Chapter 4: Turbulence at Small Scales

Part 4: Inertial Subrange

Recall:

$$K(t) = \int_0^\infty E(k,t)dk$$

$E(k,t)$ shows how the TKE is distributed among the different scales of the flow.

k^{-1} = length scale of eddy associated with wave number k.

Richardson cascade, Kolmogorov hypotheses, theory of isotropic turbulence and dimensional analysis lead to the “most famous and prominent” feature of high Re turbulence: the universal power law form of the energy spectrum in the inertial subrange.

1. **Kolmogorov’s first similarity hypothesis.**
   In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions ($l < l_e$) have a universal form that is uniquely determined by ν and ε. Note *e* = EI.

   In the universal equilibrium range, the turbulence is isotropic:
   $l < l_e, i.e., k > k_e = \frac{2\pi}{l_e}$

$$E = E(k, \varepsilon, \nu) \quad [m^3/s^2]$$

Using ε, ν to non-dimensionalize E and k:

$$E(k) = (\varepsilon\nu^5)^{1/4}\varphi(k\eta) = \eta {u_\eta}^2 \varphi(k\eta)$$

Where $u_\eta = v_d$ and $\varphi(k\eta) =$ *Kolmogorov spectrum function*.

Alternatively, using ε, k to non-dimensionalize E:

$$E(k) = \varepsilon^{2/3} k^{-5/3} \Psi(k\eta)$$

Where $\Psi(k\eta) =$ *compensated Kolmogorov spectrum function*.

$$\Psi(k\eta) = (k\eta)^{5/3}\varphi(k\eta)$$

And $k\eta > 2\pi\eta / l_e$.

2. **Kolmogorov's second similarity hypothesis**
   In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale l in the range $l_d < l < l_e$ have a universal form that is uniquely determined by ε, independent of ν.

$$\eta \ll l \ll l_0$$
$$l_d < l < l_e \sim \frac{1}{6} l_0$$

Or in terms of $k\eta$:

$$1 \gg k\eta \gg \eta / l_0$$

$$k_d \eta = \frac{2\pi\eta}{l_d} > k\eta > \frac{2\pi\eta}{l_e} = k_e \eta$$

In the inertial subrange $E(k) = f(\varepsilon)$ only; thus, for $k\eta \ll 1, \Psi$ becomes independent of $k\eta, i.e., =$ constant=C.

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$ Kolmogorov -5/3 spectrum

C~1.5 = Kolmogorov universal constant.

$E(k)$ is a power law spectrum $= CAk^{-p}$, where $p = 5/3,\ A = \varepsilon^{2/3}$.

Most turbulence data come from stationary single point time series, which are converted to spatial data using Taylor's frozen turbulence hypothesis to obtain one-dimensional spectra that can be related to the 3D spectra using theory of isotropic turbulence (tensors).

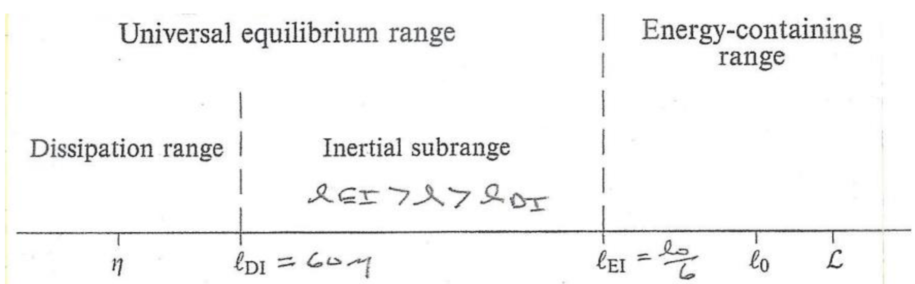

Fig. 6.1. Eddy sizes ℓ (on a logarithmic scale) at very high Reynolds number, showing the various lengthscales and ranges.

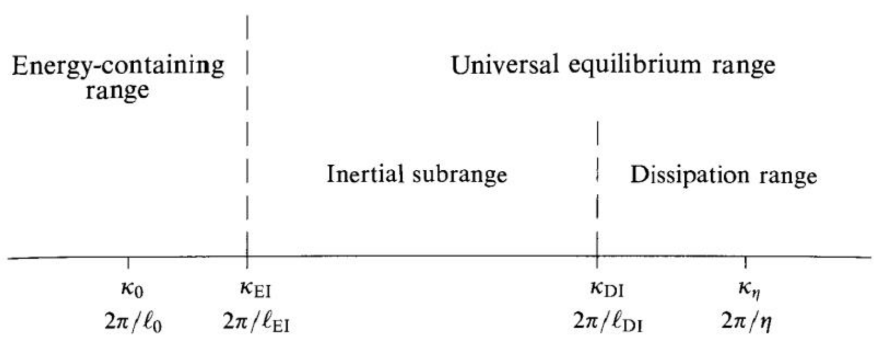

Fig. 6.12. Wavenumbers (on a logarithmic scale) at very high Reynolds number showing the various ranges.

Hidden turbulence in van Gogh's *The Starry Night*

Cite as: Phys. Fluids **36**, 095140 (2024); doi: 10.1063/5.0213627
Submitted: 12 April 2024 · Accepted: 31 July 2024 ·
Published Online: 17 September 2024

Yinxiang Ma (马寅翔),[1,a)] Wanting Cheng (程婉婷),[2] Shidi Huang (黄仕迪),[2] François G. Schmitt,[3] Xin Lin (林昕),[1] and Yongxiang Huang (黄永祥)[1,4,5]

AFFILIATIONS

[1]State Key Laboratory of Marine Environmental Science & College of Ocean and Earth Sciences, Xiamen University, Xiamen, China
[2]Center for Complex Flows and Soft Matter Research and Department of Mechanics and Aerospace Engineering, Southern University of Science and Technology, Shenzhen, Guangdong, China
[3]CNRS, Univ. Lille, Univ. Littoral Cote d'Opale, UMR 8187, LOG, Laboratoire d'Océanologie et de Géosciences, F 62930 Wimereux, France
[4]Fujian Engineering Research Center for Ocean Remote Sensing Big Data, Xiamen University, Xiamen, China
[5]Center for Marine Meteorology and Climate Change, Xiamen University, Xiamen, China

Note: This paper is part of the special topic, K. R. Sreenivasan: A Tribute on the occasion of his 75th Birthday.
[a)]**Author to whom correspondence should be addressed:** yongxianghuang@gmail.com and yongxianghuang@xmu.edu.cn

ABSTRACT

Turbulent skies have often inspired artists, particularly in the iconic swirls of Vincent van Gogh's *The Starry Night*. For an extended period, debate has raged over whether the flow pattern in this masterpiece adheres to Kolmogorov's theory of turbulence. In contrast to previous studies that examined only part of this painting, all and only the whirls/eddies in the painting are taken into account in this work, following the Richardson–Kolmogorov's cascade picture of turbulence. Consequently, the luminance's Fourier power spectrum spontaneously exhibits a characteristic $-5/3$ Kolmogorov-like power-law. This result suggests that van Gogh had a very careful observation of real flows, so that not only the sizes of whirls/eddies in *The Starry Night* but also their relative distances and intensity follow the physical law that governs turbulent flows. Moreover, a "–1"-like power-law persists in the spectrum below the scales of the smallest whirls, hinting at Batchelor-type scalar turbulence with a high Schmidt number. Our study, thus, unveils the hidden turbulence captured within *The Starry Night*.

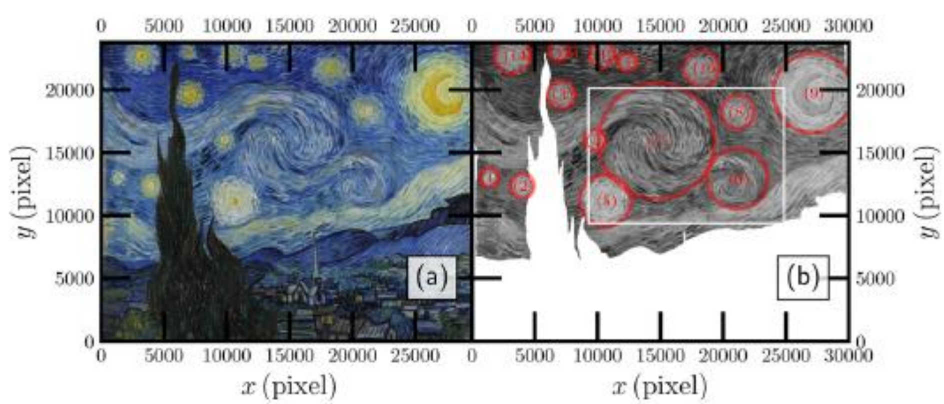

FIG. 2. (a) A high-resolution van Gogh's ***The Starry Night*** obtained from https://artsandculture.google.com with a size of 92.1 cm × 73.7 cm and 30 000 pixel ×23 756 pixel. Visually, the sky seems to be flowing with swirling eddies. (b) Gray version of the ***The Starry Night***, where the region studied by Finlay[18] is illustrated by a white square. The non-flow part is masked out manually. The whirls/eddies are recognized by naked eyes.

Luminance measures the amount of light emitted, passed through, or reflected from a particular area. It indicates how bright a surface will appear to the human eye. It is closely tied to a surface's physical characteristics and is essential because it affects how we see and understand the world.

The Wiener–Khinchine theorem:

This theorem states that, for the luminance θ (e.g., the gray-scale field Y defined above), its Fourier power spectrum $E_\theta(k)$ and the autocorrelation function $\rho_\theta(r)$ are a Fourier transform pair, which are written as follows:

$$E_\theta(k) = \int \rho_\theta(r) \exp(-j2\pi kr)\ \mathrm{d}r,\ \ \rho_\theta(r) = \int E_\theta(k) \exp(j2\pi kr)\ \mathrm{d}k, \tag{6}$$

where $j = \sqrt{-1}$ is a complex unit, $k = 1/r$ is the wavenumber, and r is the distance between two points in the physical space. The autocorrelation function is defined as $\rho_\theta(r) = \langle \theta'(x+r)\theta'(x)\rangle$, in which $\theta'(x) = \theta(x) - \langle\theta\rangle$ is the scalar variation in space and $\langle\cdot\rangle$ means ensemble average. $\rho_\theta(r)$ can be estimated when there are missing data, and in such case, an additional step is involved to correct the missing data effect; see detail of this algorithm in Ref. 45. In the case of scale invariance, one expects a power-law behavior of $E_\theta(k)$ written as follows:

$$E_\theta(k) \propto k^{-\beta_\theta}, \tag{7}$$

where $\beta_\theta > 0$ is the scaling exponent that can be determined experimentally or through theoretical considerations; for example, $\beta = 5/3$ for the velocity spectrum of high Reynolds number flows.[1,22,46]

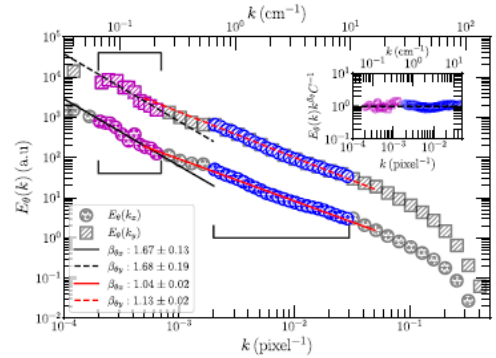

FIG. 3. Experimental Fourier power spectrum $E_\theta(k)$, where the black and red lines indicate the power-law behaviors in the ranges 6.67×10^{-2} cm^{-1} $\leq k \leq 2.33 \times 10^{-1}$ cm^{-1} (i.e., 2×10^{-4} pixel^{-1} $\leq k \leq 7 \times 10^{-4}$ pixel^{-1}) and 6.67×10^{-1} cm^{-1} $\leq k \leq 10$ cm^{-1} (i.e., 2×10^{-3} pixel^{-1} $\leq k \leq 3 \times 10^{-2}$ pixel^{-1}), respectively. For clarity, the curve $E_\theta(k_y)$ has been shifted up by multiplying a factor of 10. The inset shows the compensated curves $E_\theta(k)k^{\beta_\theta}C^{-1}$ using the corresponding scaling exponents β_θ and prefactors C to emphasize the power-law behaviors.

Before making the analysis, the original image is converted from the red-green-blue scale to the gray-scale using the following formula:

$$Y = 0.2125R + 0.7154G + 0.0721B, \tag{5}$$

where R, G, and B represent the intensity for each color channel. The function color.rgb2gray from the Python scikit-image package is utilized for this transformation, which can well preserve the flow structures.[44] In addition, the church, mountain, and village are masked out to exclude the potential influence of these non-flow-like elements, see Fig. 2(b). The so-obtained gray-scale field is subsequently treated as a passive scalar field for the following analysis.

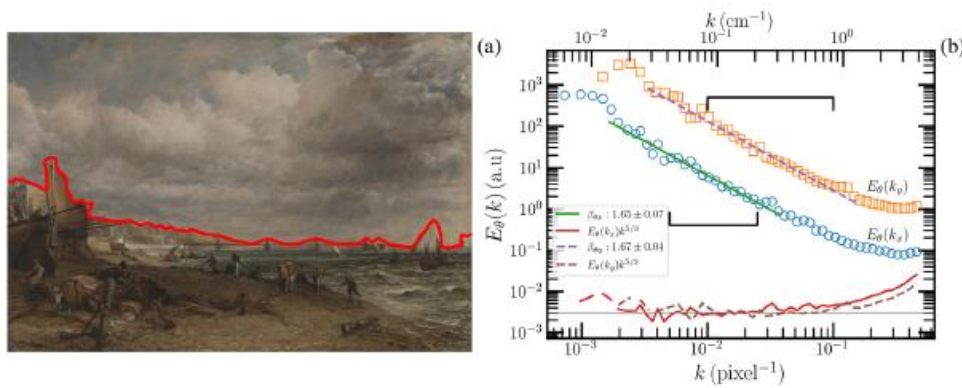

FIG. 7. (a) *Chain Pier, Brighton* painted by John Constable in 1827, obtained from https://www.tate.org.uk. The land and the cloud sky are separated by the red line. (b) Experimental Fourier power spectrum $E_\theta(k)$ of *Chain Pier, Brighton*. The green and purple dashed lines indicate power-law behaviors in the range 5×10^{-3} pixel^{-1} $\leq k \leq 2.5 \times 10^{-2}$ pixel^{-1} (i.e., 4.2×10^{-2} cm^{-1} $\leq k \leq 2.1 \times 10^{-1}$ cm^{-1}) and 10^{-2} pixel^{-1} $\leq k \leq 10^{-1}$ pixel^{-1} (i.e., 8.3×10^{-2} cm^{-1} $\leq k \leq 8.3 \times 10^{-1}$ cm^{-1}) for the data in the horizontal and vertical directions, respectively. For display clarity, the curve of $E_\theta(k_y)$ has been shifted up vertically by multiplying a factor of ten. The red solid and brown dashed lines are compensated curves $E_\theta(k)k^{5/3}$ to highlight the $-5/3$ scaling.