Chapter 4: Turbulence at Small Scales

Part 5: Relations between 1D and 3D spectra

Recall:

Note: $R_E(\tau)$ then $\hat{E}_{11}(\omega)$ can be obtained single point measurement (vs $E(\kappa, t)$ requires volume or line measurements) and if Taylor frozen turbulence hypothesis used can be transformed from time to space as approximation for spatial spectra.

Alternatively, 1D spatial spectra can be obtained directly using $f(r)$ or $g(r)$, e.g., using the former consider the Fourier transform pair:

$$E_{11}(k_1) = \frac{2}{\pi}\int_0^\infty \mathcal{R}_{11}(\hat{e_1}r_1)\cos(\kappa_1 r_1)\, dr_1$$

$$\mathcal{R}_{11}(\hat{e_1}r_1) = \frac{2}{\pi}\int_0^\infty E_{11}(\kappa_1)\cos(\kappa_1 r_1)\, d\kappa_1$$

Where $$\mathcal{R}_{11}(\hat{e_1}r_1) = \overline{u_1^2} f(r_1) = \overline{u(x)u(x+r_1)}$$

i.e.,

$$E_{11}(k_1) = \frac{2\overline{u_1^2}}{\pi}\int_0^\infty f(r_1)\cos(\kappa_1 r_1)\, dr_1$$

Note that the Kolmogorov hypotheses are for the 3D spectra $E(\kappa, t)$, which is difficult to obtain, whereas 1D spectra either point (temporal transformed to spatial using Taylor hypothesis) or line (spatial) are readily obtained; therefore, the relationship between 1D and 3D spectra is required.

Also recall, in homogeneous turbulence[1] the velocity(energy)-spectrum tensor and two-point velocity correlation tensor form a Fourier transom pair[2]

$$\mathcal{E}_{ij}(\underline{\kappa}) = \frac{1}{(2\pi)^3}\int_{\forall}\mathcal{R}_{ij}(\underline{r})e^{i\underline{\kappa}\cdot\underline{r}}\underline{dr} \quad (1)$$

$\Phi_{ij} = \mathcal{E}_{ij}\ [m^5/s^2]$

$f(t)$ implied

$$\mathcal{R}_{ij}(\underline{r}) = \int_{\forall}\mathcal{E}_{ij}(\underline{\kappa})e^{-i\underline{\kappa}\cdot\underline{r}}\underline{d\kappa} \quad (2)$$

Where $\underline{k} = (k_1, k_2, k_3)$ is the continuous wave-number vector and independent of $\underline{x}$ for homogeneous turbulence. $\mathcal{E}_{ij}(\underline{\kappa})$ is a complex quantity with the following properties:

$$\mathcal{E}_{ij}(\underline{\kappa}) = \mathcal{E}_{ji}{}^*(\underline{\kappa}) = \mathcal{E}_{ji}(-\underline{\kappa})$$

→ Due to symmetry of $\mathcal{R}_{ij}(\underline{r})$ and that $\mathcal{R}_{ij}(\underline{r})$ is real: Pope Ex.3.34

$$k_i\mathcal{E}_{ij}(\underline{\kappa}) = k_j\mathcal{E}_{ij}(\underline{\kappa}) = 0$$

→ Due to incompressibility: Pope Ex, 6.20; Proof in Appendix A.0

$$\mathcal{E}_{ij}(\underline{\kappa})Y_iY_j \geq 0 \quad all\ \ \underline{Y}$$

→ Positive semi-definite: Pope Ex. 6.21

Such that its trace is real and non-negative: $\mathcal{E}_{ii}(\underline{\kappa}) = \mathcal{E}_{ii}{}^*(\underline{\kappa}) \geq 0$.

$\mathcal{E}_{ij}(\underline{\kappa})$ represents the Reynolds stress density in $\underline{\kappa}$ space = contribution per unit ∀ in $\underline{\kappa}$ space from Fourier mode $e^{-i\underline{\kappa}\cdot\underline{r}}$ to the Reynolds stress, in particular for $\underline{r} = 0$ $\mathcal{R}_{ij}(0) = \overline{u_iu_j} = \int_{\forall}\mathcal{E}_{ij}(\underline{\kappa})\underline{d\kappa}$.

As already noted, $\mathcal{E}_{ij}$ has units of $u^2L^3 = m^5/s^2$.

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[1] For homogeneous turbulence $\mathcal{R}_{ij}(\underline{r})$ is not $f(\underline{x})$ and the information it contains can be re-expressed in terms of the wave number spectrum.
[2] Note: $\underline{dr}$ and $\underline{d\kappa}$ are not vectors but volumes in physical and wave number space.

Note:

i, j give the directions of the velocity in the physical space.

For example, $\mathcal{E}_{22}$ pertains to $u_2(\underline{x})$.

The wavenumber direction $\underline{\kappa}/|\underline{\kappa}|$ gives the direction in wave number space of the Fourier mode.

The wavenumber's magnitude determines the length scale (wavelength) of the Fourier mode: $l = 2\pi/|\underline{\kappa}|$.

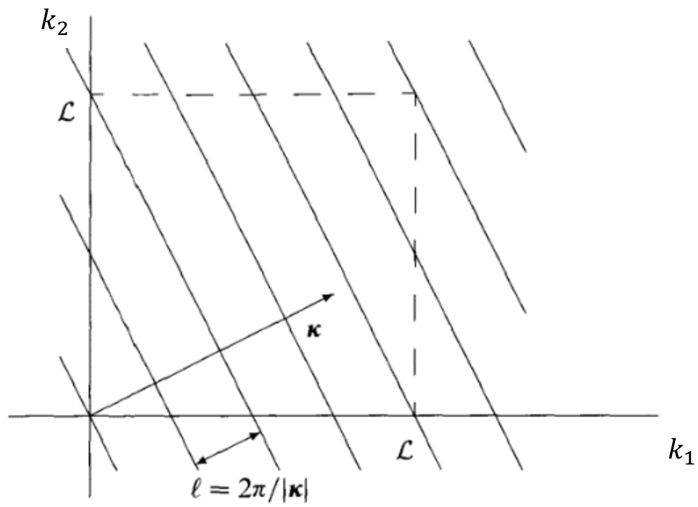


Fig. 6.8. A sketch of the Fourier mode corresponding to $\kappa = \kappa_0(4, 2, 0)$. The oblique lines show the crests, where $\Re(e^{i\kappa \cdot x}) = \cos \kappa \cdot x$ is unity.

$\mathcal{E}_{ij}(\underline{\kappa})$ also contains velocity derivative information, as per Part 1 and Pope Ex. 6.23:

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_k} = \int_\forall -i\kappa_k \mathcal{E}_{ij}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}$$

$$\frac{\partial^2 \mathcal{R}_{ij}}{\partial r_l \partial r_k} = \frac{\partial}{\partial r_l}\left[\int_\forall -i\kappa_k \mathcal{E}_{ij}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}\right]$$

$$\frac{\partial^2 \mathcal{R}_{ij}}{\partial r_l \partial r_k} = \int_\forall i^2 \kappa_k \kappa_l \mathcal{E}_{ij}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}$$

$$\overline{u_{i,k}u_{j,l}} = -\frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k \partial r_l}(0) = \int_{\forall} \kappa_k \kappa_l \mathcal{E}_{ij}(\underline{\kappa}, t) d\underline{\kappa}$$

For $j = i$ and $l = k$: $\varepsilon = \nu\overline{u_{i,k}u_{i,k}} = \int_{\forall} 2\nu\kappa^2 \frac{1}{2}\mathcal{E}_{ii}(\underline{\kappa})\underline{d\kappa} = -\frac{\partial^2 \mathcal{R}_{ii}}{\partial r_k \partial r_k}(0)$

The energy-spectrum function $E(\kappa)$ is obtained from $\mathcal{E}_{ij}(\underline{\kappa})$ by removing all directional information, i.e., the direction of the velocities is removed by considering $\frac{1}{2}\mathcal{E}_{ii}(\underline{\kappa})$ and the information about the direction of the Fourier modes is removed by integrating over all wavenumbers $\underline{\kappa}$ of magnitude $|\underline{\kappa}| = \kappa$, i.e., over $S(\kappa)$ which is a sphere centered at the origin with radius κ. $E(\kappa)$ has units m^3/s^2.

$$E(\kappa) = \oint \frac{1}{2}\mathcal{E}_{ii}(\underline{\kappa}) dS(\kappa) \quad (3)$$

$\oint dS(\kappa) = 4\pi\kappa^2$

Alternative formulation:

$$E(\kappa) = \int_{\forall} \frac{1}{2}\mathcal{E}_{ii}(\underline{\kappa})\delta(|\underline{\kappa}| - \kappa) d\kappa \quad (4)$$

Proof in Appendix A.1

The properties of $E(\kappa)$ follow from those of $\mathcal{E}_{ii}(\underline{\kappa})$: $E(\kappa)$ is real, non-negative.

Turbulent kinetic energy (Chapter 2):

$$k = \int_0^\infty E(\kappa) d\kappa$$

Dissipation (Chapter 4 Part 1):

$$\varepsilon = \int_{\forall} 2\nu\kappa^2 E(\kappa) d\kappa$$

$E(\kappa)d\kappa$ represents the contribution to k from all wavenumbers $\underline{\kappa}$ in the infinitesimal shell $\kappa \leq |\underline{\kappa}| < \kappa + d\kappa$.

In general, $\mathcal{E}_{ij}(\underline{\kappa})$ has more information than $E(\kappa)$ but for isotropic turbulence $\mathcal{E}_{ij}(\underline{\kappa})$ is completely determined by $E(\kappa)$ and directional information can only depend on $\underline{\kappa}$ and within scalar multiples. The only second order tensors that can be formed from $\underline{\kappa}$ are δ_{ij} and $\kappa_i \kappa_j$; consequently,

$$\mathcal{E}_{ij}(\underline{\kappa}) = A(\kappa)\delta_{ij} + B(\kappa)\kappa_i\kappa_j \quad (5)$$

$A(\kappa)$ and $B(\kappa)$ are determined using (1) incompressibility and (2) properties of $\oint dS(\kappa)$ and $\oint \kappa_i\kappa_j dS(\kappa)$.

(1) Apply incompressibility condition $\kappa_i\mathcal{E}_{ij}(\underline{\kappa}) = 0$ to Eq. (5).

$$\kappa_i\mathcal{E}_{ij}(\underline{\kappa}) = \kappa_i A(\kappa)\delta_{ij} + \kappa_i B(\kappa)\kappa_i\kappa_j = 0$$

$$\kappa_i A(\kappa)\delta_{ij} = -\kappa_i B(\kappa)\kappa_i\kappa_j$$

$$A(\kappa)\kappa_i\delta_{ij} = -B(\kappa)\kappa_i\kappa_i\kappa_j$$

$$A(\kappa)\kappa_j = -B(\kappa)\kappa^2\kappa_j$$

$$A(\kappa) = -B(\kappa)\kappa^2$$

$$B(\kappa) = -A(\kappa)/\kappa^2 \quad (6)$$

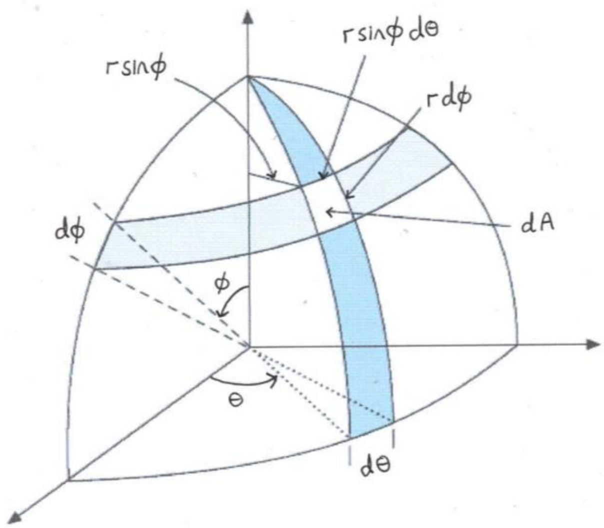
(2) Calculate the surface area of a sphere with a radius of $|\underline{\kappa}| = \kappa$:

$$\oint dS(\kappa) = \int_0^{\pi} 2\pi\kappa sin(\theta)\kappa d\theta$$

$$= -2\pi\kappa^2 cos(\theta)|_0^{\pi}$$

$$= 2\pi\kappa^2 - (-2\pi\kappa^2) = 4\pi\kappa^2$$

Alternately, the area element can be written as $dS(\kappa) = \kappa^2 sin\varphi d\theta d\varphi$ according to the following figure.



$$dA = rsin\phi d\theta \; rd\phi = r^2 sin\phi d\phi d\theta$$

$\underline{d\kappa}$ = $d\kappa_1 d\kappa_2 d\kappa_3$=d∀ (not vector)

$=\kappa^2 d\Omega d\,\kappa$

$d\Omega = \; sin\phi d\phi d\theta$

$dS(\kappa) = \kappa^2 d\Omega$

$$\oint dS(\kappa) = \int_0^{\pi}\int_0^{2\pi} \kappa^2 sin\varphi d\varphi d\theta = \int_0^{\pi} 2\pi\kappa^2 sin\varphi d\varphi = -2\pi\kappa^2 cos\varphi|_0^{\pi} = 4\pi\kappa^2$$

Note that for isotropic second order tensor, it can be written as a scalar multiple of δ_{ij} *(https://farside.ph.utexas.edu/teaching/336L/Fluid/node252.html)*,

$$\kappa_i\kappa_j = a_{ij} = \frac{1}{3}\kappa^2\delta_{ij}$$

$$\oint \kappa_i\kappa_j dS(\kappa) = \int_0^{\pi} \kappa_i\kappa_j 2\pi\kappa sin(\theta)\kappa d\theta = \int_0^{\pi} \frac{1}{3}\kappa^2\delta_{ij} 2\pi\kappa^2 sin(\theta)d\theta = \frac{4}{3}\pi\kappa^4\delta_{ij}$$

$$E(\underline{\kappa}) = \oint \tfrac{1}{2}\mathcal{E}_{ii}(\underline{\kappa})dS(\kappa) = \tfrac{1}{2}\mathcal{E}_{ii}(\underline{\kappa})\oint dS(\kappa) = \tfrac{1}{2}\mathcal{E}_{ii}(\underline{\kappa})4\pi\kappa^2 = 2\mathcal{E}_{ii}(\underline{\kappa})\pi\kappa^2 \quad (8)$$

$dS(\kappa)$ represents the surface of a sphere of radius $|\underline{\kappa}| = \kappa \rightarrow \mathcal{E}_{ii}(\underline{\kappa}) = \text{constant}$

Apply equation (5)

$$E(\underline{\kappa}) = 2\pi\kappa^2(A(\kappa)\delta_{ii} + B(\kappa)\kappa_i\kappa_i) = 2\pi\kappa^2[A(\kappa)(1+1+1) + B(\kappa)\kappa^2]$$

$$E(\kappa) = 6\pi\kappa^2 A(\kappa) + 2\pi\kappa^4 B(\kappa) \quad (9)$$

From Eq. (6) and (9), the coefficients $A(\kappa)$ and $B(\kappa)$ can be obtained,

$$A(\kappa) = \frac{E(\kappa)}{4\pi\kappa^2} \quad B(\kappa) = \frac{-E(\kappa)}{4\pi\kappa^4}$$

Substitute the coefficients $A(\kappa)$ and $B(\kappa)$ into equation (5),

$$\mathcal{E}_{ij}(\underline{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right) = \frac{E(\kappa)}{4\pi\kappa^2}P_{ij}(\kappa) \quad (10)$$

Where $P_{ij}(\kappa) = \delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}$ is referred to as the projection tensor and used later to write the Navier-Stokes equations in wavenumber space.

If $\mathcal{E}_{ij}(\underline{\kappa})$ is analytic at $\underline{\kappa}$=0 then $E(\kappa)$ varies as κ^4 for small κ (Pope Ex. 6.26); however, it's also possible that it is non-analytic with $E(\kappa)$ varying as κ^2. DNS shows both behaviors and some grid turbulence data suggests κ^2 behavior.

Recall definitions longitudinal and transverse correlation coefficients $f(r)$ and $g(r)$

$$\mathcal{R}_{11}(\hat{e_1}r_1, t) = \overline{u_1^2} f(r_1, t) \quad (11)$$

Longitudinal auto-correlation function

$$\mathcal{R}_{22}(\hat{e_1}r_1, t) = \overline{u_2^2} g(r_1, t) \quad (12)$$

Transverse auto-correlation function

One-dimensional spectra $E_{ij}(\kappa_1)$ are defined as two times the one-dimensional Fourier transform of $\mathcal{R}_{ij}(\hat{e_1}r_1)$. $E_{ij}(\kappa_1)$ has units m^3/s^2, i.e., same as $E(\kappa)$.

$$E_{ij}(\kappa_1) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathcal{R}_{ij}(\hat{e_1}r_1) e^{-i\kappa_1 r_1} dr_1 \quad (13)$$

$f(t)$ implied; m^3/s^2

Consider $i = j = 1$ as an example. $\mathcal{R}_{11}(\hat{e_1}r_1)$ is real and an even function of r_1 such that $E_{11}(\kappa_1)$ is also real and an even function of r_1.

$$E_{11}(\kappa_1) = \frac{2}{\pi}\int_{0}^{\infty} \mathcal{R}_{11}(\hat{e_1}r_1) \cos(\kappa_1 r_1)\, dr_1 \quad (14)$$

With the inversion formula

$$\mathcal{R}_{11}(\hat{e_1}r_1) = \int_{0}^{\infty} E_{11}(\kappa_1) \cos(\kappa_1 r_1)\, d\kappa_1 \quad (15)$$

The factor 2 in Eq. (13) is introduced so that (setting $r_1 = 0$ in Eq. (15)) we obtain

$$\mathcal{R}_{11}(0) = \overline{u_1^2} = \int_{0}^{\infty} E_{11}(\kappa_1) d\kappa_1$$

Recall Eq. (2) where $\mathcal{E}_{ij}(\underline{\kappa})$is the (3D) energy spectrum tensor

$$\mathcal{R}_{ij}(\underline{r}) = \int_{\forall} \mathcal{E}_{ij}(\underline{\kappa}) e^{-i\underline{\kappa}\cdot\underline{r}} \underline{d\kappa}$$

$\underline{d\kappa} = d\kappa_1 d\kappa_2 d\kappa_3$

For $i = j = 1$

$$\mathcal{R}_{11}(\hat{e_1} r_1) = \int_{-\infty}^{\infty} \left[\iint_{-\infty}^{\infty} \mathcal{E}_{11}(\underline{\kappa})\, d\kappa_2 d\kappa_3 \right] \cos(\kappa_1 r_1)\, d\kappa_1 \qquad (16)$$

$$= \int_{0}^{\infty} \left[\iint_{-\infty}^{\infty} 2\mathcal{E}_{11}(\underline{\kappa})\, d\kappa_2 d\kappa_3 \right] \cos(\kappa_1 r_1)\, d\kappa_1$$

$$= \int_{0}^{\infty} E_{11}(\kappa_1) \cos(\kappa_1 r_1)\, d\kappa_1$$

Where

$$E_{11}(\kappa_1) = 2 \iint_{-\infty}^{\infty} \mathcal{E}_{11}(\underline{\kappa})\, d\kappa_2 d\kappa_3 \qquad (17)$$

$E_{11}(\kappa_1)$ (1D spatial spectra) has contribution from all wavenumbers $\underline{\kappa}$ in the plane $\hat{e_1} \cdot \underline{\kappa} = \kappa_1$, such that $|\underline{\kappa}| > \kappa_1$, i.e., only greater than κ_1 but Fourier modes contributing to $E_{11}(\kappa_1)$ can be appreciably larger than κ_1.

The one-dimensional spectrum $E_{11}(\kappa_1)$ is related to the longitudinal auto-correlation function by

$$E_{11}(\kappa_1) = \frac{2}{\pi} \overline{u_1^2} \int_0^{\infty} f(r_1) \cos(\kappa_1 r_1)\, dr_1$$

Which is obtained from the combination of Eq. (11) and (14).

Note that for $\kappa_1 = 0$:

$$E_{11}(0) = \frac{2}{\pi}\overline{u_1^2}\int_0^{\infty} f(r_1)\,dr_1 = \frac{2}{\pi}\overline{u_1^2}L_{11}$$

Where L_{11} represents the longitudinal integral scale

$$L_{11} = \int_0^{\infty} f(r_1)\,dr_1 = \frac{\pi E_{11}(0)}{2\overline{u_1^2}}$$

Similar results can be obtained for E_{22} using the transverse auto-correlation function $g(r_1)$.

The relation between $E_{11}(\kappa_1)$ and $\mathcal{E}_{11}(\underline{\kappa})$ shown in Eq. (17) can also be written as

$$E_{11}(\kappa_1) = 2\iint_{-\infty}^{\infty} \frac{E(\kappa)}{4\pi\kappa^2}\left(1 - \frac{{\kappa_1}^2}{\kappa^2}\right) d\kappa_2 d\kappa_3 \qquad (18)$$

Where use is made of Eq. (10) for $i = j = 1$. The integration is over the plane of fixed κ_1, and the integrand is radially symmetric about the κ_1 axis.

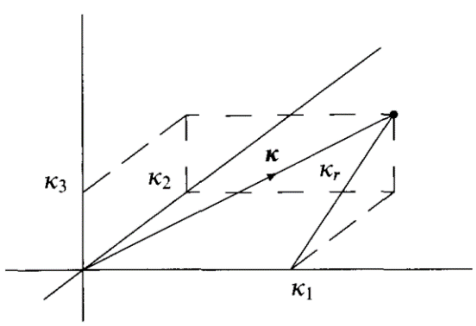
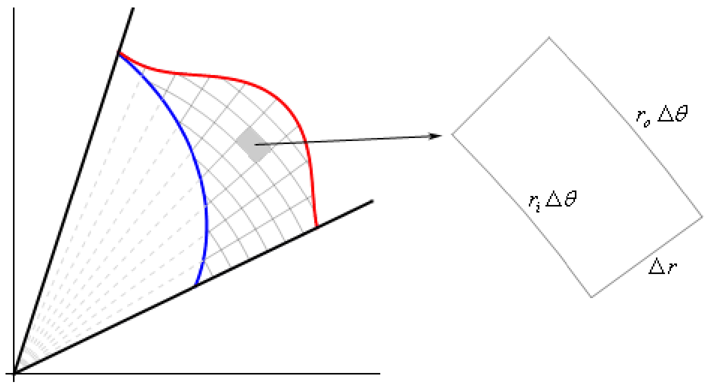


Fig. 6.10. A sketch of wavenumber space showing the definition of the radial coordinate κ_r.

Double Integrals in Polar Coordinates

https://tutorial.math.lamar.edu/classes/calciii/dipolarcoords.aspx



$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2$$

$$dA = r dr d\theta$$

$$\iint f(x,y)dA = \iint f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$E_{11}(\kappa_1) = 2\iint_{-\infty}^{\infty} \frac{E(\kappa)}{4\pi\kappa^2}\left(1 - \frac{{\kappa_1}^2}{\kappa^2}\right) d\kappa_2 d\kappa_3 \qquad (18)$$

Area element:

$$dS = d\kappa_2 d\kappa_3 = \kappa_r d\kappa_r d\theta$$

$$\iint_{-\infty}^{\infty} \frac{E(\kappa)}{2\pi\kappa^2}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa_2 d\kappa_3 =$$

$$\iint_{0,0}^{2\pi,\infty} \frac{E(\kappa)}{2\pi\kappa^2}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) \kappa_r d\kappa_r d\theta =$$

$$\int_0^{\infty} 2\pi \frac{E(\kappa)}{2\pi\kappa^2}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) \kappa_r d\kappa_r =$$

$$\int_0^{\infty} \frac{E(\kappa)}{2\pi\kappa^2}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) 2\pi\kappa_r d\kappa_r,$$

Since $\kappa_r^2 = \kappa^2 - \kappa_1^2$, changing variables for integration over κ instead of κ_r, and letting $\kappa_r = 0$ such that $\kappa = \kappa_1$ *and* $\kappa_r d\kappa_r = \kappa d\kappa$. Then Eq. (18) becomes:

$$\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{2\pi\kappa^2}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) 2\pi\kappa d\kappa$$

$$E_{11}(\kappa_1) = \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa \qquad (19)$$

Therefore, $E_{11}(\kappa_1)$ contains contribution from $\kappa > \kappa_1$, which is a phenomenon called aliasing.[3] Furthermore, $E_{11}(\kappa_1)$ is a monotonically decreasing function of κ_1, so that E_{11} is maximum at zero wavenumber, irrespective of the shape of $E(\kappa)$.

Proof in Appendix A.3

Recall

$$E_{11}(\kappa_1) = \frac{2}{\pi}\overline{u^2}\int_0^{\infty} f(r_1)cos(\kappa_1 r_1)dr_1 \qquad (20a)$$

$$E_{22}(\kappa_1) = \frac{2}{\pi}\overline{u^2}\int_0^{\infty} g(r_1)cos(\kappa_1 r_1)dr_1 \qquad (20b)$$

Note that both (17) [or (18)] and (19) are equivalent, i.e., representations in different coordinate systems and (19) is radially symmetric about the κ_1 axis. A statistically homogenous field is statistically invariant under translation. If the field is also statistically invariant under rotations and reflections of the coordinate system, then it is also statistically isotropic. Therefore, (19) is equivalent to (20a). Furthermore, inverting Eq. (19) to get Eq. (21), shows E_{11} also obeys a -5/3 law in the inertial range with different Kolmogorov constant.

[3] Aliasing in sampling is a type of measurement error that occurs when a signal is sampled at an insufficient rate, which results in a false lower frequency component, or alias, in the sampled data. In present context, κ_1 is affected by wave number > κ_1.

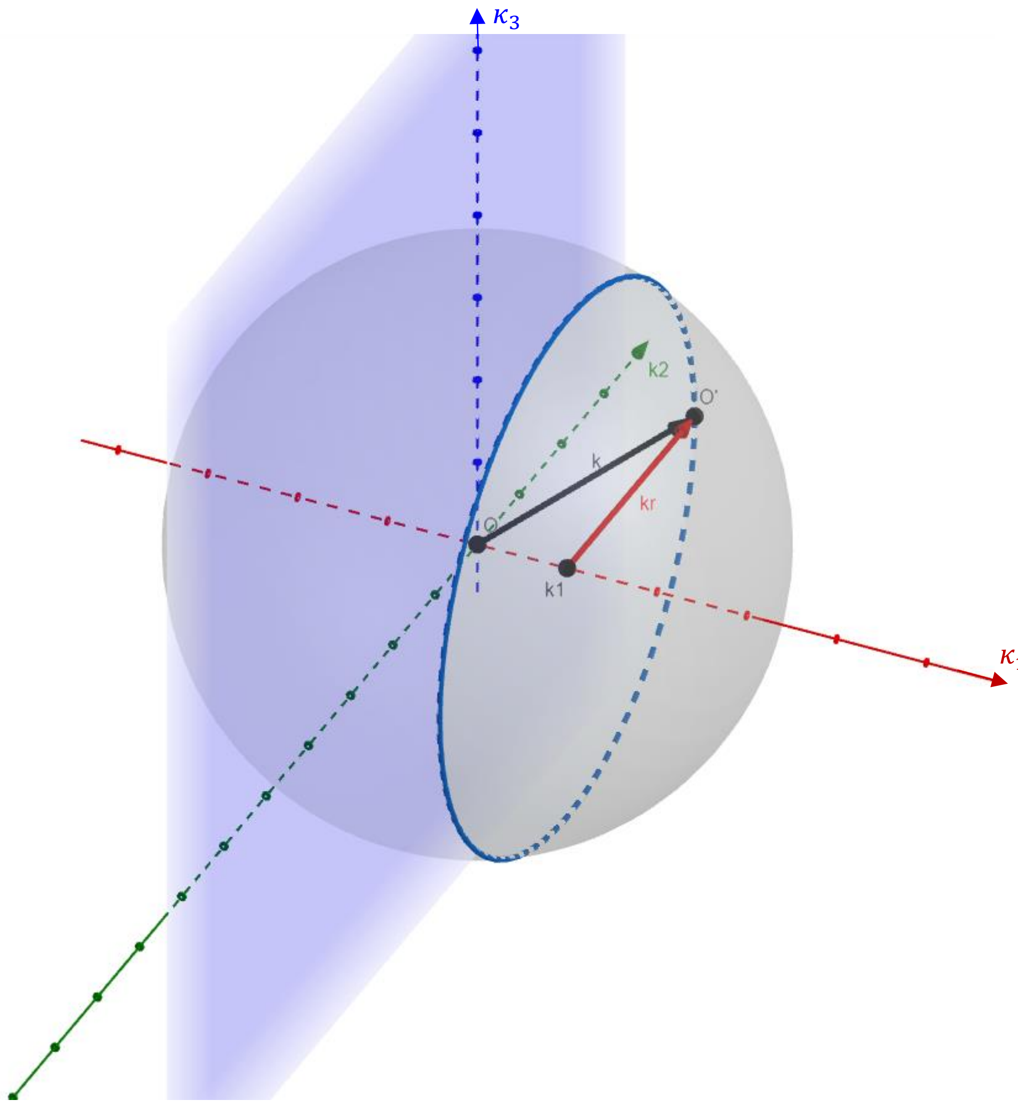


Figure 1: Representation in the wave number space of the different integrals:

- (16) integral over $\mathbb{R}^3$, equivalent to integration over sphere as $\kappa \to \infty$.
- (17) and (18) integral over $d\kappa_2 d\kappa_3$ plane intersecting κ_1, equivalent to integration over intersection between sphere and κ_1 plane as $\kappa \to \infty$.
- (19) integral over κ (black vector), when the radius of the sphere κ is $< \kappa_1$, the integral is not defined (i.e., no intersection of the radius of the sphere and the $d\kappa_2 d\kappa_3$ plane intersecting κ_1); once $\kappa = \kappa_1$ the integral is defined from κ_1 to ∞.
- (20a) integral over κ_1 axis from 0 to ∞.

Animation Play the animation for the parameter r to visualize

Inverting Eq. (19), gives the relation

$$E(\kappa) = \frac{1}{2}\kappa^3 \frac{d}{d\kappa}\left(\frac{1}{\kappa}\frac{dE_{11}(\kappa)}{d\kappa}\right) \quad (21)$$

Proof in Appendix A.2

Similarly, since (Proof Chapter 4, Part 2, A.6)

$$g(r) = f(r) + \frac{1}{2}r\frac{\partial f(r)}{\partial r}$$

then

$$E_{22}(\kappa_1) = E_{33}(\kappa_1) = \frac{1}{2}\left(E_{11}(\kappa_1) - \kappa_1\frac{dE_{11}(\kappa_1)}{d\kappa_1}\right)$$

Proof in Appendix A.4

And as per $\mathcal{R}_{22} = \mathcal{R}_{33}$, $E_{22} = E_{33}$; and are related to $E(\kappa)$ as follows:

$$E_{22}(\kappa_1) = E_{33}(\kappa_1) = \frac{1}{2}\int_{\kappa_1}^{\infty}\frac{E(\kappa)}{\kappa}\left(1 + \frac{\kappa_1^2}{\kappa^2}\right)d\kappa$$

Proof in Appendix A.5

$$E(\kappa) = -\kappa\left[\frac{dE_{22}(\kappa)}{d\kappa} + \int_{\kappa}^{\infty}\frac{1}{\kappa_1}\frac{dE_{22}(\kappa_1)}{d\kappa_1}d\kappa_1\right]$$

Proof in Appendix A.6

$$E(\kappa) = -\kappa\frac{d}{d\kappa}\left[\frac{1}{2}E_{ii}(\kappa)\right] = -\kappa\frac{d}{d\kappa}\left[\frac{1}{2}E_{11}(\kappa) + E_{22}(\kappa)\right]$$

Proof in Appendix A.7

$$L_{11} = \frac{\pi}{2\overline{u^2}}\int_0^{\infty}\frac{E(\kappa)}{\kappa}d\kappa$$

Proof in Appendix A.8

$$K = \int_0^{\infty}\left\{\frac{1}{2}E_{ii}(\kappa)\right\}d\kappa$$

Proof in Appendix A.9

Appendix A

A.0 (also see Chapter 4 Part 2 A.5)

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_j} = \overline{u_i(\underline{x})\underbrace{\frac{\partial u_j(x_j')}{\partial x_j'}}_{\boxed{\nabla\cdot\underline{u}}}\underbrace{\frac{\partial x_j'}{\partial r_j}}_{\boxed{1}}} = 0$$

(vector = divergence 2nd order tensor)

$$\mathcal{R}_{ij}(\underline{r}) = \int_{\forall} \mathcal{E}_{ij}(\underline{\kappa}) e^{-i\underline{\kappa}\cdot\underline{r}} \underline{d\kappa}$$

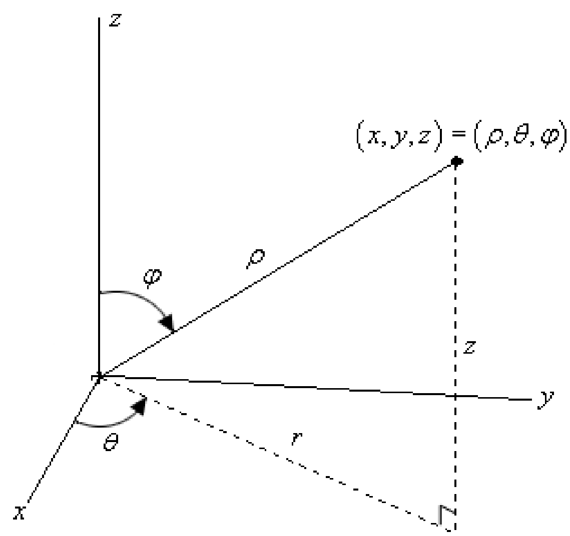
$$\frac{\partial \mathcal{R}_{ij}}{\partial r_j} = \int_{\forall} -i\kappa_j \mathcal{E}_{ij}(\underline{\kappa}) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa} = 0$$

$$\kappa_j \mathcal{E}_{ij}(\underline{\kappa}) = 0$$

$$\underline{\kappa}\cdot\underline{r} = \kappa_j r_j$$

A.1

The equivalency of equations (3) and (4) is shown as follows. Triple Integrals in Spherical Coordinates:
(https://tutorial.math.lamar.edu/classes/calciii/tisphericalcoords.aspx)



$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi$$
$$x^2 + y^2 + z^2 = \rho^2$$

$$dV = \rho^2 \sin\ \varphi d\rho d\theta d\varphi$$

$$\iiint f(x, y, z) dV = \iiint \rho^2 \sin\varphi\, f(\rho \sin\varphi \cos\theta\, , \rho \sin\varphi \sin\theta\, , \rho \cos\varphi) d\rho d\theta d\varphi$$

Using Spherical Coordinates, rewrite Eq. (4) as,

$$\mathrm{E}(k) = \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{1}{2} \mathcal{E}_{ii}(\underline{\kappa}) \delta(\rho - \kappa) \rho^2 \sin\varphi\, d\rho d\theta d\varphi$$

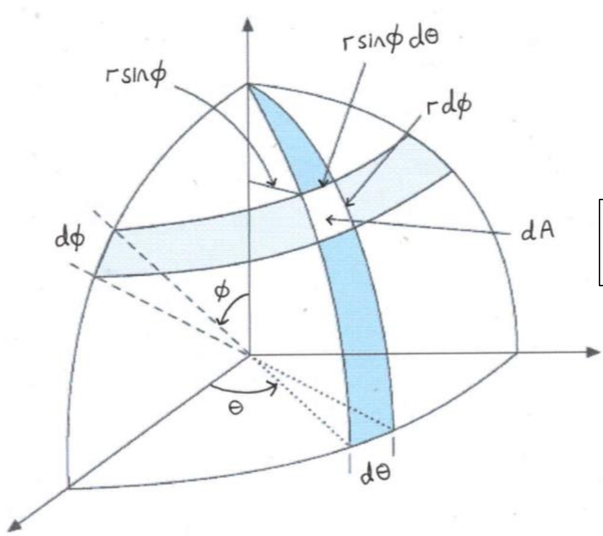
Apply

$$\int_{-\infty}^{\infty} \delta(x - a) g(x) dx = g(a)$$

$$\mathrm{E}(\kappa) = \int_0^{\pi} \int_0^{2\pi} \frac{1}{2} \mathcal{E}_{ii}(\underline{\kappa}) \kappa^2 \sin\varphi d\theta d\varphi$$

with $|\underline{\kappa}| = \kappa$.

Note the area $dS(\kappa) = \kappa^2 \sin\varphi d\theta d\varphi$



$$dA = r \sin\phi d\theta\; r d\phi = r^2 \sin\phi d\theta d\phi$$

$$E(\underline{\kappa}) = \oint \frac{1}{2} \mathcal{E}_{ii}(\underline{\kappa}) dS(\kappa)$$

A.2

$$E_{11}(\kappa_1) = \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa$$

$$\frac{dE_{11}}{d\kappa_1} = \int_{\kappa_1}^{\infty} \frac{-2\kappa_1 E(\kappa)}{\kappa^3} d\kappa = -2\kappa_1 \int_{\kappa_1}^{\infty} E(\kappa)\kappa^{-3} d\kappa \qquad (1A)$$

Using Leibniz theorem:

$$\frac{d}{da}\int_{p(a)}^{q(a)} f(x,a)dx = \int_{p}^{q} \frac{\partial f(x,a)}{\partial a} dx + f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da}$$

Where:

$$x = \kappa, a = \kappa_1$$

$$p = \kappa_1, q = \infty$$

$$\frac{d^2E_{11}}{d{\kappa_1}^2} = \underbrace{\int_{\kappa_1}^{\infty} \frac{-2E(\kappa)}{\kappa^3} d\kappa}_{\boxed{\frac{1}{\kappa_1}\frac{dE_{11}}{d\kappa_1}}} + \frac{2E(\kappa_1)}{{\kappa_1}^2}$$

$$\frac{d^2E_{11}}{d{\kappa_1}^2} = \frac{1}{\kappa_1}\frac{dE_{11}}{d\kappa_1} + \frac{2E(\kappa_1)}{{\kappa_1}^2}$$

Solving for $E(\kappa_1)$

$$E(\kappa_1) = \frac{{\kappa_1}^2}{2}\frac{d^2E_{11}}{d{\kappa_1}^2} - \frac{\kappa_1}{2}\frac{dE_{11}}{d\kappa_1}$$

Letting $\kappa_1 = \kappa$

$$E(\kappa) = \frac{1}{2}\kappa^3 \underbrace{\left[-\kappa^{-2}\frac{dE_{11}}{d\kappa} + \kappa^{-1}\frac{d^2E_{11}}{d\kappa^2}\right]}_{\boxed{\frac{d}{d\kappa}\left[\frac{1}{\kappa}\frac{dE_{11}}{d\kappa}\right]}}$$

$$E(\kappa) = \frac{1}{2}\kappa^3 \frac{d}{d\kappa}\left[\frac{1}{\kappa}\frac{dE_{11}}{d\kappa}\right]$$

A.3

$$\frac{dE_{11}}{d\kappa_1} = 0 \rightarrow \text{min or max}$$

$E_{11}(\kappa_1)$ is either a minimum or a maximum. However, as shown in Eq. 1A,

$$\frac{dE_{11}}{d\kappa_1} < 0$$

which indicates that $E_{11}(\kappa_1)$ is a decreasing function such that $k_1 = 0$ is a maximum.

A.4

$$g(r) = f(r) + \frac{1}{2} r \frac{\partial f(r)}{\partial r} \quad (2A)$$

Taking the cosine Fourier transform of Eq. (2A) yields

$$2 \int_0^\infty g(r_1) \cos(\kappa_1 r_1)\, dr_1 = 2 \int_0^\infty f(r_1) \cos(\kappa_1 r_1)\, dr_1 + \frac{1}{2} \left[\int_{-\infty}^\infty r_1 \frac{\partial f(r_1)}{\partial r_1} \cos(\kappa_1 r_1)\, dr_1 \right] \quad (3A)$$

Multiplying Eq. (3A) by $\overline{u^2}/\pi$ and using the definition of $E_{11}(\kappa_1)$ and $E_{22}(\kappa_1)$ in Eq. (20) yields

$$E_{22}(\kappa_1) = E_{11}(\kappa_1) + \frac{\overline{u^2}}{2\pi} \left[\int_{-\infty}^\infty r_1 \frac{\partial f(r_1)}{\partial r_1} \cos(\kappa_1 r_1)\, dr_1 \right] \quad (4A)$$

Considering the last term on the RHS of Eq. (4A)

$$\frac{\overline{u^2}}{2\pi} \int_{-\infty}^\infty r_1 \frac{\partial f(r_1)}{\partial r_1} \cos(\kappa_1 r_1)\, dr_1 = \frac{\overline{u^2}}{2\pi} \left[\int_{-\infty}^\infty \frac{\partial [r_1 f(r_1) \cos(\kappa_1 r_1)]}{\partial r_1} dr_1 - \int_{-\infty}^\infty f(r_1) \cos(\kappa_1 r_1)\, dr_1 - \int_{-\infty}^\infty \kappa_1 r_1 f(r_1) \sin(\kappa_1 r_1)\, dr_1 \right] \quad (5A)$$

And using the relation

$$\int_{-\infty}^{\infty} \frac{\partial[r_1 f(r_1)\cos(\kappa_1 r_1)]}{\partial r_1} dr_1 = [r_1 f(r_1)\cos(\kappa_1 r_1)]\,\big|_{\infty} - [r_1 f(r_1)\cos(\kappa_1 r_1)]\,\big|_{-\infty} = 0$$

Since $f(r)$ approaches 0 faster than $1/r$ (Pope Ex. 6.4 Solution), Eq. (5A) becomes

$$\frac{\overline{u^2}}{2\pi}\int_{-\infty}^{\infty} r_1 \frac{\partial f(r_1)}{\partial r_1}\cos(\kappa_1 r_1)\,dr_1 = -\frac{\overline{u^2}}{2\pi}\left[\int_{-\infty}^{\infty} f(r_1)\cos(\kappa_1 r_1)\,dr_1 + \int_{-\infty}^{\infty} \kappa_1 r_1 f(r_1)\sin(\kappa_1 r_1)\,dr_1\right] \qquad (6A)$$

Substituting Eq. (6A) into (4A)

$$E_{22}(\kappa_1) = E_{11}(\kappa_1) - \frac{\overline{u^2}}{2\pi}\left[\int_{-\infty}^{\infty} f(r_1)\cos(\kappa_1 r_1)\,dr_1 + \int_{-\infty}^{\infty} \kappa_1 r_1 f(r_1)\sin(\kappa_1 r_1)\,dr_1\right]$$

And using the definition of $E_{11}(\kappa_1)$

$$E_{22}(\kappa_1) = E_{11}(\kappa_1) - \frac{1}{2}E_{11}(\kappa_1) - \frac{1}{2}\kappa_1 \frac{dE_{11}(\kappa_1)}{d\kappa_1}$$

Results into

$$E_{22}(\kappa_1) = \frac{1}{2}\left(E_{11}(\kappa_1) - \kappa_1 \frac{dE_{11}(\kappa_1)}{d\kappa_1}\right) \qquad (7A)$$

A.5

$$E_{11}(\kappa_1) = \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa \qquad (8A)$$

Combining Eq. (7A) with (8A)

$$E_{22}(\kappa_1) = \left\{\left[\frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa\right] - \frac{1}{2}\kappa_1 \frac{d}{d\kappa_1}\left[\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa\right]\right\} \qquad (9A)$$

Focus on the last term in the RHS of Eq. (9A)

$$-\frac{1}{2}\kappa_1 \frac{d}{d\kappa_1}\left[\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa\right]$$

Using Leibniz theorem

$$-\frac{1}{2}\kappa_1 \left[\left(\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(-2\frac{\kappa_1}{\kappa^2}\right) d\kappa\right)\right] = \kappa_1 \left[\left(\int_{\kappa_1}^{\infty} \frac{E(\kappa)\kappa_1}{\kappa^3} d\kappa\right)\right] \qquad (10A)$$

And substituting Eq. (10A) into (9A)

$$E_{22}(\kappa_1) = \left\{\left[\frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa\right] + \kappa_1 \left[\left(\int_{\kappa_1}^{\infty} \frac{E(\kappa)\kappa_1}{\kappa^3} d\kappa\right)\right]\right\}$$

$$= \left\{\left[\frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right) d\kappa\right] + \left[\left(\int_{\kappa_1}^{\infty} \frac{E(\kappa){\kappa_1}^2}{\kappa^3} d\kappa\right)\right]\right\}$$

$$= \left\{\left[\frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 + \frac{\kappa_1^2}{\kappa^2}\right) d\kappa\right]\right\}$$

A.6

$$E_{22}(\kappa_1) = \frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa}\left(1 + \frac{\kappa_1^2}{\kappa^2}\right) d\kappa \quad 11(A)$$

Deriving (11A) with respect to κ_1 yields

$$\frac{dE_{22}(\kappa_1)}{d\kappa_1} = \frac{1}{2}\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa^3}(2\kappa_1) d\kappa - \frac{1}{2}\frac{E(\kappa_1)}{\kappa_1}\underbrace{\left(1 + \frac{\kappa_1^2}{\kappa_1^2}\right)}_{\boxed{2}}$$

$$\frac{dE_{22}(\kappa_1)}{d\kappa_1} = \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa^3}\kappa_1 d\kappa - \frac{E(\kappa_1)}{\kappa_1}$$

Where Leibniz theorem was used. The first term in the RHS can be rewritten using the fundamental theorem of calculus as

$$\int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa^3}\kappa_1 d\kappa = \int_{0}^{\infty} \frac{E(\kappa)}{\kappa^3}\kappa_1 d\kappa - \int_{0}^{\kappa_1} \frac{E(\kappa)}{\kappa^3}\kappa_1 d\kappa$$

And since $E(\kappa)$ approaches 0 faster than $\kappa \propto 1/r$ (similar reasoning Pope Ex. 6.4 Solution)

Taking a second derivative with respect to κ (assuming $\kappa = \kappa_1$) yields

$$\frac{d}{d\kappa}\left(\frac{dE_{22}(\kappa)}{d\kappa}\right) = \frac{d}{d\kappa}\int_{0}^{\infty} \frac{E(\kappa)}{\kappa^3}\kappa d\kappa - \frac{d}{d\kappa}\int_{0}^{\kappa_1} \frac{E(\kappa)}{\kappa^3}\kappa d\kappa - \frac{d}{d\kappa}\left(\frac{E(\kappa)}{\kappa}\right)$$

And since $E(\kappa)$ approaches 0 faster than $\kappa \propto 1/r$ (similar reasoning Pope Ex. 6.4 Solution)

$$\frac{d}{d\kappa}\left(\frac{dE_{22}(\kappa)}{d\kappa}\right) = \frac{E(\kappa_1)}{{\kappa_1}^2} - \frac{d}{d\kappa}\left(\frac{E(\kappa)}{\kappa}\right) \quad (12A)$$

The last term on the RHS can be decomposed applying the product rule for derivatives

$$-\frac{d}{d\kappa}\left(\frac{E(\kappa)}{\kappa}\right) = -\frac{1}{\kappa}\frac{dE(\kappa)}{d\kappa} + \frac{E(\kappa)}{\kappa^2}$$

And substituting into Eq. (12A), where we assumed that $\kappa = \kappa_1$ gives

$$\frac{d}{d\kappa}\left(\frac{dE_{22}(\kappa)}{d\kappa}\right) = -\frac{1}{\kappa}\frac{dE(\kappa)}{d\kappa}$$

Or equivalently

$$\frac{dE(\kappa)}{d\kappa} = -\kappa\frac{d}{d\kappa}\left(\frac{dE_{22}(\kappa)}{d\kappa}\right)$$

Integrating by parts

$$E(\kappa) = -\kappa\frac{dE_{22}(\kappa)}{d\kappa} + \int_0^\infty \frac{dE_{22}(\kappa)}{d\kappa}d\kappa$$

Substituting $\kappa_1 = \kappa$ in the last term of the RHS, and using the fundamental theorem of calculus

$$E(\kappa) = -\kappa\frac{dE_{22}(\kappa)}{d\kappa} + \int_0^\infty \frac{dE_{22}(\kappa_1)}{d\kappa_1}d\kappa_1 - \int_0^\kappa \frac{dE_{22}(\kappa_1)}{d\kappa_1}d\kappa_1 \quad (13A)$$

$$E(\kappa) = -\kappa\frac{dE_{22}(\kappa)}{d\kappa} + \cancel{E_{22}(\infty)} - \cancel{E_{22}(0)} - E_{22}(\kappa) + \cancel{E_{22}(0)}$$

$$E(\kappa) = -\kappa\frac{dE_{22}(\kappa)}{d\kappa} - \frac{\kappa}{\kappa}E_{22}(\kappa) = -\kappa\left(\frac{dE_{22}(\kappa)}{d\kappa} + \frac{1}{\kappa}E_{22}(\kappa)\right)$$

Focus on the last term

$$\frac{1}{\kappa}E_{22}(\kappa) = \frac{1}{\kappa_1}E_{22}(\kappa_1) = \frac{1}{\kappa_1}\int_\kappa^\infty dE_{22}(\kappa_1) = \int_\kappa^\infty \frac{1}{\kappa_1}dE_{22}(\kappa_1) = \int_\kappa^\infty \frac{1}{\kappa_1}\frac{dE_{22}}{d\kappa_1}d\kappa_1$$

Where the integrals do not start from 0 because κ_1 in undefined for $\kappa = 0$.

Therefore

$$E(\kappa) = -\kappa\left(\frac{dE_{22}(\kappa)}{d\kappa} + \int_0^\infty \frac{1}{\kappa_1}\frac{dE_{22}}{d\kappa_1}d\kappa_1\right)$$

A.7

$$E(\kappa) = \frac{1}{2}\kappa^3 \frac{d}{d\kappa}\left[\frac{1}{\kappa}\frac{dE_{11}}{d\kappa}\right] \quad (11A)$$

Assuming $\kappa_1 = \kappa$, Eq. (7A) becomes

$$E_{22}(\kappa) = E_{33}(\kappa) = \frac{1}{2}\left(E_{11}(\kappa) - \kappa\frac{dE_{11}(\kappa)}{d\kappa}\right) \quad (12A)$$

isotropy

Isolating $dE_{11}(\kappa)/d\kappa$ in Eq. (12A) yields

$$\frac{dE_{11}(\kappa)}{d\kappa} = \frac{E_{11}(\kappa) - 2E_{22}(\kappa)}{k}$$

Substituting this relation into Eq. (11A)

$$E(\kappa) = \frac{1}{2}\kappa^3 \frac{d}{d\kappa}\left[\frac{E_{11}(\kappa) - 2E_{22}(\kappa)}{k^2}\right] \quad (13A)$$

Calculating the derivative in Eq. (13A)

$$E(\kappa) = \frac{1}{2}\frac{\kappa^3}{\kappa^4}\left[\left(\frac{dE_{11}(\kappa)}{d\kappa} - 2\frac{dE_{22}(\kappa)}{d\kappa}\right)\kappa^2 - 2\big(E_{11}(\kappa) - 2E_{22}(\kappa)\big)\kappa\right]$$

$$E(\kappa) = \frac{1}{2\kappa}\left[\left(\frac{dE_{11}(\kappa)}{d\kappa} - 2\frac{dE_{22}(\kappa)}{d\kappa}\right)\kappa^2 - 2\big(E_{11}(\kappa) - 2E_{22}(\kappa)\big)\kappa\right] \quad (14A)$$

Substituting Eq. (12A) for $E_{22}(\kappa)$ into (14A) yields

$$E(\kappa) = \frac{1}{2\kappa}\left\{\left(\frac{dE_{11}(\kappa)}{d\kappa} - 2\frac{dE_{22}(\kappa)}{d\kappa}\right)k^2 - 2\left[\cancel{E_{11}(\kappa)} - \left(\cancel{E_{11}(\kappa)} - \kappa\frac{dE_{11}(\kappa)}{d\kappa}\right)\right]\kappa\right\}$$

$$E(\kappa) = \frac{1}{2}\left\{\kappa\frac{dE_{11}(\kappa)}{d\kappa} - 2\kappa\frac{dE_{22}(\kappa)}{d\kappa} - 2\kappa\frac{dE_{11}(\kappa)}{d\kappa}\right\}$$
$$= \frac{1}{2}\left\{-\kappa\frac{dE_{11}(\kappa)}{d\kappa} - 2\kappa\frac{dE_{22}(\kappa)}{d\kappa}\right\}$$

$$E(\kappa) = -\frac{1}{2}\kappa\frac{d}{d\kappa}\left(E_{11}(\kappa) + 2E_{22}(\kappa)\right)$$

$$E(\kappa) = -\kappa\frac{d}{d\kappa}\left(\frac{1}{2}E_{11}(\kappa) + E_{22}(\kappa)\right) \quad (15A)$$

Due to isotropy,

$$E_{22}(\kappa) = E_{33}(\kappa) \Rightarrow E_{22}(\kappa) = \frac{1}{2}\left(E_{22}(\kappa) + E_{33}(\kappa)\right) \quad (16A)$$

Substituting Eq. (16A) into (15A)

$$E(\kappa) = -\frac{1}{2}\kappa\frac{d}{d\kappa}\left(E_{11}(\kappa) + E_{22}(\kappa) + E_{33}(\kappa)\right) = -\frac{1}{2}\kappa\frac{dE_{ii}}{d\kappa}$$

A.8

The definition of the longitudinal integral scale is

$$L_{11} = \int_0^\infty f(r_1)dr_1 = \frac{\pi E_{11}(0)}{2\overline{u^2}} \quad (17A)$$

The relation between $E_{11}(\kappa_1)$ and $E(\kappa)$ is

$$E_{11}(\kappa_1) = \int_{\kappa_1}^\infty \frac{E(\kappa)}{\kappa}\left(1 - \frac{\kappa_1^2}{\kappa^2}\right)d\kappa \quad (18A)$$

And evaluating it at $\kappa_1 = 0$ yields

$$E_{11}(0) = \int_0^\infty \frac{E(\kappa)}{\kappa}d\kappa \quad (19A)$$

Substituting Eq. (19A) into (17A)

$$L_{11} = \frac{\pi E_{11}(0)}{2\overline{u^2}} = \frac{\pi}{2\overline{u^2}}\int_0^\infty \frac{E(\kappa)}{\kappa}d\kappa$$

A.9

$$E(\kappa) = -\frac{1}{2}\kappa\frac{dE_{ii}}{d\kappa}$$

Integrating with respect to κ between 0 and ∞

$$\int_0^\infty E(\kappa)\,d\kappa = \int_0^\infty -\kappa\frac{d}{d\kappa}\left[\frac{1}{2}E_{ii}(\kappa)\right]d\kappa$$

$$K = -\int_0^\infty \left\{\frac{d}{d\kappa}\left[\kappa\frac{1}{2}E_{ii}(\kappa)\right] - \frac{1}{2}E_{ii}(\kappa)\right\}d\kappa$$

Where:

$$\int_0^\infty \frac{d}{d\kappa}\left[\kappa\frac{1}{2}E_{ii}(\kappa)\right]d\kappa = \left[\kappa\frac{1}{2}E_{ii}(\kappa)\right]_0^\infty$$

$\left[\kappa\frac{1}{2}E_{ii}(\kappa)\right]_0 = 0$ since $\kappa = 0$ and $E_{ii}(0)$ is a finite quantity

$\left[\kappa\frac{1}{2}E_{ii}(\kappa)\right]_\infty = 0$ since $E_{ii}(\kappa)$ approaches 0 faster than $\kappa \propto 1/r$ (similar reasoning Pope Ex. 6.4 Solution)

Therefore,

$$K = \int_0^\infty \left\{\frac{1}{2}E_{ii}(\kappa)\right\}d\kappa$$

$$E_{22}(\kappa_1) = \left\{\left[\frac{1}{2}\int_{\kappa_1}^{\infty}\frac{E(\kappa)}{\kappa}\left(1+\frac{\kappa_1^2}{\kappa^2}\right)d\kappa\right]\right\}$$

$$\frac{d}{da}\int_{p(a)}^{q(a)} f(x,a)dx = \int_p^q \frac{\partial f(x,a)}{\partial a}dx + f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da}$$

$$\frac{dE_{22}}{d\kappa_1} = \frac{1}{2}\int_{\kappa_1}^{\infty}\frac{E(\kappa)}{\kappa}\left(\frac{2\kappa_1}{\kappa^2}\right)d\kappa - \frac{E(\kappa_1)}{\kappa_1}$$

$$E(\kappa_1) = \frac{1}{2}\kappa_1\int_{\kappa_1}^{\infty}\frac{E(\kappa)}{\kappa}\left(\frac{2\kappa_1}{\kappa^2}\right)d\kappa - \kappa_1\frac{dE_{22}}{d\kappa_1}$$

$$E(\kappa_1) = {\kappa_1}^2\int_{\kappa_1}^{\infty}\frac{E(\kappa)}{\kappa^3}d\kappa - \kappa_1\frac{dE_{22}}{d\kappa_1}$$