Part 6: 1D Spatial and Time Series Spectra: Highlights

1. Most often spectra are obtained from single point time series measurements and transformed to 1D spatial spectra via Taylors frozen turbulence hypothesis; or in some cases 1D spatial measurements along a line.

2. In either case, the relations between 1D and 3D spectra have shown that the Kolmogorov hypotheses are valid not only for 3D spectra as originally hypothesized, but also for their 1D counterpart.

Herein, we review several techniques for obtaining 1D spectra:

1. Temporal using autocorrelation $f(t)$ (or convolution integral) and Taylor hypotheses.
2. Spatial using even autocorrelation $f(r)$ and homogeneous isotropic turbulence assumptions.
3. Spatial using odd autocorrelation $f(r)$ and nonhomogeneous non-isotropic assumptions.
4. Power Spectral Density (PSD) approach for the Fourier transform and then invert to get the temporal autocorrelation.

A model spectrum

An analytical model spectrum is used for the evaluation of the Kolmogorov hypotheses and the experimentally (or numerically) obtained spectrums:

$$E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3}f_L(\kappa L)f_\eta(\kappa\eta) \quad (2)$$

Where f_L and f_η are specified non-dimensional functions. The function f_L determines the shape of the energy-containing range, $f_L \to 1$ for $\kappa L \to \infty$, i.e., small $l = 2\pi/\kappa$. The function f_η determines the shape of the dissipation range, $f_\eta \to 1$ for $\kappa\eta \to 0$, i.e., large $l = 2\pi/\kappa$.

In the inertial subrange, both f_L and $f_\eta \rightarrow 1$ so that the Kolmogorov -5/3 spectrum is obtained with the constant C recovered.

The specification of f_L is

$$f_L(\kappa L) = \left(\frac{\kappa L}{[(\kappa L)^2 + c_L]^{\frac{1}{2}}} \right)^{\frac{5}{3}+p_0} \qquad (3)$$

p_0 is taken to be 2, and c_L is a positive constant. Clearly, $f_L \rightarrow 1$ for large κL, while the exponent $\frac{5}{3} + p_0$ leads to $E(\kappa) \propto \kappa^{p_0} = \kappa^2$ for small κL. Or for $p_0 = 4$ leads to $E(\kappa) \propto \kappa^4$ for small κL.[1]

The specification of f_η is

$$f_\eta(\kappa\eta) = exp\left\{-\beta \left\{\left[(\kappa\eta)^4 + {c_\eta}^4\right]^{1/4} - c_\eta\right\}\right\} \qquad (4)$$

Where β and c_η are positive constants. For $c_\eta = 0$: $f_\eta(\kappa\eta) = \exp(-\beta\kappa\eta)$. In either case, exponential decay is exhibited for large $\kappa\eta$.

Since the velocity field $\underline{u}(\underline{x})$ is infinitely differentiable, for large κ, the energy spectrum decays more rapidly than any power of κ, thus, exponential decay is used, as suggested by Kraichnan. Experiments support exponential decay with $\beta = 5.2$. However, the simplified exponential form with $c_\eta = 0$ departs from unity too rapidly for small $\kappa\eta$ and the value of β is constrained to be $\beta = 2.1$. These deficiencies are remedied by Eq. (4).

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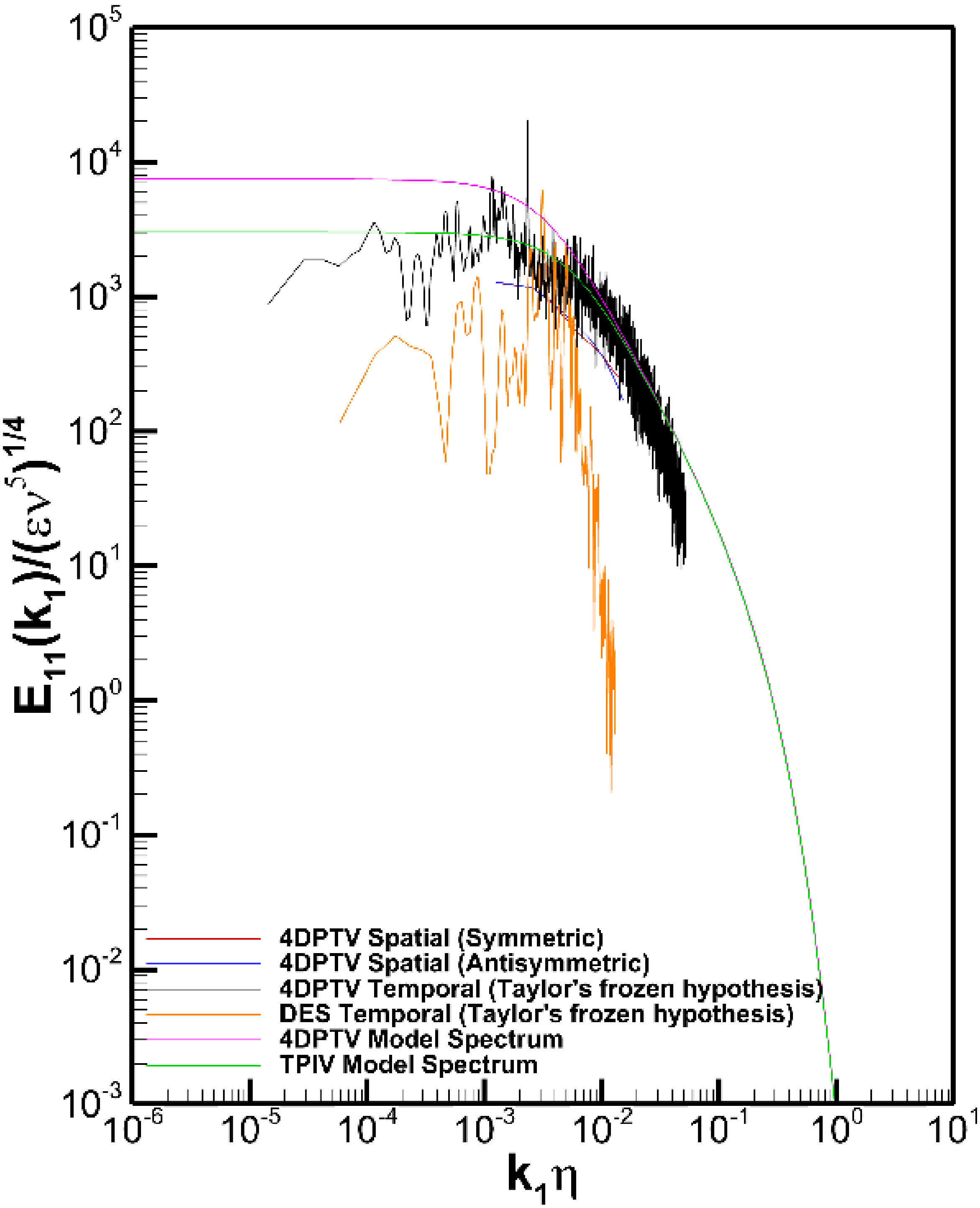
[1] Chapter 4 Part 5 pg. 7: If $\mathcal{E}_{ij}(\underline{\kappa})$ is analytic at $\underline{\kappa}$=0 then $E(\kappa)$ varies as κ^4 for small κ (Pope Ex. 6.26); however, it's also possible that it is non-analytic with $E(\kappa)$ varying as κ^2. DNS shows both behaviors and some grid turbulence data suggests κ^2 behavior.



Frederick Stern, Yugo Sanada, Zachary Starman, Shanti Bhushan, Christian Milano, "4DPTV Measurements and DES of the Turbulence Structure and Vortex-Vortex Interaction for 5415 Sonar Dome Vortices," 35th Symposium on Naval Hydrodynamics, Nantes, France, 7 July - 12 July 2024. Turbulence Analysis of SDVP for β=10° at x/L=0.12. 1D longitudinal velocity spectra shown for Kolmogorov scaling using macro-scale values.

Method 3: Energy spectrum from temporal autocorrelation $f(\tau)$

1. Calculate temporal autocorrelation.

$$R_E(\tau) = \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2}}$$

2. Obtain Fourier transform of $R_E(\tau)$.

$$\hat{R}_E(2\pi\omega) = 2\int_0^\infty R_E(\tau)\cos(2\pi\omega\tau)\,d\tau$$

(Note: ω: Frequency

3. Calculate the time micro and macro/integral scales.

$$\tau_E = [-2/f''(0)]^{1/2}, \qquad T = \int_0^\infty f(\tau)d\tau$$

4. Using Taylor hypothesis, calculate the Taylor microscale, dissipation, and Kolmogorov scale.

$$\lambda_f = \overline{U}\tau_E, \qquad \varepsilon = 30\nu\frac{\overline{u^2}}{\lambda_f^2}, \qquad \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

5. Calculate the 1D energy spectrum in time from the Fourier transform of $R_E(\tau)$.

$$\hat{E}_{11}(\omega) = 2\overline{u^2}\hat{R}_E(2\pi\omega)$$

6. Calculate the 1D energy spectrum in space from the 1D energy spectrum in time.

$$E_{11}(\kappa_1) = \frac{\overline{U}}{2\pi}\hat{E}_{11}(\omega)$$

7. Plot $E_{11}(\kappa_1)/(\varepsilon\nu^5)^{1/4}$ vs $\kappa_1\eta$

Details Method 3

Taylor's frozen turbulence hypothesis relates time sequence of streamwise velocity data to data distributed along a straight line in the flow direction as if the turbulent velocity field at a given instant in time convects downstream at the local mean velocity, i.e., as if it were frozen.

$$u(x, t+\tau) = u\left(x - \overline{U}\tau, t\right) \quad (1)$$

Where $x - \overline{U}\tau$ represents the upstream point, so that

$$\overline{u(x,t)u(x,t+\tau)} = \overline{u(x,t)u(x-\overline{U}\tau,t)} \quad (2)$$

Combining with $f(r)$ and $R_E(\tau)$ definitions

$$f(r) = \frac{\overline{u(x)u(x+r)}}{\overline{u^2(x)}} \quad (3)$$

$$R_E(\tau) = \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2}} \quad (4)$$

Taking $r = -\overline{U}\tau$ in Eq. (3)

$$f\left(-\overline{U}\tau\right) = \frac{\overline{u(x)u\left(x-\overline{U}\tau\right)}}{\overline{u^2(x)}}$$

Comparing with Eq. (4) and adding t dependence to Eq. (3) and x dependence to Eq. (4)

$$\underbrace{\frac{\overline{u(x,t)u(x,t+\tau)}}{\overline{u^2}}}_{R_E(\tau)} = \underbrace{\frac{\overline{u(x,t)u\left(x-\overline{U}\tau,t\right)}}{\overline{u^2}}}_{f\left(-\overline{U}\tau\right)} \quad (5)$$

Assuming zero separation $(x + r = 0,)$ and zero time-delay $(t + \tau = 0)$ and using the symmetry of $R_E(\tau)$ and $f(r)$ yields

$$x = \pm r, \qquad t = \pm\tau$$

Using these relations in Eqs. (3) and (4), including their t and x dependence

$$f(r = x) = \frac{\overline{u(x,t)u(0,t)}}{\overline{u^2}} \quad (6)$$

$$R_E(\tau = t) = \frac{\overline{u(x,t)u(x,0)}}{\overline{u^2}} \quad (7)$$

Comparing the RHS of Eq. (5) and (6), shows that

$$x = \overline{U}\tau \Rightarrow \tau = \frac{x}{\overline{U}} \quad (8)$$

Substituting Eq. (8) into (7) and equating to (6) yields

$$R_E\left(\tau = t = \frac{x}{\overline{U}}\right) = \frac{\overline{u\left(x, \frac{x}{\overline{U}}\right)u(x,0)}}{\overline{u^2}} = f(r = x) = \frac{\overline{u(x,t)u(0,t)}}{\overline{u^2}}$$

$$R_E\left(\frac{x}{\overline{U}}\right) = f(x)$$

Using the definitions of $E_{11}(k_1)$ and $\hat{R}_E(\omega')$

$$E_{11}(k_1) = \frac{\overline{u^2}}{\pi}\int_{-\infty}^{\infty} e^{-ik_1x} f(x)dx \qquad \hat{R}_E(\omega') = \int_{-\infty}^{\infty} e^{-i\tau\omega'} R_E(\tau)d\tau$$

$\omega' = 2\pi\omega$
ω Hz
ω' rad/s

$$f(x) = R_E\left(\frac{x}{\overline{U}}\right)$$

$$= \frac{\overline{u^2}}{\pi}\int_{-\infty}^{\infty} e^{-ik_1x} R_E\left(\frac{x}{\overline{U}}\right)dx$$

$$dx = \overline{U}d\tau$$

$$= \frac{\overline{u^2}}{\pi}\int_{-\infty}^{\infty} e^{-ik_1\overline{U}\tau} R_E(\tau)\overline{U}d\tau$$

From the comparison of the exponentials highlighted in yellow, the following relationship is obtained:
$k_1\overline{U}\tau = 2\pi\omega\tau \Rightarrow \omega = k_1\overline{U}/2\pi.$

$$= \frac{\overline{u^2}\overline{U}}{\pi}\int_{-\infty}^{\infty} e^{-i\tau\omega'} R_E(\tau)d\tau = \frac{\overline{u^2}\overline{U}}{\pi}\hat{R}_E(\omega') \quad (9)$$

Recall (Chapter 2)

$$\hat{E}_{11}(\omega) = 2\overline{u^2}\hat{R}_E(2\pi\omega) \quad (10)$$

Combining Eqs. (9) and (10), to obtain a relation between $\hat{E}_{11}(\omega)$ and $E_{11}(k_1)$:

$$E_{11}(k_1) = \frac{\overline{u^2}\overline{U}}{\pi}\frac{\hat{E}_{11}(\omega)}{2\overline{u^2}} = \frac{\overline{U}}{2\pi}\hat{E}_{11}(\omega)$$

Or equivalently

$$\hat{E}_{11}(\omega) = \frac{2\pi}{\overline{U}}E_{11}\left(\frac{2\pi\omega}{\overline{U}}\right)$$

Thus, $E_{11}(k_1)$ can be determined from measurements of $\hat{E}_{11}(\omega) = \hat{E}_{11}\left(\frac{k_1\overline{U}}{2\pi}\right)$.
Measured time spectra via Taylor hypothesis provide $E_{11}(k_1)$

Taylor micro-scale

$$f(x) = R_E\left(\tau = \frac{x}{\overline{U}}\right)$$

$$f' = R_{E,\tau}\frac{d\tau}{dx} = R_E{}'/\overline{U}$$

$$f'' = R_E{}''/\overline{U}^2$$

$$\lambda_f{}^2 = -\frac{2}{f''} = -\frac{2\overline{U}^2}{R_E{}''} = \overline{U}^2\lambda_t{}^2$$

Taylor macro-scale

$$\Lambda_f = \int_0^\infty f(x)dx$$

$$= \int_0^\infty R_E(\tau)\,\overline{U}d\tau$$

$$\Lambda_f = \overline{U}\Lambda_t$$