shown in Eq. (17).

Relationship between the eigenvalues and eigenvectors of $\boldsymbol{\varepsilon_{ij}}$ and $\boldsymbol{\varepsilon_{ij}^A}$

Representing the dissipation tensor in its principal axes gives:

$$\hat{\varepsilon}_{ij} = \nu \begin{pmatrix} \lambda_{\varepsilon_1} & 0 & 0 \\ 0 & \lambda_{\varepsilon_2} & 0 \\ 0 & 0 & \lambda_{\varepsilon_3} \end{pmatrix}$$

And combining this with Eq. (9) gives:

$$\hat{\varepsilon}_{ij}^A = \nu \begin{pmatrix} \frac{\lambda_{\varepsilon_1}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3} & 0 & 0 \\ 0 & \frac{\lambda_{\varepsilon_2}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3} & 0 \\ 0 & 0 & \frac{\lambda_{\varepsilon_3}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3} \end{pmatrix} \quad (20)$$

Now, it is possible to calculate the principal invariants of $\hat{\varepsilon}_{ij}^A$, using Eqs. (11), (12), and (13).

$$I_\varepsilon = trace\left(\hat{\varepsilon}_{ij}^A\right) = \frac{\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \lambda_{\varepsilon_3}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - 1$$

$$II_\varepsilon = \frac{1}{2}\left\{\left[trace\left(\hat{\varepsilon}_{ij}^{A}\right)\right]^2 - trace\left(\hat{\varepsilon}_{ij}^{A^2}\right)\right\}$$

$$= \left(\frac{\lambda_{\varepsilon_1}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3}\right)\left(\frac{\lambda_{\varepsilon_3}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3}\right) + \left(\frac{\lambda_{\varepsilon_2}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3}\right)\left(\frac{\lambda_{\varepsilon_3}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3}\right) + \left(\frac{\lambda_{\varepsilon_1}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3}\right)\left(\frac{\lambda_{\varepsilon_2}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{3}\right)$$

$$= \frac{\lambda_{\varepsilon_1}\lambda_{\varepsilon_3} + \lambda_{\varepsilon_2}\lambda_{\varepsilon_3} + \lambda_{\varepsilon_1}\lambda_{\varepsilon_2}}{4\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}^2} - \frac{1}{3}\frac{\left(\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \lambda_{\varepsilon_3}\right)}{\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} + \frac{1}{3}$$

$$III_\varepsilon = \det\left(\hat{\varepsilon}_{ij}^{A}\right)$$

$$= \left(\frac{\lambda_{\varepsilon_1}\lambda_{\varepsilon_2}\lambda_{\varepsilon_3}}{8\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}^3} - \frac{\left(\lambda_{\varepsilon_1}\lambda_{\varepsilon_2} + \lambda_{\varepsilon_1}\lambda_{\varepsilon_3} + \lambda_{\varepsilon_2}\lambda_{\varepsilon_3}\right)}{4\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}^2} + \frac{\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \lambda_{\varepsilon_3}}{\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}} - \frac{1}{27}\right)$$

The resulting characteristic polynomial is:

$$\lambda^3 - I_\varepsilon\lambda^2 + II_\varepsilon\lambda - III_\varepsilon = 0$$

And solving for the eigenvalues:

$$\lambda_{\varepsilon_1}^{A} = -\frac{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} - 3\lambda_{\varepsilon_1}}{6\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}}$$

$$\lambda_{\varepsilon_2}^{A} = -\frac{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} - 3\lambda_{\varepsilon_2}}{6\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}}$$

$$\lambda_{\varepsilon_3}^{A} = -\frac{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} - 3\lambda_{\varepsilon_3}}{6\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}}$$

Which can be rewritten as:

$$\lambda_{\varepsilon_1}^{A} = -\frac{1}{3} + \frac{\lambda_{\varepsilon_1}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}}$$

$$\lambda_{\varepsilon_2}^{A} = -\frac{1}{3} + \frac{\lambda_{\varepsilon_2}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}}$$

$$\lambda_{\varepsilon_3}^{A} = -\frac{1}{3} + \frac{\lambda_{\varepsilon_3}}{2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2}}$$

Moreover, the trace of ε_{ij} is an invariant, per Eq. (11), i.e., it has the same value in any reference system.

$$2\overline{\left(\frac{\partial u_k}{\partial x_k}\right)^2} = 2\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v}{\partial y}\right)^2} + 2\overline{\left(\frac{\partial w}{\partial z}\right)^2} = \lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \lambda_{\varepsilon_3}$$

Finally, an expression relating ${\lambda_{\varepsilon}}_i^A$ to λ_{ε_i} can be obtained, i.e., relating the eigenvalues of the normalized anisotropic dissipation tensor and the eigenvalues of the dissipation tensor:

$$ {\lambda_{\varepsilon}}_i^A = -\frac{1}{3} + \frac{\lambda_{\varepsilon_i}}{\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \lambda_{\varepsilon_3}} $$

The elements on the diagonal of $\hat{\varepsilon}_{ij}^A$ are equal to the eigenvalues ${\lambda_{\varepsilon}}_i^A$, since the diagonal representation of the normalized anisotropic tensor was used, as shown in Eq. (20).

Lumley Triangle

The anisotropy stress tensor has zero trace, i.e., $b_{ii} = 0$ and consequently it has two independent invariants and two independent eigenvalues, as shown above.

These two invariants can be taken as the quantities ξ and η, or II_b and III_b:

$$ 6\eta^2 = -2\mathit{II}_b = b_{ii}^2 = b_{ij}b_{ji} $$

$$ 6\xi^3 = 3\mathit{III}_b = b_{ii}^2 = b_{ij}b_{jk}b_{ki} $$

Problem 11.4 Pope

ξ and η are related to the eigenvalues of $\hat{b}_{ij}$:

$$ \eta^2 = \frac{1}{3}\left({\lambda_b}_1^2 + \lambda_{b_1}\lambda_{b_2} + {\lambda_b}_2^2\right) $$

$$ \xi^2 = -\frac{1}{2}\lambda_{b_1}\lambda_{b_2}\left(\lambda_{b_1} + \lambda_{b_2}\right) $$

Similarly, for II_b and III_b:

$$ \mathit{II}_b = -3\eta^2 = -\left({\lambda_b}_1^2 + \lambda_{b_1}\lambda_{b_2} + {\lambda_b}_2^2\right) $$

$$ \mathit{III}_b = 2\xi^3 = -\lambda_{b_1}\lambda_{b_2}\left(\lambda_{b_1} + \lambda_{b_2}\right) $$

Therefore, the state of anisotropy of the Reynolds stresses can be characterized by two invariants, i.e., ξ and η, or II_b and III_b.

At any point and time in turbulent flows, ξ and η (Π_b and III_b) can be calculated from the Reynolds stress tensor and plotted on a $\xi - \eta$ (III_b-$-\Pi_b$) plane. Some special states of the Reynolds stress tensor correspond to points and curves in this plane and generate what is known as the Lumley triangle.

Every Reynolds stress that can occur in a turbulent flow (i.e., that is realizable) corresponds to a point in the Lumley triangle, whereas points outside correspond to unrealizable turbulent states.

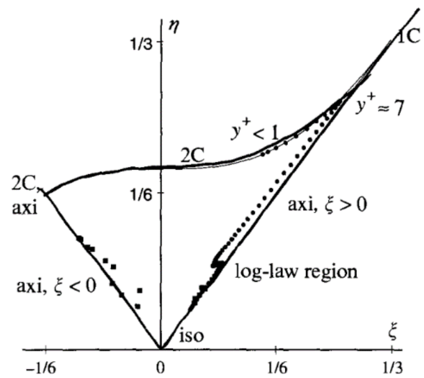

Fig. 11.1. The Lumley triangle on the plane of the invariants ξ and η of the Reynolds-stress anisotropy tensor. The lines and vertices correspond to special states (see Table 11.1). Circles: from DNS of channel flow (Kim *et al.* 1987). Squares: from experiments on a turbulent mixing layer (Bell and Mehta 1990). 1C, one-component; 2C, two-component.

Lumley triangle in $\xi - \eta$ plane

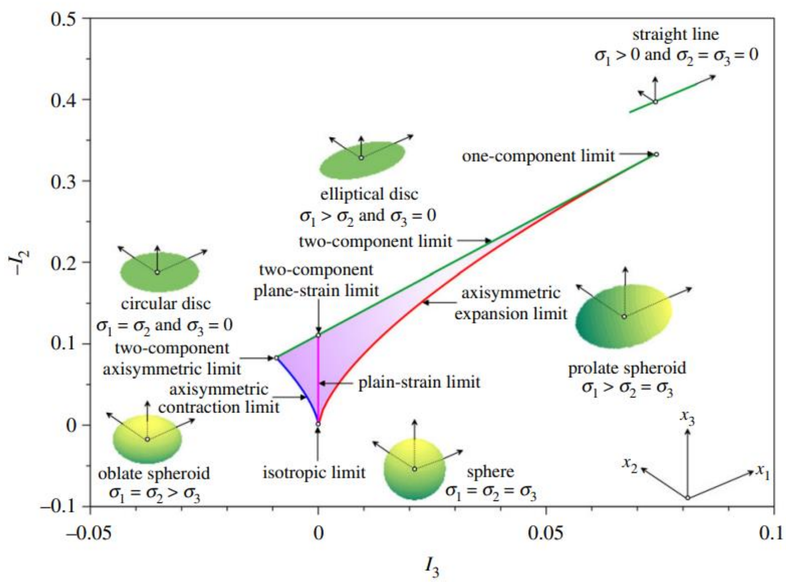

Figure 5. AIM on $-I_2I_3$ frame illustrating all possible realizations of turbulence states. (Online version in colour.)

Lumley triangle in $III_b - -\Pi_b$ plane, σ represent the RS principal stresses.

State of turbulence	Invariants	Eigenvalues of b_{ij}	Shape of RS ellipsoid	Designation in Fig. 11.1
Isotropic	$\xi = \eta = 0$	$\lambda_{b_1} = \lambda_{b_2} = \lambda_{b_3} = 0$	Sphere	iso
Two-component axisymmetric	$\xi = -\frac{1}{6}$	$\lambda_{b_1} = \lambda_{b_2} = \frac{1}{6}$	Disk	2C, axi
One-component	$\xi = \frac{1}{3}, \eta = \frac{1}{3}$	$\lambda_{b_1} = \frac{2}{3}$ $\lambda_{b_2} = \lambda_{b_3} = -\frac{1}{3}$	Line	1C
Axisymmetric (one large eigenvalue)	$\eta = \xi$	$-\frac{1}{3} \le \lambda_{b_1} = \lambda_2 \le 0$	Prolate spheroid	axi, $\xi < 0$
Axisymmetric (one small eigenvalue)	$\eta = -\xi$	$0 \le \lambda_{b_1} = \lambda_{b_2} \le \frac{1}{6}$	Oblate spheroid	axi, $\xi > 0$
Two-component	$\eta = \left(\frac{1}{27} + 2\xi^3\right)^{\frac{1}{2}}$ $F(\xi, \eta) = 0$	$\lambda_{b_1} + \lambda_{b_2} = \frac{1}{3}$	Ellipse	2C

Table 11.1

Moreover, the special states of the Reynolds stress tensor can be related to the shape of the Reynolds stress ellipsoid defined as:

$$E(\underline{V}, \alpha) \le 1 \quad \text{Pope 11.8}$$

Where:

$$E(\underline{V}, \alpha) \equiv \frac{1}{\alpha^2} \mathcal{R}_{ij}^{-1} (U_i - \overline{U_i})(U_j - \overline{U_j})$$

i.e.,

$$\frac{1}{\alpha^2} \mathcal{R}_{ij}^{-1} (U_i - \overline{U_i})(U_j - \overline{U_j}) \le 1$$

And writing the Reynolds stress in their principal axes:

$$\mathcal{R}_{ij}^{-1} = \begin{bmatrix} 1/\lambda_{u_1} & 0 & 0 \\ 0 & 1/\lambda_{u_2} & 0 \\ 0 & 0 & 1/\lambda_{u_3} \end{bmatrix}$$

Gives:

$$\frac{1}{\alpha^2}\begin{bmatrix} 1/\lambda_{u_1} & 0 & 0 \\ 0 & 1/\lambda_{u_2} & 0 \\ 0 & 0 & 1/\lambda_{u_3} \end{bmatrix}\left(U_i - \overline{U_i}\right)\left(U_j - \overline{U_j}\right) \leq 1$$

$$\frac{1}{\alpha^2}\begin{bmatrix} \dfrac{U_1 - \overline{U_1}}{\lambda_{u_1}} \\ \dfrac{U_2 - \overline{U_2}}{\lambda_{u_2}} \\ \dfrac{U_3 - \overline{U_3}}{\lambda_{u_3}} \end{bmatrix}\left(U_j - \overline{U_j}\right) \leq 1$$

$$\frac{1}{\alpha^2}\left[\left(\frac{U_1 - \overline{U_1}}{\sqrt{\lambda_{u_1}}}\right)^2 + \left(\frac{U_2 - \overline{U_2}}{\sqrt{\lambda_{u_2}}}\right)^2 + \left(\frac{U_3 - \overline{U_3}}{\sqrt{\lambda_{u_3}}}\right)^2\right] \leq 1$$

And assuming $\alpha = 1$:

$$\left(\frac{U_1 - \overline{U_1}}{\sqrt{\lambda_{u_1}}}\right)^2 + \left(\frac{U_2 - \overline{U_2}}{\sqrt{\lambda_{u_2}}}\right)^2 + \left(\frac{U_3 - \overline{U_3}}{\sqrt{\lambda_{u_3}}}\right)^2 \leq 1$$

Represents the equation of an ellipsoid centered in $\left(\overline{U_1}, \overline{U_2}, \overline{U_3}\right)$ with the half-lengths of the principal axes being$\sqrt{\lambda_{u_i}}$, $i = (1,2,3)$.

Therefore, the ellipsoid represents the volume in velocity space of the realizable fluctuations of the turbulence. If $\underline{U}$ is joint normally distributed, the probability of $\underline{U}$ lying withing the ellipsoid is:

$$P(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\alpha} s^2 e^{-s^2/2} ds = 1 - Q\left(\frac{3}{2}, \frac{1}{2}\alpha^2\right) \quad (21)$$

"From the perspective of the shape of turbulent eddies, the bottom, left and right vertices of the Lumley triangle correspond to three-dimensional isotropic turbulence (−I2 = I3 = 0), two-component two-dimensional turbulence and one-component two-dimensional turbulence, respectively. On the other hand, from the perspective of the shape of the ellipsoid formed by the principal components of the Reynolds stresses, the characteristics of the realizations of turbulence states are briefly given in table 1. The bottom vertex of the Lumley triangle corresponds to the isotropic limit (figure 5), where the stress ellipsoid appears to be a sphere ($\sigma1 = \sigma2 = \sigma3$). Here, σ1−3 are the principal components of Reynolds stresses with respect to a set of coordinates (x1, x2, x3). The left-curved side of the Lumley triangle corresponding to the axisymmetric contraction limit makes the stress ellipsoid an oblate spheroid, because one principal component is smaller than the other two equal components ($\sigma1 = \sigma2 > \sigma3$). However, at the left vertex that refers to a two-component axisymmetric limit, the stress ellipsoid becomes a circular disc owing to two equal principal components with a vanishing third component ($\sigma1 = \sigma2$ and $\sigma3 = 0$). The right-curved side of the Lumley triangle corresponding to the axisymmetric expansion limit represents a prolate spheroid, because one principal component is larger than the other two equal components ($\sigma1 > \sigma2 = \sigma3$). Note that both axisymmetric contraction and expansion limits are symmetric about the line $I3 = 0$, termed the plane-strain limit (figure 5). Furthermore, on the top-linear side of the Lumley triangle representing the two-component limit, the stress ellipsoid forms an elliptical disc. The reason is that one principal component is larger than the other one together with a third component that disappears ($\sigma1 > \sigma2$ and $\sigma3 = 0$). The plane-strain limit intersects the two-component limit at a point, termed the two-component plane-strain limit. Moreover, the right vertex of the Lumley triangle signifies the one-component limit, where the stress ellipsoid appears to be a straight line owing to one finite component along with two other vanishing components ($\sigma1 > 0$, $\sigma2 = \sigma3 = 0$).”

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